

# An All-But-One Entropic Uncertainty Relation, and Application to Password-Based Identification

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## Abstract

Entropic uncertainty relations are quantitative characterizations of Heisenberg's uncertainty principle, which make use of an entropy measure to quantify uncertainty. In quantum cryptography, they are often used as convenient tools in security proofs.

We propose a new entropic uncertainty relation. It is the first such uncertainty relation that lower bounds the uncertainty in the measurement outcome for *all but one* choice for the measurement from an *arbitrarily large* (but specifically chosen) set of possible measurements, and, at the same time, uses the *min-entropy* as entropy measure, rather than the Shannon entropy. This makes it especially suited for quantum cryptography.

As application, we propose a new *quantum identification scheme* in the bounded-quantum-storage model. Because the scheme requires a perfectly single-qubit source to operate securely, it is currently mainly of theoretical interest. Our new uncertainty relation forms the core of the new scheme's security proof in the bounded-quantum-storage model. In contrast to the original quantum identification scheme proposed by Damgård *et al.*, our new scheme also offers some security in case the bounded-quantum-storage assumption fails to hold. Specifically, our scheme remains secure against an adversary that has unbounded storage capabilities but is restricted to non-adaptive single-qubit operations. The scheme by Damgård *et al.*, on the other hand, completely breaks down under such an attack.

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# 1 Introduction

## 1.1 A New Uncertainty Relation

In this work, we propose and prove a new general entropic uncertainty relation. Uncertainty relations are quantitative characterizations of the uncertainty principle of quantum mechanics, which expresses that for certain pairs of measurements, there exists no state for which the measurement outcome is determined for *both* measurements: at least one of the outcomes must be somewhat uncertain. *Entropic* uncertainty relations express this uncertainty in at least one of the measurement outcomes by means of an entropy measure, usually the Shannon entropy. Our new entropic uncertainty relation distinguishes itself from previously known uncertainty relations by the following collection of features:

1. It uses the *min-entropy* as entropy measure, rather than the Shannon entropy. Such an uncertainty relation is sometimes also called a *high-order* entropic uncertainty relation.<sup>1</sup> Since privacy amplification needs a lower bound on the min-entropy, high-order entropic uncertainty relations are useful tools in quantum cryptography.
2. It lower bounds the uncertainty in the measurement outcome for *all but one* measurement, chosen from an *arbitrary* (and arbitrarily large) family of possible measurements. This is clearly *stronger* than typical entropic uncertainty relations that lower bound the uncertainty on *average* (over the choice of the measurement).
3. The measurements can be chosen to be qubit-wise measurements, in the computational or Hadamard basis, and thus the uncertainty relation is applicable to practical schemes (which can be implemented using current technology).

To the best of our knowledge, no previous entropic uncertainty relation satisfies (1) and (2) simultaneously, let alone in combination with (3). Indeed, as pointed out in a recent overview article by Wehner and Winter [WW10], little is known about entropic uncertainty relations for more than two measurement outcomes, and even less when additionally considering min-entropy.

To explain our new uncertainty relation, we find it helpful to first discuss a simpler variant, which does not satisfy (1), and which follows trivially from known results. Fix an arbitrary family  $\{\mathcal{B}_1, \dots, \mathcal{B}_m\}$  of bases for a given quantum system (i.e., Hilbert space). The *maximum overlap* of such a family is defined as

$$c := \max\{|\langle \phi | \psi \rangle| : |\phi\rangle \in \mathcal{B}_j, |\psi\rangle \in \mathcal{B}_k, 1 \leq j < k \leq m\},$$

and let  $d := -\log(c^2)$ . Let  $\rho$  be an arbitrary quantum state of that system, and let  $X$  denote the measurement outcome when  $\rho$  is measured in one of the bases. We model the choice of the basis by a random variable  $J$ , so that  $H(X|J=j)$  denotes the Shannon entropy of the measurement outcome when  $\rho$  is measured in basis  $\mathcal{B}_j$ . It follows immediately from Maassen and Uffink's uncertainty relation [MU88] that

$$H(X|J=j) + H(X|J=k) \geq -\log(c^2) = d \quad \forall j \neq k.$$

As a direct consequence, there exists a choice  $j'$  for the measurement so that  $H(X|J=j) \geq \frac{d}{2}$  for all  $j \in \{1, \dots, m\}$  with  $j \neq j'$ . In other words, for any state  $\rho$  there exists

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<sup>1</sup>This is because the min-entropy coincides with the Rényi entropy  $H_\alpha$  of high(est) order  $\alpha = \infty$ . In comparison, the Shannon entropy coincides with the Rényi entropy of (relatively) low order  $\alpha = 1$ .

$j'$  so that unless the choice for the measurement coincides with  $j'$ , which happens with probability at most  $\max_j P_J(j)$ , there is at least  $d/2$  bits of entropy in the outcome  $X$ .

Our new high-order entropic uncertainty relation shows that this very statement essentially still holds when we replace Shannon by min-entropy, except that  $j'$  becomes randomized: for any  $\rho$ , there exists a *random variable*  $J'$ , independent of  $J$ , such that<sup>2</sup>

$$H_{\min}(X|J=j, J'=j') \gtrsim \frac{d}{2} \quad \forall j \neq j' \in \{1, \dots, m\}$$

no matter what the distribution of  $J$  is. Thus, unless the measurement  $J$  coincides with  $J'$ , there is roughly  $d/2$  bits of min-entropy in the outcome  $X$ . Furthermore, since  $J'$  is *independent* of  $J$ , the probability that  $J$  coincides with  $J'$  is at most  $\max_j P_J(j)$ , as is the case for a fixed  $J'$ .

Note that we have no control over (the distribution of)  $J'$ . We can merely guarantee that it exists and is independent of  $J$ . It may be insightful to interpret  $J'$  as a *virtual guess* for  $J$ , guessed by the party that prepares  $\rho$ , and whose goal is to have little uncertainty in the measurement outcome  $X$ . The reader may think of the following specific way of preparing  $\rho$ : sample  $j'$  according to some arbitrary distribution  $J'$ , and then prepare the state as the, say, first basis vector of  $\mathcal{B}_{j'}$ . If the resulting mixture  $\rho$  is then measured in some basis  $\mathcal{B}_j$ , sampled according to an arbitrary (independent) distribution  $J$ , then unless  $j = j'$  (i.e., our guess for  $j$  was correct), there is obviously lower bounded uncertainty in the measurement outcome  $X$  (assuming a non-trivial maximum overlap). Our uncertainty relation can be understood as saying that for *any* state  $\rho$ , no matter how it is prepared, there exists such a (virtual) guess  $J'$ , which exhibits this very behavior: if it differs from the actual choice for the measurement then there is lower bounded uncertainty in the measurement outcome  $X$ . As an immediate consequence, we can for instance say that  $X$  has min-entropy at least  $d/2$ , except with a probability that is given by the probability of guessing  $J$ , e.g., except with probability  $1/m$  if the measurement is chosen uniformly at random from the family. This is clearly the best we can hope for.

We stress that because the min-entropy is more conservative than the Shannon entropy, our high-order entropic uncertainty relation does not follow from its simpler Shannon-entropy version. Neither can it be deduced in an analogous way; the main reason being that for fixed pairs  $j \neq k$ , there is no strong lower bound on  $H_{\min}(X|J=j) + H_{\min}(X|J=k)$ , in contrast to the case of Shannon entropy. More precisely and more generally, the *average* uncertainty  $\frac{1}{|J|} \sum_j H_{\min}(X|J=j)$  does not allow a lower bound higher than  $\log |J|$ . To see this, consider the following example for  $|J| = 2$  (the example can easily be extended to arbitrary  $|J|$ ). Suppose that  $\rho$  is the uniform mixture of two pure states, one giving no uncertainty when measured in basis  $j$ , and the other giving no uncertainty when measured in basis  $k$ . Then,  $\frac{1}{2}H_{\min}(X|J=j) + \frac{1}{2}H_{\min}(X|J=k) = 1$ . Because of a similar reason, we cannot hope to get a good bound for all but a *fixed* choice of  $j'$ ; the probabilistic nature of  $J'$  is necessary (in general). Hence, compared to bounding the average uncertainty, the all-but-one form of our uncertainty relation not only makes our uncertainty relation stronger in that uncertainty for all-but-one implies uncertainty on average (yet not vice versa), but it also allows for *more* uncertainty.

By using asymptotically good error-correcting codes, one can construct families of bases that have a large value of  $d$ , and thus for which our uncertainty relation guarantees a large amount of min-entropy (we discuss this in more detail in Section 3.2). These families consist of qubit-wise measurements in the computational or the Hadamard basis, hence these measurements can be performed with current technology.

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<sup>2</sup>The rigorous version of the approximate inequality  $\gtrsim$  is stated in Theorem 9.

The proof of our new uncertainty relation comprises a rather involved probability reasoning to prove the existence of the random variable  $J'$  and builds on earlier work presented in [Sch07].

## 1.2 Quantum Identification with “Hybrid” Security

As an application of our entropic uncertainty relation, we propose a new *quantum identification protocol*. Informally, the goal of (password-based) identification is to prove knowledge of a possibly low-entropy password  $w$ , without giving away any information on  $w$  (beyond what is unavoidable). In [DFSS07], Damgård *et al.* showed the existence of such an identification protocol in the *bounded-quantum-storage model* (BQSM). This means that the proposed protocol involves the communication of qubits, and security is proven against any dishonest participant that can store only a limited number of these qubits (whereas legitimate participants need no quantum storage at all to honestly execute the protocol).

Our uncertainty relation gives us the right tool to prove security of the new quantum identification protocol in the BQSM. The distinguishing feature of our new protocol is that it also offers some security in case the assumption underlying the BQSM fails to hold. Indeed, we additionally prove security of our new protocol against a dishonest server that has unbounded quantum-storage capabilities and can reliably store all the qubits communicated during an execution of the protocol, but is restricted to non-adaptive single-qubit operations and measurements.<sup>3</sup> This is in sharp contrast to protocol QID by Damgård *et al.*, which completely breaks down against a dishonest server that can store all the communicated qubits in a quantum memory and postpone the measurements until the user announces the correct measurement bases. On the downside, our protocol only offers security in case of a perfectly single-qubit (e.g. single-photon) source, because multi-qubit emissions reveal information about  $w$ . Hence, given the immature state of single-qubit-source technology at the time of this writing, our protocol is currently mainly of theoretical interest.

We want to stress that proving security of our protocol in this *single-qubit-operations model* (SQOM) is non-trivial. Indeed, as we will see, standard tools like privacy amplification are not applicable. Our proof relies on a certain minimum-distance property of random binary matrices and makes use of Diaconis and Shahshahani’s XOR inequality (Theorem 1, see also [Dia88]).

## 1.3 Related Work

The study of *entropic* uncertainty relations, whose origin dates back to 1957 with the work of Hirschman [HJ57], has received a lot of attention over the last decade due to their various applications in quantum information theory. We refer the reader to [WW10] for a recent overview on entropic uncertainty relations. Most of the known entropic uncertainty relations are of the form

$$\frac{1}{|J|} \sum_j H_\alpha(X|J=j) \geq h,$$

where  $H_\alpha$  is the Rényi entropy.<sup>4</sup> I.e., most uncertainty relations only give a lower bound on the entropy of the measurement outcome  $X$  *on average* over the (random) choice

<sup>3</sup>It is known that *some* restriction is necessary (see [DFSS07]).

<sup>4</sup>The Rényi entropy [Rén61] is defined as  $H_\alpha(X) := \frac{1}{1-\alpha} \log \sum_x P_X(x)^\alpha$ . Nevertheless, for most known uncertainty relations  $\alpha = 1$ , i.e. the Shannon entropy.

of the measurement. As argued in Section 1.1, the bound  $h$  on the *min*-entropy can be at most  $\log |J|$ , no matter the range of  $X$ . Furthermore, an uncertainty relation of this form only guarantees that there is uncertainty in  $X$  for *some* measurement(s), but does not specify precisely for how many, and certainly it does not guarantee uncertainty for *all but one* measurements. The same holds for the high-order entropic uncertainty relation from [DFR<sup>+</sup>07], which considers an exponential number of measurement settings and guarantees that except with negligible probability over the (random) choice of the measurement, there is lower-bounded min-entropy in the outcome. On the other hand, the high-order entropic uncertainty relation from [DFSS05] only considers *two* measurement settings and guarantees lower-bounded min-entropy with probability (close to)  $\frac{1}{2}$ .

The uncertainty relation we know of that comes closest to ours is Lemma 2.13 in [FHS11]. Using our notation, it shows that  $X$  is  $\epsilon$ -close to having roughly  $d/2$  bits of min-entropy (i.e., the same bound we get), but only for all but an  $\epsilon$ -fraction of all the  $m$  possible choices for the measurement  $j$ , where  $\epsilon$  is about  $\sqrt{2/m}$ .

With respect to our application, backing up the security of the identification protocol by Damgård *et al.* [DFSS07] against an adversary that can overcome the quantum-memory bound assumed by the BQSM was also the goal of [DFL<sup>+</sup>09]. However, the solution proposed there relies on an unproven computational-hardness assumption, and as such, strictly speaking, can be broken by an adversary in the SQOM, i.e., by storing qubits and measuring them later qubit-wise and performing (possibly infeasible) classical computations. On the other hand, by *assuming* a lower bound on the hardness of the underlying computational problem against quantum machines, the security of the protocol in [DFL<sup>+</sup>09] holds against an adversary with much more quantum computing power than our protocol in the SQOM, which restricts the adversary to single-qubit operations.

We hope that with future research on this topic, new quantum identification (or other cryptographic) protocols will be developed with security in the same spirit as our protocol, but with a more relaxed restriction on the adversary's quantum computation capabilities, for instance that he can only perform a limited number of quantum computation steps, and in every step he can only act on a limited number of qubits coherently.

## 2 Preliminaries

### 2.1 Basic Notation

Sets as well as families are written using a calligraphic font, e.g.  $\mathcal{A}, \mathcal{X}$ , and we write  $|\mathcal{A}|$  etc. for the cardinality. We use  $[n]$  as a shorthand for  $\{1, \dots, n\}$ .

For an  $n$ -bit vector  $v = (v_1, \dots, v_n)$  in  $\{0, 1\}^n$ , we write  $|v|$  for its Hamming weight, and, for any subset  $\mathcal{I} \subseteq [n]$ , we write  $v_{\mathcal{I}}$  for the restricted vector  $(v_i)_{i \in \mathcal{I}} \in \{0, 1\}^{|\mathcal{I}|}$ . For two vectors  $v, w \in \{0, 1\}^n$ , the *Schur product* is defined as the element-wise product  $v \odot w := (v_1 w_1, v_2 w_2, \dots, v_n w_n) \in \{0, 1\}^n$ , and the *inner product* between  $v$  and  $w$  is given by  $v \cdot w := v_1 w_1 \oplus \dots \oplus v_n w_n \in \{0, 1\}$ , where the addition is modulo 2. We write  $\text{span}(F)$  for the *row span* of a matrix  $F$ ; the set of vectors obtained by making all possible linear combinations (modulo 2) of the rows of  $F$ , i.e. the set  $\{sF : \forall s \in \{0, 1\}^\ell\}$ , where  $s$  should be interpreted as a row vector and  $sF$  denotes a vector-matrix product.

## 2.2 Probability Theory

A finite probability space is a non-empty finite set  $\Omega$  together with a function  $\Pr : \Omega \rightarrow \mathbb{R}$  such that  $\Pr(\omega) \geq 0 \quad \forall \omega \in \Omega$  and  $\sum_{\omega \in \Omega} \Pr(\omega) = 1$ . An *event* is a subset of  $\Omega$ . A *random variable* is a function  $X : \Omega \rightarrow \mathcal{X}$  from a finite probability space  $(\Omega, \Pr)$  to a finite set  $\mathcal{X}$ . We denote random variables as capital letters, for example  $X, Y, Z$ . The *distribution* of  $X$ , which we denote as  $P_X$ , is given by  $P_X(x) = \Pr[X = x] = \Pr[\{\omega \in \Omega : X(\omega) = x\}]$ . The joint distribution of two (or more) random variables  $X$  and  $Y$  is denoted by  $P_{XY}$ , i.e.,  $P_{XY}(x, y) = \Pr[X = x \wedge Y = y]$ . Specifically, we write  $U_{\mathcal{X}}$  for the uniform probability distribution over  $\mathcal{X}$ . Usually, we leave the probability space  $(\Omega, \Pr)$  implicit, and understand random variables to be defined by their joint distribution, or by some “experiment” that uniquely determines their joint distribution.

Random variables  $X$  and  $Y$  are *independent* if  $P_{XY} = P_X P_Y$  (which should be understood as  $P_{XY}(x, y) = P_X(x)P_Y(y) \quad \forall x \in \mathcal{X}, y \in \mathcal{Y}$ ). The random variables  $X, Y$  and  $Z$  form a (first-order) Markov chain, denoted by  $X \leftrightarrow Y \leftrightarrow Z$ , if  $P_{XZ|Y} = P_{X|Y}P_{Z|Y}$ . The *statistical distance* (also known as variational distance) between distributions  $P_X$  and  $P_Y$  is written as  $\text{SD}(P_X, P_Y) := \frac{1}{2} \|P_X - P_Y\|_1$ .

The *bias* of a binary random variable  $X$  is defined as  $\text{bias}(X) := |P_X(0) - P_X(1)|$ . This also naturally defines the bias of  $X$  conditioned on an event  $\mathcal{E}$  as  $\text{bias}(X|\mathcal{E}) := |P_{X|\mathcal{E}}(0) - P_{X|\mathcal{E}}(1)|$ . The bias thus ranges between 0 and 1 and can be understood as a degree of predictability of a bit: if the bias is small then the bit is close to random, and if the bias is large (i.e. approaches 1) then the bit has essentially no uncertainty. For a sum of two independent binary random variables  $X_1$  and  $X_2$ , the bias of the sum is the product of the individual biases, i.e.  $\text{bias}(X_1 \oplus X_2) = \text{bias}(X_1)\text{bias}(X_2)$ .

**Theorem 1** (Diaconis and Shahshahani’s Information-Theoretic XOR Lemma). *Let  $X$  be a random variable over  $\mathcal{X} := \{0, 1\}^n$  with distribution  $P_X$ . Then, the following holds,*

$$\text{SD}(P_X, U_{\mathcal{X}}) \leq \frac{1}{2} \left[ \sum_{f \in \{0,1\}^n \setminus \{0^n\}} \text{bias}(f \cdot X)^2 \right]^{\frac{1}{2}}.$$

The original version of Theorem 1 appeared in [Dia88], where it is expressed in the language of representation theory. The version above is due to [NN93].

**Theorem 2** (Hoeffding’s Inequality). *Let  $X_1, X_2, \dots, X_n$  be independent binary random variables, each distributed according to the Bernoulli distribution with parameter  $\mu$ , and let  $\bar{X} := n^{-1} \sum_{i \in [n]} X_i$ . Then for  $0 < t < 1 - \mu$*

$$\Pr[\bar{X} - \mu \geq t] \leq \exp(-2nt^2).$$

For a proof, the reader is referred to [Hoe63].

## 2.3 Quantum Systems and States

We assume that the reader is familiar with the basic concepts of quantum information theory; the main purpose of this section is to fix some terminology and notation. A quantum system  $A$  is associated with a complex Hilbert space,  $\mathcal{H} = \mathbb{C}^d$ , its *state space*. By default, we write  $\mathcal{H}_A$  for the state space of system  $A$ , and  $\rho_A$  (respectively  $|\varphi_A\rangle$ ) in case

of a pure state) for the state of  $A$ . We write  $\mathcal{D}(\mathcal{H})$  for the set of all density matrices on Hilbert space  $\mathcal{H}$ .

The state space of a *bipartite* quantum system  $AB$ , consisting of two (or more) subsystems, is given by  $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ . If the state of  $AB$  is given by  $\rho_{AB}$  then the state of subsystem  $A$ , when treated as a stand-alone system, is given by the *partial trace*  $\rho_A = \text{tr}_B(\rho_{AB})$ , and correspondingly for  $B$ . *Measuring* a system  $A$  in basis  $\{|i\rangle\}_{i \in I}$ , where  $\{|i\rangle\}_{i \in I}$  is an orthonormal basis of  $\mathcal{H}_A$ , means applying the measurement described by the projectors  $\{|i\rangle\langle i|\}_{i \in I}$ , such that outcome  $i \in I$  is observed with probability  $p_i = \text{tr}(|i\rangle\langle i|\rho_A)$  (respectively  $p_i = |\langle i|\varphi_A\rangle|^2$  in case of a pure state). If  $A$  is a subsystem of a bipartite system  $AB$ , then it means applying the measurement described by the projectors  $\{|i\rangle\langle i| \otimes \mathbb{I}_B\}_{i \in I}$ , where  $\mathbb{I}_B$  is the identity operator on  $\mathcal{H}_B$ .

A *qubit* is a quantum system  $A$  with state space  $\mathcal{H}_A = \mathbb{C}^2$ . The *computational basis*  $\{|0\rangle, |1\rangle\}$  (for a qubit) is given by  $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , and the *Hadamard basis* by  $\{H|0\rangle, H|1\rangle\}$ , where  $H$  denotes the 2-dimensional *Hadamard matrix*  $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ . We also call the computational basis the *plus* basis and associate it with the ‘+’-symbol, and we call the Hadamard basis the *times* basis and associate it with the ‘×’-symbol. For bit vectors  $x = (x_1, \dots, x_n) \in \{0, 1\}^n$  and  $v = (v_1, \dots, v_n) \in \{+, \times\}^n$  we then write  $|x\rangle_v = |x_1\rangle_{v_1} \otimes \dots \otimes |x_n\rangle_{v_n}$  where  $|x_i\rangle_+ := |x_i\rangle$  and  $|x_i\rangle_\times := H|x_i\rangle$ .

Subsystem  $X$  of a bipartite quantum system  $XE$  is called *classical*, if the state of  $XE$  is given by a density matrix of the form

$$\rho_{XE} = \sum_{x \in \mathcal{X}} P_X(x) |x\rangle\langle x| \otimes \rho_E^x,$$

where  $\mathcal{X}$  is a finite set of cardinality  $|\mathcal{X}| = \dim(\mathcal{H}_X)$ ,  $P_X : \mathcal{X} \rightarrow [0, 1]$  is a probability distribution,  $\{|x\rangle\}_{x \in \mathcal{X}}$  is some fixed orthonormal basis of  $\mathcal{H}_X$ , and  $\rho_E^x$  is a density matrix on  $\mathcal{H}_E$  for every  $x \in \mathcal{X}$ . Such a state, called *hybrid* or *cq*- (for classical-quantum) state, can equivalently be understood as consisting of a *random variable*  $X$  with distribution  $P_X$ , taking on values in  $\mathcal{X}$ , and a system  $E$  that is in state  $\rho_E^x$  exactly when  $X$  takes on the value  $x$ . This formalism naturally extends to two (or more) classical systems  $X, Y$  etc. For any event  $\mathcal{E}$  (defined by  $P_{\mathcal{E}|X}(x) = \Pr[\mathcal{E}|X = x]$  for all  $x$ ), we may write

$$\rho_{XE|\mathcal{E}} := \sum_x P_{X|\mathcal{E}}(x) |x\rangle\langle x| \otimes \rho_E^x.$$

If the state of  $XE$  satisfies  $\rho_{XE} = \rho_X \otimes \rho_E$ , where  $\rho_X = \text{tr}_E(\rho_{XE}) = \sum_x P_X(x) |x\rangle\langle x|$  and  $\rho_E = \text{tr}_X(\rho_{XE}) = \sum_x P_X(x) \rho_E^x$ , then  $X$  is *independent* of  $E$ , and thus no information on  $X$  can be obtained from system  $E$ . Moreover, if  $\rho_{XE} = \frac{1}{|\mathcal{X}|} \mathbb{I}_X \otimes \rho_E$ , where  $\mathbb{I}_X$  denotes the identity on  $\mathcal{H}_X$ , then  $X$  is *random-and-independent* of  $E$ . We also want to be able to express that a random variable  $X$  is (close) to being independent of a quantum system  $E$  when given a random variable  $Y$ . Formally, this is expressed by saying that  $\rho_{XYE}$  equals  $\rho_{X \leftrightarrow Y \leftrightarrow E}$ , where

$$\rho_{X \leftrightarrow Y \leftrightarrow E} := \sum_{x, y} P_{XY}(x, y) |x\rangle\langle x| \otimes |y\rangle\langle y| \otimes \rho_E^y.$$

This notion, called *conditional independence*, for the quantum setting was introduced in [DFSS07].

For a matrix  $\rho$ , the trace norm is defined as  $\|\rho\|_1 := \text{tr}\sqrt{\rho\rho^*}$ , where  $\rho^*$  denotes the Hermitian transpose of  $\rho$ .

**Definition 3.** The *trace distance* between two density matrices  $\rho, \sigma \in \mathcal{D}(\mathcal{H})$  is defined as  $\delta(\rho, \sigma) := \frac{1}{2} \|\rho - \sigma\|_1$ .

If two states  $\rho$  and  $\sigma$  are  $\varepsilon$ -close in trace distance, i.e.  $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$ , we use  $\rho \approx_\varepsilon \sigma$  as shorthand. In case of classical states, the trace distance coincides with the statistical distance. Moreover, the trace distance between two states cannot increase when applying the same quantum operation (i.e., CPTP map) to both states. As a consequence, if  $\rho \approx_\varepsilon \sigma$  then the states cannot be distinguished with statistical advantage better than  $\varepsilon$ .

**Definition 4.** For a density matrix  $\rho_{XE} \in \mathcal{D}(\mathcal{H}_X \otimes \mathcal{H}_E)$  with classical  $X$ , the *distance to uniform* of  $X$  given  $E$  is defined as

$$d_{\text{unif}}(X|E) := \frac{1}{2}\|\rho_{XE} - \rho_U \otimes \rho_E\|_1,$$

where  $\rho_U := \frac{1}{\dim(\mathcal{H}_X)}\mathbb{I}_X$ .

## 2.4 Min-Entropy and Privacy Amplification

We make use of Renner's notion of the *conditional min-entropy*  $H_{\min}(\rho_{AB}|B)$  of a system  $A$  conditioned on another system  $B$  [Ren05]. If the state  $\rho_{AB}$  is clear from the context, we may write  $H_{\min}(A|B)$  instead of  $H_{\min}(\rho_{AB}|B)$ . The formal definition is given by  $H_{\min}(\rho_{AB}|B) := \sup_{\sigma_B} \max\{h \in \mathbb{R} : 2^{-h} \cdot \mathbb{I}_A \otimes \sigma_B - \rho_{AB} \geq 0\}$  where the supremum is over all density matrices  $\sigma_B$  on  $\mathcal{H}_B$ . If  $\mathcal{H}_B$  is the trivial space  $\mathbb{C}$ , we obtain the unconditional min-entropy of  $\rho_A$ , denoted as  $H_{\min}(\rho_A)$ , which simplifies to  $H_{\min}(\rho_A) = -\log \lambda_{\max}(\rho_A)$ , where  $\lambda_{\max}(\rho_A)$  is the largest eigenvalue of  $\rho_A$ .

We will need the following chain rule.

**Lemma 5.** For any density matrix  $\rho$  on  $\mathcal{H}_{XYE}$  with classical  $X$  and  $Y$  it holds that

$$H_{\min}(X|YE) \geq H_{\min}(X|Y) - H_{\max}(E).$$

The proof can be found in Appendix B.

For the special case of a hybrid state  $\rho_{XE} \in \mathcal{D}(\mathcal{H}_X \otimes \mathcal{H}_E)$  with classical  $X$ , it is shown in [KRS09] that the conditional min-entropy of a quantum state coincides with the negative logarithm of the *guessing probability conditional on quantum side information*

$$p_{\text{guess}}(X|E) := \max_{\{M_x\}} \sum_x P_X(x) \text{tr}(M_x \rho_E^x),$$

where the latter is the probability that the party holding  $\mathcal{H}_E$  guesses  $X$  correctly using the POVM  $\{M_x\}_x$  on  $\mathcal{H}_E$  that maximizes  $p_{\text{guess}}$ . Thus,

$$H_{\min}(X|E) = -\log p_{\text{guess}}(X|E). \quad (1)$$

For random variables  $X$  and  $Y$ , we have that  $p_{\text{guess}}(X|Y)$  simplifies to

$$p_{\text{guess}}(X|Y) = \sum_y P_Y(y) p_{\text{guess}}(X|Y=y) = \sum_y P_Y(y) \max_x P_{X|Y}(x|y).$$

Finally, we make use of Renner's privacy amplification theorem [RK05, Ren05], as given below. Recall that a function  $g : \mathcal{R} \times \mathcal{X} \rightarrow \{0,1\}^\ell$  is called a *universal* (hash) function, if for the random variable  $R$ , uniformly distributed over  $\mathcal{R}$ , and for any distinct  $x, y \in \mathcal{X}$ :  $\Pr[g(R, x) = g(R, y)] \leq 2^{-\ell}$ .

**Theorem 6** (Privacy amplification). *Let  $\rho_{XE}$  be a hybrid state with classical  $X$ . Let  $g : \mathcal{R} \times \mathcal{X} \rightarrow \{0, 1\}^\ell$  be a universal hash function, and let  $R$  be uniformly distributed over  $\mathcal{R}$ , independent of  $X$  and  $E$ . Then  $K = g(R, X)$  satisfies*

$$d_{\text{unif}}(K|RE) \leq \frac{1}{2} \cdot 2^{-\frac{1}{2}(H_{\min}(X|E) - \ell)}.$$

Informally, Theorem 6 states that if  $X$  contains sufficiently more than  $\ell$  bits of entropy when given  $E$ , then  $\ell$  nearly random-and-independent bits can be extracted from  $X$ .

### 3 The All-But-One Entropic Uncertainty Relation

Throughout this section,  $\{\mathcal{B}_1, \dots, \mathcal{B}_m\}$  is an arbitrary but fixed family of bases for the state space  $\mathcal{H}$  of a quantum system. For simplicity, we restrict our attention to an  $n$ -qubit system, such that  $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$  for  $n \in \mathbb{N}$ , but our results immediately generalize to arbitrary quantum systems. We write the  $2^n$  basis vectors of the  $j$ -th basis  $\mathcal{B}_j$  as  $\mathcal{B}_j = \{|x\rangle_j : x \in \{0, 1\}^n\}$ . Let  $c$  be the maximum overlap of  $\{\mathcal{B}_1, \dots, \mathcal{B}_m\}$ , i.e.,

$$c := \max\{|\langle x|_j|y\rangle_k| : x, y \in \{0, 1\}^n, 1 \leq j < k \leq m\}.$$

In order to obtain our entropic uncertainty relation that lower bounds the min-entropy of the measurement outcome for all but one measurement, we first show an uncertainty relation that expresses uncertainty by means of the probability measure of given sets.

**Theorem 7** (Theorem 4.18 in [Sch07]). *Let  $\rho$  be an arbitrary state of  $n$  qubits. For  $j \in [m]$ , let  $Q^j(\cdot)$  be the distribution of the outcome when  $\rho$  is measured in the  $\mathcal{B}_j$ -basis, i.e.,  $Q^j(x) = \langle x|_j \rho |x\rangle_j$  for any  $x \in \{0, 1\}^n$ . Then, for any family  $\{\mathcal{L}^j\}_{j \in [m]}$  of subsets  $\mathcal{L}^j \subset \{0, 1\}^n$ , it holds that*

$$\sum_{j \in [m]} Q^j(\mathcal{L}^j) \leq 1 + c(m-1) \cdot \max_{j \neq k \in [m]} \sqrt{|\mathcal{L}^j| |\mathcal{L}^k|}.$$

A special case of Theorem 7, obtained by restricting the family of bases to the specific choice  $\{\mathcal{B}_+, \mathcal{B}_\times\}$  with  $\mathcal{B}_+ = \{|x\rangle : x \in \{0, 1\}^n\}$  and  $\mathcal{B}_\times = \{H^{\otimes n}|x\rangle : x \in \{0, 1\}^n\}$  (i.e. either the computational or Hadamard basis for all qubits), is an uncertainty relation that was proven and used in the original paper about the BQSM [DFSS05]. The proof of Theorem 7 goes along similar lines as the proof in the journal version of [DFSS05] for the special case outlined above. It is based on the norm inequality

$$\|A_1 + \dots + A_m\| \leq 1 + (m-1) \cdot \max_{j \neq k \in [m]} \|A_j A_k\|,$$

which holds for arbitrary orthogonal projectors  $A_1, \dots, A_m$ . Recall that for a linear operator  $A$  on the complex Hilbert space  $\mathcal{H}$ , the *operator norm* is defined as  $\|A\| := \sup \|A|\psi\rangle\|$ , where the supremum is over all norm-1  $|\psi\rangle \in \mathcal{H}$ ; this is identical to  $\|A\| := \sup |\langle \varphi|A|\psi\rangle|$ , where the supremum is over all norm-1  $|\varphi\rangle, |\psi\rangle \in \mathcal{H}$ . Furthermore,  $A$  is called an *orthogonal projector* if  $A^2 = A$  and  $A^* = A$ . The proof of this norm inequality can be found in Appendix A. The proof of Theorem 7 is given here.

*Proof of Theorem 7.* For  $j \in [m]$ , we define the orthogonal projectors  $A^j := \sum_{x \in \mathcal{L}^j} |x\rangle_j \langle x|_j$ . Using the spectral decomposition of  $\rho = \sum_w \lambda_w |\varphi_w\rangle \langle \varphi_w|$  and the linearity of the trace, we

have

$$\begin{aligned} \sum_{j \in [m]} Q^j(\mathcal{L}^j) &= \sum_{j \in [m]} \text{tr}(A^j \rho) = \sum_{j \in [m]} \sum_w \lambda_w \text{tr}(A^j |\varphi_w\rangle\langle\varphi_w|) = \sum_w \lambda_w \left( \sum_{j \in [m]} \langle\varphi_w| A^j |\varphi_w\rangle \right) \\ &= \sum_w \lambda_w \langle\varphi_w| \left( \sum_{j \in [m]} A^j \right) |\varphi_w\rangle \leq \left\| \sum_{j \in [m]} A^j \right\| \leq 1 + (m-1) \cdot \max_{j \neq k \in [m]} \|A^j A^k\|, \end{aligned}$$

where the last inequality is the norm inequality (Proposition 26 in Appendix A). To conclude, we show that  $\|A^j A^k\| \leq c\sqrt{|\mathcal{L}^j||\mathcal{L}^k|}$ . Let us fix  $j \neq k \in [m]$ . Note that by the restriction on the overlap of the family of bases  $\{\mathcal{B}_j\}_{j \in [m]}$ , we have that  $|\langle x|_j|y\rangle_k| \leq c$  holds for all  $x, y \in \{0, 1\}^n$ . Then, with the sums over  $x$  and  $y$  understood as over  $x \in \mathcal{L}^j$  and  $y \in \mathcal{L}^k$ , respectively,

$$\begin{aligned} \|A^j A^k|\psi\rangle\|^2 &= \left\| \sum_x |x\rangle_j \langle x|_j \sum_y |y\rangle_k \langle y|_k |\psi\rangle \right\|^2 = \left\| \sum_x |x\rangle_j \sum_y \langle x|_j|y\rangle_k \langle y|_k|\psi\rangle \right\|^2 \\ &= \sum_x \left| \sum_y \langle x|_j|y\rangle_k \langle y|_k|\psi\rangle \right|^2 \leq \sum_x \left( \sum_y |\langle x|_j|y\rangle_k \langle y|_k|\psi\rangle| \right)^2 \\ &\leq c^2 \sum_x \left( \sum_y |\langle y|_k|\psi\rangle| \right)^2 \leq c^2 |\mathcal{L}^j| |\mathcal{L}^k|. \end{aligned}$$

The third equality follows from Pythagoras, the first inequality holds by triangle inequality, the second inequality by the bound on  $|\langle x|_j|y\rangle_k|$ , and the last follows from Cauchy-Schwarz. This implies  $\|A^j A^k\| \leq c\sqrt{|\mathcal{L}^j||\mathcal{L}^k|}$  and finishes the proof.  $\square$

In the same spirit as in (the journal version of) [DFSS05], we reformulate above uncertainty relation in terms of a “good event”  $\mathcal{E}$ , which occurs with reasonable probability, and if it occurs, the measurement outcomes have high min-entropy. The statement is obtained by choosing the sets  $\mathcal{L}^j$  in Theorem 7 appropriately.

Because we now switch to entropy notation, it will be convenient to work with a measure of overlap between bases that is logarithmic in nature and *relative* to the number  $n$  of qubits. Hence, we define

$$\delta := -\frac{1}{n} \log c^2.$$

We will later see that for “good” choices of bases,  $\delta$  stays constant for growing  $n$ .

**Corollary 8.** *Let  $\rho$  be an arbitrary  $n$ -qubit state, let  $J$  be a random variable over  $[m]$  (with arbitrary distribution  $P_J$ ), and let  $X$  be the outcome when measuring  $\rho$  in basis  $\mathcal{B}_J$ .<sup>5</sup> Then, for any  $0 < \epsilon < \delta/4$ , there exists an event  $\mathcal{E}$  such that*

$$\sum_{j \in [m]} \Pr[\mathcal{E}|J=j] \geq (m-1) - (2m-1) \cdot 2^{-\epsilon n}$$

and

$$H_{\min}(X|J=j, \mathcal{E}) \geq \left( \frac{\delta}{2} - 2\epsilon \right) n$$

for  $j \in [m]$  with  $P_{J|\mathcal{E}}(j) > 0$ .

---

<sup>5</sup>I.e.,  $P_{X|J}(x|j) = Q^j(x)$ , using the notation from Theorem 7.

*Proof.* For  $j \in [m]$  define

$$\mathcal{S}^j := \{x \in \{0, 1\}^n : Q^j(x) \leq 2^{-(\delta/2-\epsilon)n}\}$$

to be the sets of strings with small probabilities and denote by  $\mathcal{L}^j := \overline{\mathcal{S}^j}$  their complements<sup>6</sup>. Note that for all  $x \in \mathcal{L}^j$ , we have that  $Q^j(x) > 2^{-(\delta/2-\epsilon)n}$  and therefore  $|\mathcal{L}^j| < 2^{(\delta/2-\epsilon)n}$ . It follows from Theorem 7 that

$$\sum_{j \in [m]} Q^j(\mathcal{S}^j) = \sum_{j \in [m]} (1 - Q^j(\mathcal{L}^j)) \geq m - (1 + (m-1) \cdot 2^{-\epsilon n}) = (m-1) - (m-1)2^{-\epsilon n}.$$

We define  $\mathcal{E} := \{X \in \mathcal{S}^J \wedge Q^J(\mathcal{S}^J) \geq 2^{-\epsilon n}\}$  to be the event that  $X \in \mathcal{S}^J$  and at the same time the probability that this happens is not too small. Then  $\Pr[\mathcal{E}|J=j] = \Pr[X \in \mathcal{S}^j \wedge Q^j(\mathcal{S}^j) \geq 2^{-\epsilon n}|J=j]$  either vanishes (if  $Q^j(\mathcal{S}^j) < 2^{-\epsilon n}$ ) or else equals  $Q^j(\mathcal{S}^j)$ . In either case,  $\Pr[\mathcal{E}|J=j] \geq Q^j(\mathcal{S}^j) - 2^{-\epsilon n}$  holds and thus the first claim follows by summing over  $j \in [m]$  and using the derivation above. Furthermore, let  $p = \max_j P_J(j)$ , then  $\Pr[\bar{\mathcal{E}}] = \sum_{j \in [m]} P_J(j) \Pr[\bar{\mathcal{E}}|J=j] \leq p \sum_{j \in [m]} \Pr[\bar{\mathcal{E}}|J=j] \leq p(m - (\sum_{j \in [m]} Q^j(\mathcal{S}^j) - 2^{-\epsilon n})) \leq p(1 + (2m-1) \cdot 2^{-\epsilon n})$ , and  $\Pr[\mathcal{E}] \geq (1-p) - p(2m-1) \cdot 2^{-\epsilon n}$

Regarding the second claim, in case  $J=j$ , we have

$$\begin{aligned} H_{\min}(X|J=j, \mathcal{E}) &= -\log \left( \max_{x \in \mathcal{S}^j} \frac{Q^j(x)}{Q^j(\mathcal{S}^j)} \right) \\ &\geq -\log \left( \frac{2^{-(\delta/2-\epsilon)n}}{Q^j(\mathcal{S}^j)} \right) = (\delta/2 - \epsilon)n + \log(Q^j(\mathcal{S}^j)). \end{aligned}$$

As  $Q^j(\mathcal{S}^j) \geq 2^{-\epsilon n}$  by definition of  $\mathcal{E}$ , we have  $H_{\min}(X|J=j, \mathcal{E}) \geq (\delta/2 - 2\epsilon)n$ .  $\square$

### 3.1 Main Result and Its Proof

We are now ready to state and prove our new all-but-one entropic uncertainty relation.

**Theorem 9.** *Let  $\rho$  be an arbitrary  $n$ -qubit state, let  $J$  be a random variable over  $[m]$  (with arbitrary distribution  $P_J$ ), and let  $X$  be the outcome when measuring  $\rho$  in basis  $\mathcal{B}_J$ . Then, for any  $0 < \epsilon < \delta/4$ , there exists a random variable  $J'$  with joint distribution  $P_{JJ'X}$  such that (1)  $J$  and  $J'$  are independent and (2) there exists an event  $\Psi$  with  $\Pr[\Psi] \geq 1 - 2 \cdot 2^{-\epsilon n}$  such that<sup>7</sup>*

$$H_{\min}(X|J=j, J'=j', \Psi) \geq \left(\frac{\delta}{2} - 2\epsilon\right)n - 1$$

for all  $j, j' \in [m]$  with  $j \neq j'$  and  $P_{JJ'|\Psi}(j, j') > 0$ .

Note that, as phrased, Theorem 9 requires that  $J$  is fixed and known, and only then the existence of  $J'$  can be guaranteed. This is actually not necessary. By looking at the proof, we see that  $J'$  can be defined simultaneously in all  $m$  probability spaces  $P_{X|J=j}$  with  $j \in [m]$ , without having assigned a probability distribution to  $J$  yet, so that the resulting random variable  $J'$  we obtain by assigning an *arbitrary* probability distribution  $P_J$  to  $J$ , satisfies the claimed properties. This in particular implies that the (marginal) distribution of  $J'$  is fully determined by  $\rho$ .

<sup>6</sup>Here's the mnemonic:  $\mathcal{S}$  for the strings with *S*mall probabilities,  $\mathcal{L}$  for *L*arge.

<sup>7</sup>Instead of introducing such an event  $\Psi$ , we could also express the min-entropy bound by means of the *smooth* min-entropy of  $X$  given  $J=j$  and  $J'=j'$ .

The idea of the proof of Theorem 9 is to (try to) define the random variable  $J'$  in such a way that the event  $J \neq J'$  coincides with the “good event”  $\mathcal{E}$  from Corollary 8. It then follows immediately from Corollary 8 that  $H_{\min}(X|J = j, J' \neq J) \geq (\delta/2 - 2\epsilon)n$ , which is already close to the actual min-entropy bound we need to prove. This approach dictates that if the event  $\mathcal{E}$  does not occur, then  $J'$  needs to *coincide* with  $J$ . Vice versa, if  $\mathcal{E}$  does occur, then  $J'$  needs to be *different* to  $J$ . However, it is a priori unclear *how* to choose  $J'$  different to  $J$  in case  $\mathcal{E}$  occurs. There is only one way to set  $J'$  to be equal to  $J$ , but there are many ways to set  $J'$  to be different to  $J$  (unless  $m = 2$ ). It needs to be done in such a way that without conditioning on  $\mathcal{E}$  or its complement,  $J$  and  $J'$  are independent.

Somewhat surprisingly, it turns out that the following does the job. To simplify this informal discussion, we assume that the sum of the  $m$  probabilities  $\Pr[\mathcal{E}|J = j]$  from Corollary 8 equals  $m - 1$  exactly. It then follows that the corresponding complementary probabilities,  $\Pr[\bar{\mathcal{E}}|J = j]$  for the  $m$  different choices of  $j \in [m]$ , add up to 1 and thus form a probability distribution.  $J'$  is now chosen, in the above spirit depending on the event  $\mathcal{E}$ , so that its marginal distribution  $P_{J'}$  coincides with this probability distribution:  $P_{J'}(j') = \Pr[\bar{\mathcal{E}}|J = j']$  for all  $j' \in [m]$ . Thus, in case the event  $\mathcal{E}$  occurs,  $J'$  is chosen according to this distribution but conditioned on being different to the value  $j$ , taken on by  $J$ . The technical details, and how to massage the argument in case the sum of the  $\Pr[\mathcal{E}|J = j]$ 's is not exactly  $m - 1$ , are worked out in the proof below.

*Proof of Theorem 9.* From Corollary 8 we know that for any  $0 < \epsilon < \delta/4$ , there exists an event  $\mathcal{E}$  such that  $\sum_{j \in [m]} \Pr[\mathcal{E}|J = j] = m - 1 - \alpha$ , and thus  $\sum_{j \in [m]} \Pr[\bar{\mathcal{E}}|J = j] = 1 + \alpha$ , for  $-1 \leq \alpha \leq (2m - 1)2^{-\epsilon n}$ . We make a case distinction between  $\alpha = 0$ ,  $\alpha > 0$  and  $\alpha < 0$ ; we start with the case  $\alpha = 0$ , we subsequently prove the other two cases by reducing them to the case  $\alpha = 0$  by “inflating” and “deflating” the event  $\mathcal{E}$  appropriately. The approach for the case  $\alpha = 0$  is to define  $J'$  in such way that  $\mathcal{E} \iff J \neq J'$ , i.e., the event  $J \neq J'$  coincides with the event  $\mathcal{E}$ . The min-entropy bound from Corollary 8 then immediately translates to  $H_{\min}(X|J = j, J' \neq J) \geq (\delta/2 - 2\epsilon)n$ , and to  $H_{\min}(X|J = j, J' = j') \geq (\delta/2 - 2\epsilon)n$  for  $j' \neq j$  with  $P_{JJ'}(j, j') > 0$ , as we will show. What is not obvious about the approach is how to define  $J'$  when it is supposed to be different from  $J$ , i.e., when the event  $\mathcal{E}$  occurs, so that in the end  $J$  and  $J'$  are independent.

Formally, we define  $J'$  by means of the following conditional probability distributions:

$$P_{J'|JX\bar{\mathcal{E}}}(j'|j, x) := \begin{cases} 1 & \text{if } j = j' \\ 0 & \text{if } j \neq j' \end{cases} \quad \text{and} \quad P_{J'|JX\mathcal{E}}(j'|j, x) := \begin{cases} 0 & \text{if } j = j' \\ \frac{\Pr[\bar{\mathcal{E}}|J = j']}{\Pr[\mathcal{E}|J = j]} & \text{if } j \neq j' \end{cases}$$

We assume for the moment that the denominator in the latter expression does not vanish for any  $j$ ; we take care of the case where it does later. Trivially,  $P_{J'|JX\bar{\mathcal{E}}}$  is a proper distribution, with non-negative probabilities that add up to 1, and the same holds for  $P_{J'|JX\mathcal{E}}$ :

$$\sum_{j' \in [m]} P_{J'|JX\bar{\mathcal{E}}} = \sum_{j' \in [m] \setminus \{j\}} P_{J'|JX\bar{\mathcal{E}}} = \sum_{j' \in [m] \setminus \{j\}} \frac{\Pr[\bar{\mathcal{E}}|J = j']}{\Pr[\mathcal{E}|J = j]} = 1$$

where we used that  $\sum_{j \in [m]} \Pr[\bar{\mathcal{E}}|J = j] = 1$  (because  $\alpha = 0$ ) in the last equality. Furthermore, it follows immediately from the definition of  $J'$  that  $\bar{\mathcal{E}} \implies J = J'$  and  $\mathcal{E} \implies J \neq J'$ . Hence,  $\mathcal{E} \iff J \neq J'$ , and thus the bound from Corollary 8 translates to  $H_{\min}(X|J = j, J' \neq J) \geq (\delta/2 - 2\epsilon)n$ . It remains to argue that  $J'$  is independent of  $J$ , and that the bound also holds for  $H_{\min}(X|J = j, J' = j')$  whenever  $j \neq j'$ .

The latter follows immediately from the fact that conditioned on  $J \neq J'$  (which is equivalent to  $\mathcal{E}$ ),  $X, J$  and  $J'$  form a Markov chain  $X \leftrightarrow J \leftrightarrow J'$ , and thus, given  $J = j$ , additionally conditioning on  $J' = j'$  does not change the distribution of  $X$ . For the independence of  $J$  and  $J'$ , consider the joint probability distribution of  $J$  and  $J'$ , given by

$$\begin{aligned} P_{JJ'}(j, j') &= P_{J'J\mathcal{E}}(j', j) + P_{J'J\bar{\mathcal{E}}}(j', j) \\ &= P_J(j)\Pr[\mathcal{E}|J = j]P_{J'|J\mathcal{E}}(j'|j) + P_J(j)\Pr[\bar{\mathcal{E}}|J = j]P_{J'|J\bar{\mathcal{E}}}(j'|j) \\ &= P_J(j)\Pr[\bar{\mathcal{E}}|J = j'], \end{aligned}$$

where the last equality follows by separately analyzing the cases  $j = j'$  and  $j \neq j'$ . It follows immediately that the marginal distribution of  $J'$  is  $P_{J'}(j') = \sum_j P_{JJ'}(j, j') = \Pr[\bar{\mathcal{E}}|J = j']$ , and thus  $P_{JJ'} = P_J \cdot P_{J'}$ .

What is left to do for the case  $\alpha = 0$  is to deal with the case where there exists  $j^*$  with  $\Pr[\mathcal{E}|J = j^*] = 0$ . Since  $\sum_{j \in [m]} \Pr[\bar{\mathcal{E}}|J = j] = 1$ , it holds that  $\Pr[\bar{\mathcal{E}}|J = j] = 0$  for  $j \neq j^*$ . This motivates to define  $J'$  as  $J' := j^*$  with probability 1. Note that this definition directly implies that  $J'$  is independent from  $J$ . Furthermore, by the above observations:  $\mathcal{E} \iff J \neq J'$ . This concludes the case  $\alpha = 0$ .

Next, we consider the case  $\alpha > 0$ . The idea is to “inflate” the event  $\mathcal{E}$  so that  $\alpha$  becomes 0, i.e., to define an event  $\mathcal{E}'$  that contains  $\mathcal{E}$  (meaning that  $\mathcal{E} \implies \mathcal{E}'$ ) so that  $\sum_{j \in [m]} \Pr[\mathcal{E}'|J = j] = m - 1$ , and to define  $J'$  as in the case  $\alpha = 0$  (but now using  $\mathcal{E}'$ ). Formally, we define  $\mathcal{E}'$  as the disjoint union  $\mathcal{E}' = \mathcal{E} \vee \mathcal{E}_\circ$  of  $\mathcal{E}$  and an event  $\mathcal{E}_\circ$ . The event  $\mathcal{E}_\circ$  is defined by means of  $\Pr[\mathcal{E}_\circ|\mathcal{E}, J = j, X = x] = 0$ , so that  $\mathcal{E}$  and  $\mathcal{E}_\circ$  are indeed disjoint, and  $\Pr[\mathcal{E}_\circ|J = j, X = x] = \alpha/m$ , so that indeed

$$\sum_{j \in [m]} \Pr[\mathcal{E}'|J = j] = \sum_{j \in [m]} (\Pr[\mathcal{E}|J = j] + \Pr[\mathcal{E}_\circ|J = j]) = (m - 1 - \alpha) + \alpha = m - 1.$$

We can now apply the analysis of the case  $\alpha = 0$  to conclude the existence of  $J'$ , independent of  $J$ , such that  $J \neq J' \iff \mathcal{E}'$  and thus  $(J \neq J') \wedge \bar{\mathcal{E}}_\circ \iff \mathcal{E}' \wedge \bar{\mathcal{E}}_\circ \iff \mathcal{E}$ . Setting  $\Psi := \bar{\mathcal{E}}_\circ$ , it follows that

$$H_{\min}(X|J = j, J \neq J', \Psi) = H_{\min}(X|J = j, \mathcal{E}) \geq (\delta/2 - 2\epsilon)n,$$

where  $\Pr[\Psi] = 1 - \Pr[\mathcal{E}_\circ] = 1 - \alpha/m \geq 1 - (2m - 1)2^{-\epsilon n}/m \geq 1 - 2 \cdot 2^{-\epsilon n}$ . Finally, using similar reasoning as in the case  $\alpha = 0$ , it follows that the same bound holds for  $H_{\min}(X|J = j, J' = j', \Psi)$  whenever  $j \neq j'$ . This concludes the case  $\alpha > 0$ .

Finally, we consider the case  $\alpha < 0$ . The approach is the same as above, but now  $\mathcal{E}'$  is obtained by “deflating”  $\mathcal{E}$ . Specifically, we define  $\mathcal{E}'$  by means of  $\Pr[\mathcal{E}'|\bar{\mathcal{E}}, J = j, X = x] = \Pr[\mathcal{E}'|\bar{\mathcal{E}}] = 0$ , so that  $\mathcal{E}'$  is contained in  $\mathcal{E}$ , and  $\Pr[\mathcal{E}'|\mathcal{E}, J = j, X = x] = \Pr[\mathcal{E}'|\mathcal{E}] = \frac{m-1}{m-1-\alpha}$ , so that

$$\sum_{j \in [m]} \Pr[\mathcal{E}'|J = j] = \sum_{j \in [m]} \Pr[\mathcal{E}'|\mathcal{E}] \cdot \Pr[\mathcal{E}|J = j] = m - 1.$$

Again, from the  $\alpha = 0$  case we obtain  $J'$ , independent of  $J$ , such that the event  $J \neq J'$  is equivalent to the event  $\mathcal{E}'$ .

It follows that

$$\begin{aligned} H_{\min}(X|J = j, J \neq J') &= H_{\min}(X|J = j, \mathcal{E}') = H_{\min}(X|J = j, \mathcal{E}', \mathcal{E}) \\ &\geq H_{\min}(X|J = j, \mathcal{E}) - \log(P[\mathcal{E}'|\mathcal{E}, J = j]) \geq (\delta/2 - 2\epsilon)n - 1, \end{aligned}$$

where the second equality holds because  $\mathcal{E}' \implies \mathcal{E}$ , the first inequality holds because additionally conditioning on  $\mathcal{E}'$  increases the probabilities of  $X$  conditioned on  $J = j$  and  $\mathcal{E}$  by at most a factor  $1/P[\mathcal{E}'|\mathcal{E}, J = j]$ , and the last inequality holds by Corollary 8) and because  $P[\mathcal{E}'|\mathcal{E}, J = j] = \frac{m-1}{m-1-\alpha} \geq \frac{1}{2}$ , where the latter holds since  $\alpha \geq -1$ . Finally, using similar reasoning as in the previous cases, it follows that the same bound holds for  $H_{\min}(X|J = j, J' = j')$  whenever  $j \neq j'$ . This concludes the proof.  $\square$

## 3.2 Constructing Good Families of Bases

Here, we discuss some interesting choices for the family  $\{\mathcal{B}_1, \dots, \mathcal{B}_m\}$  of bases. We say that such a family is “good” if  $\delta = -\frac{1}{n} \log(c^2)$  converges to a strictly positive constant as  $n$  tends to infinity. There are various ways to construct such families. For example, a family obtained through sampling according to the Haar measure will be good with overwhelming probability (a precise statement, in which “good” means  $\delta = 0.9$ , can be found at the very end of the proof of Theorem 2.5 of [FHS11]). The best possible constant  $\delta = 1$  is achieved for a family of *mutually unbiased bases*. However, for arbitrary quantum systems (i.e., not necessarily multi-qubit systems) it is not well understood how large such a family may be, beyond that its size cannot exceed the dimension plus 1.

In the upcoming section, we will use the following simple and well-known construction. For an arbitrary binary code  $\mathcal{C} \subset \{+, \times\}^n$  of size  $m$ , minimum distance  $d$  and encoding function  $\mathbf{c} : [m] \rightarrow \mathcal{C}$ , we can construct a family  $\{\mathcal{B}_1, \dots, \mathcal{B}_m\}$  of bases as follows. We identify the  $j$ th codeword, i.e.  $\mathbf{c}(j) = (c_1, \dots, c_n)$  for  $j \in [m]$ , with the basis  $\mathcal{B}_j = \{|x\rangle_{\mathbf{c}(j)} : x \in \{0, 1\}^n\} = \{(H^{c_1} \otimes \dots \otimes H^{c_n})|x\rangle : x \in \{0, 1\}^n\}$ . In other words,  $\mathcal{B}_j$  measures qubit-wise in the computational or the Hadamard basis, depending on the corresponding coordinate of  $\mathbf{c}(j)$ . It is easy to see that the maximum overlap  $c$  of the family obtained this way is directly related to the minimum distance of  $\mathcal{C}$ , namely  $\delta = -\frac{1}{n} \log(c^2)$  coincides with the relative minimal distance  $d/n$  of  $\mathcal{C}$ . Hence, choosing an asymptotically good code immediately yields a good family of bases.

## 4 Application: A New Quantum Identification Scheme

Our main application of the new uncertainty relation is in proving security of a new identification scheme in the quantum setting. The goal of (password-based) identification is to “prove” knowledge of a password  $w$  (or some other low-entropy key, like a PIN) without giving  $w$  away. More formally, given a user  $\mathbf{U}$  and a server  $\mathbf{S}$  that hold a pre-agreed password  $w \in \mathcal{W}$ ,  $\mathbf{U}$  wants to convince  $\mathbf{S}$  that he indeed knows  $w$ , but in such a way that he gives away as little information on  $w$  as possible in case he is actually interacting with a dishonest server  $\mathbf{S}^*$ .

In [DFSS07], Damgård *et al.* showed the existence of a secure identification scheme in the *bounded-quantum-storage* model. The scheme involves the communication of qubits, and is secure against an arbitrary dishonest server  $\mathbf{S}$  that has limited quantum storage capabilities and can only store a certain fraction of the communicated qubits, whereas the security against a dishonest user  $\mathbf{U}^*$  holds unconditionally.

On the negative side, it is known that *without* any restriction on (one of) the dishonest participants, secure identification is impossible (even in the quantum setting). Indeed, if a quantum scheme is unconditionally secure against a dishonest user, then unavoidably it can be broken by a dishonest server with unbounded quantum-storage and unbounded quantum-computing power; this follows essentially from [Lo97] (see also [DFSS07]). Thus,

the best one can hope for (for a scheme that is unconditionally secure against a dishonest user) is that in order to break it, unbounded quantum storage *and* unbounded quantum-computing power is *necessary* for the dishonest server. This is not the case for the scheme of [DFSS07]: storing all the communicated qubits as they are, and measuring them qubit-wise in one or the other basis at the end, completely breaks the scheme. Thus, no quantum computing power at all is necessary to break the scheme, only sufficient quantum storage.

In this section, we propose a new identification scheme, which can be regarded as a first step towards closing the above gap. Like the scheme from [DFSS07], our new scheme is secure against an unbounded dishonest user and against a dishonest server with limited quantum storage capabilities. The new uncertainty relation forms the main ingredient in the user-security proof in the BQSM. Furthermore, and in contrast to [DFSS07], a minimal amount of quantum computation power is *necessary* to break the scheme, beyond sufficient quantum storage. Indeed, next to the security against a dishonest server with bounded quantum storage, we also prove—in Section 5—security against a dishonest server that can store all the communicated qubits, but is restricted to measure them qubit-wise (in arbitrary qubit bases) at the end of the protocol execution. Thus, beyond sufficient quantum storage, quantum computation that involves *pairs* of qubits is necessary (and in fact sufficient) to break the new scheme.

Restricting the dishonest server to qubit-wise measurements may look restrictive; however, we stress that in order to break the scheme, the dishonest server needs to store many qubits *and* perform quantum operations on them that go beyond single-qubit operations; this may indeed be considerably more challenging than storing many qubits and measuring them qubit-wise. Furthermore, it turns out that proving security against such a dishonest server that is restricted to qubit-wise measurements is already challenging; indeed, standard techniques do not seem applicable here. Therefore, handling a dishonest server that can, say, act on *blocks* of qubits, must be left to future research.

## 4.1 Security Definitions

We first formalize the security properties we want to achieve. We borrow the definitions from [DFSS07], which are argued to be “the right ones” in [FS09].

**Definition 10** (Correctness). An identification protocol is said to be  $\varepsilon$ -correct if, after an execution by honest  $U$  and honest  $S$ ,  $S$  accepts with probability  $1 - \varepsilon$ .

**Definition 11** (User security). An identification protocol for two parties  $U$ ,  $S$  is  $\varepsilon$ -secure for the user  $U$  against (dishonest) server  $S^*$  if the following holds: If the initial state of  $S^*$  is independent of  $W$ , then its state  $E$  after execution of the protocol is such that there exists a random variable  $W'$  that is independent of  $W$  and such that

$$\rho_{WW'E|W \neq W'} \approx_{\varepsilon} \rho_{W \leftrightarrow W' \leftrightarrow E|W \neq W'}.$$

**Definition 12** (Server security). An identification protocol for two parties  $U$ ,  $S$  is  $\varepsilon$ -secure for the server  $S$  against (dishonest) user  $U^*$  if the following holds: whenever the initial state of  $U^*$  is independent of  $W$ , then there exists a random variable  $W'$  (possibly  $\perp$ ) that is independent of  $W$  such that if  $W \neq W'$  then  $S$  accepts with probability at most  $\varepsilon$ . Furthermore, the common state  $\rho_{WE}$  after execution of the protocol (including  $S$ 's announcement to accept or reject) satisfies

$$\rho_{WW'E|W \neq W'} \approx_{\varepsilon} \rho_{W \leftrightarrow W' \leftrightarrow E|W \neq W'}.$$

We will prove the user-security of the protocol in two different models, in which different assumptions are made. Because these assumptions are in some sense “orthogonal”, the hope is that if security would break down in one model to a failing assumption, the protocol is still secure by the other model.

## 4.2 Description of the New Quantum Identification Scheme

Let  $\mathcal{C} \subset \{+, \times\}^n$  be a binary code with minimum distance  $d$ , and let  $\mathfrak{c} : \mathcal{W} \rightarrow \mathcal{C}$  be its encoding function. Let  $m := |\mathcal{W}|$ , and typically,  $m < 2^n$ . Let  $\mathcal{F}$  be the class of all linear functions from  $\{0, 1\}^n$  to  $\{0, 1\}^\ell$ , where  $\ell < n$ , represented as  $\ell \times n$  binary matrices. It is well-known that this class is two-universal. Furthermore, let  $\mathcal{G}$  be a strongly two-universal class of hash functions from  $\mathcal{W}$  to  $\{0, 1\}^\ell$ . Protocol Q-ID is shown below.

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### Protocol Q-ID

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1. U picks  $x \in \{0, 1\}^n$  independently and uniformly at random and sends  $|x\rangle_{\mathfrak{c}(w)}$  to S.
  2. S measures in basis  $\mathfrak{c}(w)$ . Let  $x'$  be the outcome.
  3. U picks  $f \in \mathcal{F}$  independently and uniformly at random and sends it to S
  4. S picks  $g \in \mathcal{G}$  independently and uniformly at random and sends it to U
  5. U computes and sends  $z := f(x) \oplus g(w)$  to S
  6. S accepts if and only if  $z = z'$  where  $z' := f(x') \oplus g(w)$
- 

Our scheme is quite similar to the scheme in [DFSS07]. The difference is that in our scheme, both parties, U and S, use  $\mathfrak{c}(w)$  as basis for preparing/measuring the qubits in step (1) and (2), whereas in [DFSS07], only S uses  $\mathfrak{c}(w)$  and U uses a *random* basis  $\theta \in \{+, \times\}^n$  instead, and then U communicates  $\theta$  to S and all the positions where  $\theta$  and  $\mathfrak{c}(w)$  differ are dismissed. Thus, in some sense, our new scheme is more natural since why should U use a random basis when he knows the right basis (i.e., the one that S uses)? In [DFSS07], using a random basis (for U) was crucial for their proof technique, which is based on an entropic uncertainty relation of a certain form, which asks for a random basis. However, using a random basis, which then needs to be announced, renders the scheme insecure against a dishonest server  $S^*$  that is capable of storing all the communicated qubits and then measure them in the right basis once it has been announced. Our new uncertainty relation applies to the case where an  $n$ -qubit state is measured in a basis that is sampled from a code  $\mathcal{C}$ , and thus is applicable to the new scheme where U uses basis  $\mathfrak{c}(w) \in \mathcal{C}$ . Since this basis is common knowledge (to the honest participants), it does not have to be communicated, and as such a straightforward store-and-then-measure attack as above does not apply.

A downside of our scheme is that security only holds in case of a perfect quantum source, which emits exactly one qubit when triggered. Indeed, a multi-photon emission enables a dishonest server  $S^*$  to learn information on the basis used, and thus gives away information on the password  $w$  in our scheme. As such, our scheme is currently mainly of theoretical interest.

It is straightforward to verify that (in the ideal setting with perfect sources, no noise, etc.) Q-ID satisfies the correctness property (Definition 10) perfectly, i.e.  $\varepsilon = 0$ . In the remaining sections, we prove (unconditional) security against a dishonest user, and we prove security against two kinds of restricted dishonest servers. First, against a dishonest

server that has limited quantum storage capabilities, and then against a dishonest server that can store an unbounded number of qubits, but can only store and measure them qubit-wise.

### 4.3 (Unconditional) Server Security

First, we claim security of Q-ID against an arbitrary dishonest user  $U^*$  (that is merely restricted by the laws of quantum mechanics).

**Theorem 13.** *Q-ID is  $\varepsilon$ -secure for the server with  $\varepsilon = \binom{m}{2}2^{-\ell}$ .*

*Proof.* Clearly, from the steps (1) to (5) in the protocol Q-ID,  $U^*$  learns no information on  $W$  at all. The only information he may learn is by observing whether  $S$  accepts or not in step (6). Therefore, in order to prove server security, it suffices to show the existence of a random variable  $W'$ , independent of  $W$ , with the property that  $S$  rejects whenever  $W' \neq W$  (except with probability  $\frac{1}{2}m(m-1)2^{-\ell}$ ).

We may assume that  $\mathcal{W} = \{1, \dots, m\}$ . Let  $\rho_{WX'FGZE}$  be the state describing the password  $W$ , the variables  $X', F, G$  and  $Z$  occurring in the protocol from the server's point of view, and  $U^*$ 's quantum state  $E$  before observing  $S$ 's decision to accept or reject. For any  $w \in \mathcal{W}$ , consider the state  $\rho_{X'FGZE|W=w}^w := \rho_{X'FGZE|W=w}$ . Note that the reduced state  $\rho_{FGZE}^w$  is the same for any  $w \in \mathcal{W}$ ; this follows from the assumption that  $U^*$ 's initial state is independent of  $W$  and because  $F, G$  and  $Z$  are produced independently of  $W$ . We may thus write  $\rho_{X'FGZE}^w$  as  $\rho_{X'_wFGZE}$ , and we can “glue together” the states  $\rho_{X'_wFGZE}$  for all choices of  $w$ . This means, there exists a state  $\rho_{X'_1 \dots X'_m FGZE_1 \dots E_m}$  that correctly reduces to  $\rho_{X'_wFGZE_w} = \rho_{X'_wFGZE}$  for any  $w \in \mathcal{W}$ , and conditioned on  $FGZ$ , we have that  $X'_i E_i$  is independent of  $X'_j E_j$  for any  $i \neq j \in \mathcal{W}$ . It is easy to see that for any  $i \neq j \in \mathcal{W}$ ,  $G$  is independent of  $X'_i, X'_j$  and  $F$ . Therefore, by the strong two-universality of  $G$ , for any  $i \neq j$  it holds that  $Z'_i \neq Z'_j$  except with probability  $2^{-\ell}$ , where  $Z'_w = FX'_w + G(w)$  for any  $w$ . Therefore, by the union bound,  $Z'_1, \dots, Z'_m$  are pairwise distinct and thus  $Z$  can coincide with at most one of the  $Z'_w$ 's, except with probability  $\varepsilon = \frac{1}{2}m(m-1)2^{-\ell}$ . Let  $W'$  be defined such that  $Z = Z'_{W'}$ ; if there is no such  $Z'_w$  then we let  $W' = \perp$ , and if there are more than one then we let it be the first. Recall, the latter can happen with probability at most  $\varepsilon$ . We now extend the state  $\rho_{X'_1 \dots X'_m FGZW'E_1 \dots E_m}$  by  $W$ , chosen independently according to  $P_W$ . Clearly  $W'$  is independent of  $W$ . Furthermore, except with probability at most  $\varepsilon$ , if  $W \neq W'$  then  $Z \neq Z'_W$ . Finally note that  $\rho_{X'_W FGZW'WE_W}$  is such that  $\rho_{X'_W FGZW'WE_W} = \sum_w P_W(w) \rho_{X'_w FGZE_w} \otimes |w\rangle\langle w| = \sum_w P_W(w) \rho_{X'_w FGZE}^w \otimes |w\rangle\langle w| = \rho_{X'FGZWE}$ . Thus, also with respect to the state  $\rho_{X'FGZWE}$  there exist  $W'$ , independent of  $W$ , such that if  $W' \neq W$  then  $Z \neq Z'$  except with probability at most  $\varepsilon$ . This was to be shown.  $\square$

### 4.4 User Security in the Bounded-Quantum-Storage Model

Next, we consider a dishonest server  $S^*$ , and first prove security of Q-ID in the *bounded-quantum-storage model*. In this model, as introduced in [DFSS05], it is assumed that the adversary (here  $S^*$ ) cannot store more than a fixed number of qubits, say  $q$ . The security proof of Q-ID in the bounded-quantum-storage model is very similar to the corresponding proof in [DFSS07] for their scheme, except that we use the new uncertainty relation from Section 3. Furthermore, since our uncertainty relation (Theorem 9) already guarantees the existence of the random variable  $W'$  as required by the security property, no *entropy-splitting* as in [DFSS07] is needed.

In the following, let  $\delta := d/n$ , i.e. the relative minimum distance of  $\mathcal{C}$ .

**Theorem 14.** *Let  $S^*$  be a dishonest server whose quantum memory is at most  $q$  qubits at step 3 of Q-ID. Then, for any  $0 < \kappa < \delta/4$ , Q-ID is  $\varepsilon$ -secure for the user with*

$$\varepsilon = 2^{-\frac{1}{2}((\delta/2-2\kappa)n-1-q-\ell)} + 4 \cdot 2^{-\kappa n}.$$

*Proof.* We consider and analyze a purified version of Q-ID where in step (1) instead of sending  $|X\rangle_c$  to  $S^*$  for a uniformly distributed  $X$ ,  $U$  prepares a fully entangled state  $2^{-n/2} \sum_x |x\rangle|x\rangle$  and sends the second register to  $S^*$  while keeping the first. Then, in step (3) when the memory bound has applied,  $U$  measures his register in the basis  $\mathfrak{c}(W)$  in order to obtain  $X$ . Note that this procedure produces exactly the same common state as in the original (non-purified) version of Q-ID. Thus, we may just as well analyze this purified version.

The state of  $S^*$  consists of his initial state and his part of the EPR pairs, and may include an additional ancilla register. Before the memory bound applies,  $S^*$  may perform any unitary transformation on his composite system. When the memory bound is applied (just before step (3) is executed in Q-ID),  $S^*$  has to measure all but  $q$  qubits of his system. Let the classical outcome of this measurement be denoted by  $y$ , and let  $E'$  be the remaining quantum state of at most  $q$  qubits. The common state has collapsed to a  $(n+q)$ -qubit state and depends on  $y$ ; the analysis below holds for any  $y$ . Next,  $U$  measures his  $n$ -qubit part of the common state in basis  $\mathfrak{c}(W)$ ; let  $X$  denote the classical outcome of this measurement. By our new uncertainty relation (Theorem 9) and subsequently applying the min-entropy chain rule that is given in Lemma 5 (to take the  $q$  stored qubits into account) it follows that there exists  $W'$ , independent of  $W$ , and an event  $\Psi$  that occurs at least with probability  $1 - 2 \cdot 2^{-\kappa n}$ , such that

$$H_{\min}(X|E', W = w, W' = w', \Psi) \geq (\delta/2 - 2\kappa)n - 1 - q.$$

for any  $w, w'$  such that  $w \neq w'$ . Because  $U$  chooses  $F$  independently at random from a 2-universal family, privacy amplification guarantees that

$$d_{\text{unif}}(F(X)|E'F, W = w, W' = w') \leq \varepsilon' := \frac{1}{2} \cdot 2^{-\frac{1}{2}((\delta/2-2\kappa)n-1-q-\ell)} + 2 \cdot 2^{-\kappa n},$$

for any  $w, w'$  such that  $w \neq w'$ . Recall that  $Z = F(X) \oplus G(W)$ . By security of the one-time pad it follows that

$$d_{\text{unif}}(Z|E'FG, W = w, W' = w') \leq \varepsilon', \quad (2)$$

for any  $w, w'$  such that  $w \neq w'$ . To prove the claim, we need to bound,

$$\begin{aligned} & \delta(\rho_{WW'E|W \neq W'}, \rho_{W \leftrightarrow W' \leftrightarrow E|W \neq W'}) \\ &= \frac{1}{2} \|\rho_{WW'E'FGZ|W \neq W'} - \rho_{W \leftrightarrow W' \leftrightarrow E'FGZ|W \neq W'}\|_1 \\ &\leq \frac{1}{2} \|\rho_{WW'E'FGZ|W \neq W'} - \rho_{WW'E'FG|W \neq W'} \otimes 2^{-\ell} \mathbb{I}\|_1 \\ &\quad + \frac{1}{2} \|\rho_{WW'E'FG|W \neq W'} \otimes 2^{-\ell} \mathbb{I} - \rho_{W \leftrightarrow W' \leftrightarrow E'FGZ|W \neq W'}\|_1 \end{aligned} \quad (3)$$

where the equality follows by definition of trace distance (Definition 3) and the fact that the output state  $E$  is obtained by applying a unitary transformation to the set of registers

$(E', F, G, W', Z)$ . The inequality is the triangle inequality; in the remainder of the proof, we will show that both terms in (3) are upper bounded by  $\varepsilon'$ .

$$\begin{aligned} & \frac{1}{2} \|\rho_{WW'E'FGZ|W \neq W'} - \rho_{WW'E'FG|W \neq W'} \otimes 2^{-\ell} \mathbb{I}\|_1 \\ &= \sum_{w \neq w'} P_{WW'|W \neq W'}(w, w') d_{\text{unif}}(Z|E'FG, W = w, W' = w') \leq \varepsilon', \end{aligned}$$

where the latter inequality follows from (2). For the other term, we reason as follows:

$$\begin{aligned} & \frac{1}{2} \|\rho_{WW'E'FG|W \neq W'} \otimes 2^{-\ell} \mathbb{I} - \rho_{W \leftrightarrow W' \leftrightarrow E'FGZ|W \neq W'}\|_1 \\ &= \frac{1}{2} \sum_{w \neq w'} P_{WW'|W \neq W'}(w, w') \|\rho_{E'FG|W \neq W'}^{w, w'} \otimes 2^{-\ell} \mathbb{I} - \rho_{E'FGZ|W \neq W'}^{w'}\|_1 \\ &= \frac{1}{2} \sum_{w \neq w'} P_{WW'|W \neq W'}(w, w') \|\rho_{E'FG|W \neq W'}^{w, w'} \otimes 2^{-\ell} \mathbb{I} \\ &\quad - \sum_{\substack{w'' \\ \text{s.t. } w'' \neq w'}} P_{W|W', W \neq W'}(w''|w') \rho_{E'FGZ|W \neq W'}^{w'', w'}\|_1 \\ &= \frac{1}{2} \sum_{w'} P_{W'|W \neq W'}(w') \left\| \sum_{\substack{w \\ \text{s.t. } w \neq w'}} P_{W|W', W \neq W'}(w|w') \rho_{E'FG|W \neq W'}^{w, w'} \otimes 2^{-\ell} \mathbb{I} \right. \\ &\quad \left. - \sum_{\substack{w'' \\ \text{s.t. } w'' \neq w'}} P_{W|W', W \neq W'}(w''|w') \rho_{E'FGZ|W \neq W'}^{w'', w'} \sum_{\substack{w \\ \text{s.t. } w \neq w'}} P_{W|W', W \neq W'}(w|w') \right\|_1 \\ &= \frac{1}{2} \sum_{w \neq w'} P_{WW'|W \neq W'}(w, w') \|\rho_{E'FG|W \neq W'}^{w, w'} \otimes 2^{-\ell} \mathbb{I} - \rho_{E'FGZ|W \neq W'}^{w, w'}\|_1 \\ &= \sum_{w \neq w'} P_{WW'|W \neq W'}(w, w') d_{\text{unif}}(Z|E'FG, W = w, W' = w') \leq \varepsilon', \end{aligned}$$

where the first equality follows by definition of conditional independence and by a basic property of the trace distance; the third and fourth equality follow by linearity of the trace distance. The inequality on the last line follows from (2). This proves the claim.  $\square$

## 5 User Security in the Single-Qubit-Operations Model

We now consider a dishonest server  $S^*$  that can store an unbounded number of qubits. Clearly, against such a  $S^*$ , Theorem 14 provides no security guarantee anymore. We show here that there is still *some* level of security left. Specifically, we show that Q-ID is still secure against a dishonest server  $S^*$  that can reliably store all the communicated qubits and measure them qubit-wise and non-adaptively at the end of the protocol. This feature distinguishes our identification protocol from the protocol from [DFSS07], which completely breaks down against such an attack.

### 5.1 The Model

Formally, a dishonest server  $S^*$  in the SQOM is modeled as follows.

1.  $S^*$  may reliably store the  $n$ -qubit state  $|x\rangle_{c(w)} = |x_1\rangle_{c(w)_1} \otimes \cdots \otimes |x_n\rangle_{c(w)_n}$  received in step (1) of Q-ID.

2. At the end of the protocol, in step (5),  $S^*$  chooses an arbitrary sequence  $\theta = (\theta_1, \dots, \theta_n)$ , where each  $\theta_i$  describes an arbitrary orthonormal basis of  $\mathbb{C}^2$ , and measures each qubit  $|x_i\rangle_{c(w)_i}$  in basis  $\theta_i$  to observe  $Y_i \in \{0, 1\}$ . Hence, we assume that  $S^*$  measures all qubits at the end of the protocol.
3. The choice of  $\theta$  may depend on all the classical information gathered during the execution of the protocol, but we assume a *non-adaptive* setting where  $\theta_i$  does not depend on  $Y_j$  for  $i \neq j$ , i.e.,  $S^*$  has to choose  $\theta$  entirely before performing any measurement.

Considering complete projective measurements acting on individual qubits, rather than general single-qubit POVMs, may be considered a restriction of our model. Nonetheless, general POVM measurements can always be described by projective measurements on a bigger system. In this sense, restricting to projective measurements is consistent with the requirement of single-qubit operations. It seems non-trivial to extend our security proof to general single-qubit POVMs.

The restriction to non-adaptive measurements (item 3) is rather strong, even though the protocol from [DFSS07] already breaks down in this non-adaptive setting. The restriction was introduced as a stepping stone towards proving the adaptive case. Up to now, we have unfortunately not yet succeeded in doing so, hence we leave the adaptive case for future research.

We also leave for future research the case of a less restricted dishonest server  $S^*$  that can do measurements on blocks that are less stringently bounded in size. Whereas the adaptive versus non-adaptive issue appears to be a proof-technical problem (Q-ID looks secure also against an adaptive  $S^*$ ), allowing measurements on larger blocks will require a new protocol, since Q-ID becomes insecure when  $S^*$  can do measurements on blocks of size 2, as we show in Section 5.5.

## 5.2 No Privacy Amplification

One might expect that proving security of Q-ID in the SQOM, i.e., against a dishonest server  $S^*$  that is restricted to single-qubit operations should be straightforward, but actually the opposite is true, for the following reason. Even though it is not hard to show that after his measurements,  $S^*$  has lower bounded uncertainty in  $x$  (except if he was able to guess  $w$ ), it is not clear how to conclude that  $f(x)$  is close to random so that  $z$  does not reveal a significant amount of information about  $w$ . The reason is that standard privacy amplification fails to apply here. Indeed, the model allows  $S^*$  to postpone the measurement of all qubits to step (5) of the protocol. The hash function  $f$ , however, is chosen and sent already in step (3). This means that  $S^*$  can choose his measurements in step (5) depending on  $f$ . As a consequence, the distribution of  $x$  from the point of view of  $S^*$  may depend on the choice of the hash function  $f$ , in which case the privacy-amplification theorem does not give any guarantees.

## 5.3 Single-Qubit Measurements

Consider an arbitrary sequence  $\theta = (\theta_1, \dots, \theta_n)$  where each  $\theta_i$  describes an orthonormal basis of  $\mathbb{C}^2$ . Let  $|\psi\rangle$  be an  $n$ -qubit system of the form

$$|\psi\rangle = |x\rangle_b = H^{b_1}|x_1\rangle \otimes \dots \otimes H^{b_n}|x_n\rangle,$$

where  $x$  and  $b$  are arbitrary in  $\{0, 1\}^n$ . Measuring  $|\psi\rangle$  qubit-wise in basis  $\theta$  results in a measurement outcome  $Y = (Y_1, \dots, Y_n) \in \{0, 1\}^n$ . Suppose that  $x$ ,  $b$  and  $\theta$  are in fact realizations of the random variables  $X$ ,  $B$  and  $\Theta$  respectively. It follows immediately from the product structure of the state  $|\psi\rangle$  that

$$P_{Y|XB\Theta}(y|x, b, \theta) = \prod_{i=0}^n P_{Y_i|X_i B_i \Theta_i}(y_i|x_i, b_i, \theta_i),$$

i.e. the random variables  $Y_i$  are statistically independent conditioned on arbitrary fixed values for  $X_i$ ,  $B_i$  and  $\Theta_i$  but such that  $P_{X_i B_i \Theta_i}(x_i, b_i, \theta_i) > 0$ .

**Lemma 15.** *The distribution  $P_{Y_i|X_i B_i \Theta_i}(y_i|x_i, b_i, \theta_i)$  exhibits the following symmetries:*

$$P_{Y_i|X_i B_i \Theta_i}(0|0, b_i, \theta_i) = P_{Y_i|X_i B_i \Theta_i}(1|1, b_i, \theta_i)$$

and

$$P_{Y_i|X_i B_i \Theta_i}(0|1, b_i, \theta_i) = P_{Y_i|X_i B_i \Theta_i}(1|0, b_i, \theta_i)$$

for all  $i \in [n]$ , for all  $b_i$  and  $\theta_i$  with  $P_{X_i B_i \Theta_i}(\xi, b_i, \theta_i) > 0$  for all  $\xi \in \{0, 1\}$ .

The proof can found in Appendix C. The symmetry characterized in Lemma 15 coincides with that of the *binary symmetric channel*, i.e. we can view  $Y$  as a “noisy version” of  $X$ , where this noise—produced by the measurement—is independent of  $X$ .

Formally, we can write  $Y$  as

$$Y = X \oplus \Delta, \tag{4}$$

where the random variable  $\Delta = (\Delta_1, \dots, \Delta_n) \in \{0, 1\}^n$  thus represents the error between the random variable  $X \in \{0, 1\}^n$  that is “encoded” in the quantum state and the measurement outcome  $Y \in \{0, 1\}^n$ . By substituting (4) in Lemma 15, we get the following corollary.

**Corollary 16** (Independence Between  $\Delta$  and  $X$ ). *For every  $i \in [n]$  it holds that*

$$P_{\Delta_i|X_i B_i \Theta_i}(\delta_i|x_i, b_i, \theta_i) = P_{\Delta_i|B_i \Theta_i}(\delta_i|b_i, \theta_i)$$

for all  $\delta_i \in \{0, 1\}$  and for all  $x_i, b_i$  and  $\theta_i$  such that  $P_{X_i B_i \Theta_i}(x_i, b_i, \theta_i) > 0$ .

Furthermore, since the random variables  $Y_i$  are statistically independent conditioned on fixed values for  $X_i$ ,  $B_i$  and  $\Theta_i$ , it follows that the  $\Delta_i$  are statistically independent conditioned on fixed values for  $B_i$  and  $\Theta_i$ .

**Definition 17** (Quantized Basis). For any orthonormal basis  $\theta_i = \{|v_1\rangle, |v_2\rangle\}$  on  $\mathbb{C}^2$ , we define the *quantized basis* of  $\theta_i$  as

$$\hat{\theta}_i := j^* \in \{0, 1\}, \quad \text{where } j^* \in \arg \max_{j \in \{0, 1\}} \max_{k \in \{1, 2\}} |\langle v_k | H^j | 0 \rangle|.$$

If both  $j \in \{0, 1\}$  attain the maximum, then  $j^*$  is chosen arbitrarily. The quantized basis of the sequence  $\theta = (\theta_1, \dots, \theta_n)$  is naturally defined as the element-wise application of the above, resulting in  $\hat{\theta} \in \{0, 1\}^n$ .

We will use the bias as a measure for the predictability of  $\Delta_i$ .

**Theorem 18.** *When measuring the qubit  $H^{b_i}|x_i\rangle$  for any  $x_i, b_i \in \{0, 1\}$  in any orthonormal basis  $\theta_i$  on  $\mathbb{C}^2$  for which the quantized basis  $\hat{\theta}_i$  is the complement of  $b_i$ , i.e.  $\hat{\theta}_i = b_i \oplus 1$ , then the bias of  $\Delta_i \in \{0, 1\}$ , where  $\Delta_i = Y_i \oplus x_i$  and  $Y_i \in \{0, 1\}$  is the measurement outcome, is upper bounded by*

$$\text{bias}(\Delta_i) \leq \frac{1}{\sqrt{2}}.$$

Since the theorem holds for any  $x_i \in \{0, 1\}$  and since Corollary 16 guarantees that  $\Delta_i$  is independent from an arbitrary random variable  $X_i$ , the theorem also applies when we replace  $x_i$  by the random variable  $X_i$ .

In order to prove Theorem 18, we need the following lemma.

**Lemma 19.** *If, for any orthonormal basis  $\theta_i$  on  $\mathbb{C}^2$ , there exists a bit  $b_i \in \{0, 1\}$  so that when measuring the qubit  $H^{b_i}|x_i\rangle$  for any  $x_i \in \{0, 1\}$  in the basis  $\theta_i$  to obtain  $Z_i \in \{0, 1\}$  it holds that*

$$\text{bias}(Z_i) \geq 1/\sqrt{2},$$

*then it holds that when measuring the qubit  $H^{b_i \oplus 1}|x_i\rangle$  in the basis  $\theta_i$  to obtain  $Y_i \in \{0, 1\}$ ,*

$$\text{bias}(Y_i) \leq 1/\sqrt{2}.$$

*Proof.* First note that for any  $x_i, b_i \in \{0, 1\}$  and any orthonormal basis  $\theta_i$  on  $\mathbb{C}^2$ , measuring a state  $H^{b_i}|x_i\rangle$  in  $\theta_i = \{|v\rangle, |w\rangle\}$  where  $|v\rangle = \alpha|0\rangle + \beta|1\rangle$  and  $|w\rangle = \beta|0\rangle - \alpha|1\rangle$  gives the same outcome distribution (up to permutations) as when measuring one of the basis states of  $\theta_i$  (when viewed as a quantum state), say  $|w\rangle$ , using the basis  $\{H^{b_i}|x_i\rangle, H^{b_i}|x_i \oplus 1\rangle\}$ . To see why this holds, note that it follows immediately that  $|\langle w|H^{b_i}|x_i\rangle|^2 = |\langle x_i|H^{b_i}|w\rangle|^2$ . Furthermore, we have already shown in the proof of Lemma 15 that

$$|\langle v|H^{b_i}|x_i\rangle|^2 = |\langle w|H^{b_i}|x_i \oplus 1\rangle|^2$$

holds.

Hence, we can apply Theorem 7 with  $\rho = |w\rangle\langle w|$  (this implies that  $n = 1$ ),  $m = 2$  and  $\mathcal{B}_0$  and  $\mathcal{B}_1$  are the computational and Hadamard basis respectively. The maximum overlap between those bases is  $c = 1/\sqrt{2}$ . Theorem 7 gives us that

$$p_{\max}^{\{|0\rangle, |1\rangle\}} + p_{\max}^{\{|+\rangle, |-\rangle\}} \leq 1 + \frac{1}{\sqrt{2}},$$

where  $p_{\max}^{\{|0\rangle, |1\rangle\}}$  and  $p_{\max}^{\{|+\rangle, |-\rangle\}}$  respectively denote the maximum probability in the distribution obtained by measuring in the computational and Hadamard basis. By simple manipulations we can write this as a bound on the sum of the biases:

$$\begin{aligned} \frac{2}{\sqrt{2}} &\geq (2p_{\max}^{\{|0\rangle, |1\rangle\}} - 1) + (2p_{\max}^{\{|+\rangle, |-\rangle\}} - 1) \\ &= \text{bias}(Y_i) + \text{bias}(Z_i). \end{aligned} \tag{5}$$

From this relation, the claim follows immediately.  $\square$

Following [Sch07], we want to remark that both biases in (5) are equal to  $1/\sqrt{2}$  when  $\theta_i$  is the *Breidbart basis*, which is the basis that is precisely “in between” the computational and the Hadamard basis:<sup>8</sup>

$$|v\rangle = \cos(\frac{\pi}{8})|0\rangle + \sin(\frac{\pi}{8})|1\rangle \quad \text{and} \quad |w\rangle = \sin(\frac{\pi}{8})|0\rangle - \cos(\frac{\pi}{8})|1\rangle.$$

<sup>8</sup>In [Sch07], the corresponding state is called the “Hadamard-invariant state.”

*Proof of Theorem 18 .* Let  $\theta_i = \{|v_0\rangle, |v_1\rangle\}$ . We will make a case distinction based on the value of

$$\mu := \max_{k \in \{0,1\}} |\langle v_k | H^{\hat{\theta}_i} | 0 \rangle|. \quad (6)$$

If  $\mu \leq \cos(\pi/8)$ , then we also have that  $\max_{k \in \{0,1\}} |\langle v_k | H^{b_i} | x_i \rangle| \leq \cos(\pi/8)$  where  $b_i = \hat{\theta}_i \oplus 1$ , this holds by definition of the quantized basis (Definition 17). Then, the probability of obtaining outcome  $Y_i = k^*$ , where  $k^* \in \{0, 1\}$  achieves the maximum in (6), is bounded by

$$P_{Y_i}(k^*) = |\langle v_{k^*} | H^{b_i} | x_i \rangle|^2 \leq \cos^2(\pi/8) = \frac{1}{2} + \frac{1}{2\sqrt{2}}.$$

Hence,

$$\text{bias}(\Delta_i) = \text{bias}(Y_i) = |P_{Y_i}(k^*) - (1 - P_{Y_i}(k^*))| = |2P_{Y_i}(k^*) - 1| \leq \frac{1}{\sqrt{2}}.$$

If  $\mu > \cos(\pi/8)$ , then when measuring the state  $H^{\hat{\theta}_i} | x_i \rangle$  in  $\theta_i$  to obtain  $Z_i \in \{0, 1\}$ , we have that  $\text{bias}(Z_i) > 1/\sqrt{2}$  (this follows from similar computations as performed above). We now invoke Lemma 19 to conclude that when measuring the state  $H^{b_i} | x_i \rangle$  in  $\theta_i$  to obtain  $Y_i$ ,  $\text{bias}(\Delta_i) = \text{bias}(Y_i) < \frac{1}{\sqrt{2}}$ .  $\square$

## 5.4 User Security of Q-ID

We are now ready to state and prove the security of Q-ID against a dishonest user in the SQOM.

**Theorem 20** (User Security). *Let  $S^*$  be a dishonest server with unbounded quantum storage that is restricted to non-adaptive single-qubit operations, as specified in Section 5.1. Then, for any  $0 < \beta < \frac{1}{4}$ , user security (as defined in Definition 11) holds with*

$$\varepsilon \leq \frac{1}{2} 2^{\frac{1}{2}\ell - \frac{1}{4}(1-\beta)d} + \binom{m}{2} 2^{2\ell} \exp(-2d\beta^2)$$

Note that  $d$  is typically linear in  $n$  whereas  $\ell$  is chosen independently of  $n$ , hence the expression above is negligible in  $d$ .

To prove Theorem 20 we need the following technical lemma and corollary. Recall that  $\mathcal{F}$  denotes the class of all linear functions from  $\{0, 1\}^n$  to  $\{0, 1\}^\ell$ , where  $\ell < n$ , represented as binary  $\ell \times n$  matrices.

**Lemma 21.** *Let  $n, k$  and  $\ell$  be arbitrary positive integers, let  $0 < \beta < \frac{1}{4}$  and let  $\mathcal{I} \subset [n]$  such that  $|\mathcal{I}| \geq k$ , and let  $F$  be uniform over  $\mathcal{F} = \{0, 1\}^{\ell \times n}$ . Then, it holds except with probability  $2^{2\ell} \exp(-2k\beta^2)$  (the probability is over the random matrix  $F$ ) that*

$$|(f \odot g)_{\mathcal{I}}| > (\frac{1}{4} - \beta)k \quad \forall f, g \in \text{span}(F) \setminus \{0^n\}$$

*Proof.* Without loss of generality, we will assume that  $|\mathcal{I}| = k$ . Now take arbitrary but non-zero vectors  $r, s \in \{0, 1\}^\ell$  and let  $V := rF$  and  $W := sF$ . We will analyze the case  $r \neq s$ ; the case  $r = s$  is similar but simpler. Because each element of  $F$  is an independent random bit, and  $r$  and  $s$  are non-zero and  $r \neq s$ ,  $V$  and  $W$  are independent and uniformly distributed  $n$ -bit vectors with expected relative Hamming weight  $1/2$ . Hence, on average  $|(V \odot W)_{\mathcal{I}}|$  equals  $k/4$ . Furthermore, using Hoeffding's inequality (Theorem 2), we may conclude that

$$\Pr \left[ \frac{k}{4} - |(V \odot W)_{\mathcal{I}}| > \beta k \right] = \Pr \left[ |(V \odot W)_{\mathcal{I}}| < (\frac{1}{4} - \beta)k \right] \leq \exp(-2k\beta^2).$$

Finally, the claim follows by applying the union bound over the choice of  $r$  and  $s$  (each  $2^\ell$  possibilities).  $\square$

Recall that  $\mathcal{C}$  is a binary code with minimum distance  $d$ ,  $\mathbf{c}(\cdot)$  its encoding function, and that  $m := |\mathcal{W}|$ .

**Corollary 22.** *Let  $0 < \beta < \frac{1}{4}$ , and let  $F$  be uniformly distributed over  $\mathcal{F}$ . Then,  $F$  has the following property except with probability  $\binom{m}{2}2^{2\ell} \exp(-2d\beta^2)$ : for any string  $s \in \{0, 1\}^n$  (possibly depending on the choice of  $F$ ), there exists at most one  $\tilde{c} \in \mathcal{C}$  such that for any code word  $c \in \mathcal{C}$  different from  $\tilde{c}$ , it holds that*

$$|f \odot (c \oplus s)| \geq \frac{1}{2}(\frac{1}{4} - \beta)d \quad \forall f \in \text{span}(F) \setminus \{0^n\}$$

We prove the statement by arguing for two  $\tilde{c}$ 's and showing that they must be identical. In the proof, we will make use of the two following propositions.

**Proposition 23.**  $|a| \geq |a \odot b|$  for all  $a, b \in \{0, 1\}^n$ .

*Proof.* Follows immediately.  $\square$

**Proposition 24.**  $|a \odot b| + |a \odot c| \geq |a \odot (b \oplus c)|$  for all  $a, b, c \in \{0, 1\}^n$ .

*Proof.*  $|a \odot (b \oplus c)| = |a \odot b \oplus a \odot c| \leq |a \odot b| + |a \odot c|$ , where the equality is the distributivity of the Schur product, and the inequality is the triangle inequality for the Hamming weight.  $\square$

*Proof of Corollary 22.* By Lemma 21 with  $\mathcal{I} := \{i \in [n] : c_i \neq c'_i\}$  for  $c, c' \in \mathcal{C}$ , and by applying the union bound over all possible pairs  $(c, c')$ , we obtain that except with probability  $\binom{m}{2}2^{2\ell} \exp(-2d\beta^2)$  (over the choice of  $F$ ), it holds that

$$|f \odot g \odot (c \oplus c')| > (\frac{1}{4} - \beta)d \tag{7}$$

for all  $f, g \in \text{span}(F) \setminus \{0^n\}$  and all  $c, c' \in \mathcal{C}$  with  $c \neq c'$ .

Now, for such an  $F$ , and for every choice of  $s \in \{0, 1\}^n$ , consider  $\tilde{c}_1, \tilde{c}_2 \in \mathcal{C}$  and  $f_1, f_2 \in \text{span}(F) \setminus \{0^n\}$  such that

$$|f_1 \odot (\tilde{c}_1 \oplus s)| < \frac{1}{2}(\frac{1}{4} - \beta)d \quad \text{and} \quad |f_2 \odot (\tilde{c}_2 \oplus s)| < \frac{1}{2}(\frac{1}{4} - \beta)d.$$

We will show that this implies  $\tilde{c}_1 = \tilde{c}_2$ , which proves the claim. Indeed, we can write

$$\begin{aligned} (\frac{1}{4} - \beta)d &> |f_1 \odot (\tilde{c}_1 \oplus s)| + |f_2 \odot (\tilde{c}_2 \oplus s)| \\ &\geq |f_1 \odot f_2 \odot (\tilde{c}_1 \oplus s)| + |f_1 \odot f_2 \odot (\tilde{c}_2 \oplus s)| \geq |f_1 \odot f_2 \odot (\tilde{c}_1 \oplus \tilde{c}_2)| \end{aligned}$$

where the second inequality is Proposition 23 applied twice and the third inequality is Proposition 24. This contradicts (7) unless  $\tilde{c}_1 = \tilde{c}_2$ .  $\square$

Now we are ready to prove Theorem 20. In the proof, when  $F \in \mathcal{F}$  acts on an  $n$ -bit vector  $x \in \{0, 1\}^n$ , we prefer the notation  $F(x)$  over matrix-product notation  $Fx$ .<sup>9</sup>

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<sup>9</sup>When using matrix-product notation ambiguities could arise, e.g. in subscripts of probability distributions like  $P_{FX}$ : then it is not clear whether this means the joint distribution of  $F$  and  $X$  or the distribution of  $F$  acting on  $X$ ?

*Proof of Theorem 20.* Consider an execution of Q-ID, with a dishonest server  $S^*$  as described in Section 5.1. We let  $W, X$  and  $Z$  be the random variables that describe the values  $w, x$  and  $z$  occurring in the protocol.

From Q-ID's description, we see that  $F$  is uniform over  $\mathcal{F}$ . Hence, by Corollary 22 it will be "good" (in the sense that the bound from Corollary 22 holds) except with probability  $\binom{m}{2}2^{2\ell}\exp(-2d\beta^2)$ . From here, we consider a fixed choice for  $F$  and condition on the event that it is "good," we thus book-keep the probability that  $F$  is "bad" and take it into account at the end of the analysis. Although we have fixed  $F$ , we will keep using capital notation for it, to emphasize that  $F$  is a matrix. We also fix  $G = g$  for an arbitrary  $g$ ; the analysis below holds for any such choice.

Let  $\Theta$  describe the qubit-wise measurement performed by  $S^*$  at the end of the execution, and  $Y$  the corresponding measurement outcome. By the non-adaptivity restriction and by the requirement in Definition 11 that  $S^*$  is initially independent of  $W$ , we may conclude that, once  $G$  and  $F$  are fixed,  $\Theta$  is a function of  $Z$ . (Recall that  $Z = F(X) \oplus g(W)$ .)

We will define  $W'$  with the help of Corollary 22. Let  $\hat{\Theta}$  be the quantized basis of  $\Theta$ , as defined in Definition 17. Given a fixed value  $\theta$  for  $\Theta$ , and thus a fixed value  $\hat{\theta}$  for  $\hat{\Theta}$ , we set  $s$ , which is a variable that occurs in Corollary 22, to  $s = \hat{\theta}$ . Corollary 22 now guarantees that there exists *at most one*  $\tilde{c}$ . If  $\tilde{c}$  indeed exists, then we choose  $w'$  such that  $\mathfrak{c}(w') = \tilde{c}$ . Otherwise, we pick  $w' \in \mathcal{W}$  arbitrarily (any choice will do). Note that this defines the random variable  $W'$ , and furthermore note that  $Z \rightarrow \Theta \rightarrow \hat{\Theta} \rightarrow W'$  forms a Markov chain. Moreover, by the choice of  $w'$  it immediately follows from Corollary 22 that for all  $w \neq w'$  and for all  $f \in \text{span}(F) \setminus \{0^n\}$  it holds that

$$|f \odot (\mathfrak{c}(w) \oplus \hat{\theta})| \geq \frac{1}{2}(\frac{1}{4} - \beta)d. \quad (8)$$

We will make use of this bound later in the proof.

Since the model (Section 5.1) enforces the dishonest server to measure all qubits at the end of the protocol, the system  $E = (Y, Z, \Theta)$  is classical and hence the trace-distance-based user-security definition (Definition 11) simplifies to a bound on the statistical distance between distributions. I.e., it is sufficient to prove that

$$\text{SD}(P_{EW|W'=w', W' \neq W}, P_{W|W'=w', W' \neq W} P_{E|W'=w', W' \neq W}) \leq \varepsilon$$

holds for any  $w'$ . Consider the distribution that appears above as the first argument to the statistical distance, i.e.  $P_{EW|W'=w', W' \neq W}$ . By substituting  $E = (Y, Z, \Theta)$ , it factors as follows<sup>10</sup>

$$\begin{aligned} P_{YZ\Theta W|W', W' \neq W'} &= P_{W|W', W' \neq W'} P_{Z\Theta|W W', W' \neq W'} P_{Y|Z\Theta W W', W' \neq W'} \\ &= P_{W|W', W' \neq W'} P_{Z\Theta|W', W' \neq W'} P_{Y|F(X)\Theta W W', W' \neq W'}, \end{aligned} \quad (9)$$

where the equality  $P_{Z\Theta|W W', W' \neq W'} = P_{Z\Theta|W', W' \neq W'}$  holds by the following argument:  $Z$  is independent of  $W$  (since  $F(X)$  acts as one-time pad) and  $Z \rightarrow \Theta \rightarrow W'$  is a Markov chain, and  $S^*$  (who computes  $\Theta$  from  $Z$ ) is initially independent of  $W$  by Definition 11, hence  $W$  is independent of  $Z, \Theta$  and  $W'$ , which implies the above equality. The equality  $P_{Y|Z\Theta W W', W' \neq W'} = P_{Y|F(X)\Theta W W', W' \neq W'}$  holds by the observation that given  $W, Z$  is uniquely determined by  $F(X)$  and vice versa.

In the remainder of this proof we will show that

$$d_{\text{unif}}(Y|F(X) = u, \Theta = v, W = w, W' = w') \leq \frac{1}{2}2^{\frac{\ell}{2} - \frac{1}{4}(4 - \beta)d},$$

<sup>10</sup>Note that we shorten notation here by omitting the parentheses containing the function arguments. The quantification is over all inputs for which all involved conditional probabilities are well-defined.

for all  $u, v, w$  such that  $w \neq w'$ , where  $w'$  is determined by  $v$ . This then implies that the rightmost factor in (9) is essentially independent of  $W$ , and concludes the proof.

To simplify notation, we define  $\mathcal{E}$  to be the event

$$\mathcal{E} := \{F(X) = u, \Theta = v, W = w, W' = w'\}$$

for fixed but arbitrary choices  $u, v$  and  $w$  such that  $w \neq w'$ , where  $w'$  is determined by  $v$ . We show closeness to the uniform distribution by using the XOR inequality from Diaconis *et al.* (Theorem 1), i.e., we use the inequality

$$d_{\text{unif}}(Y|\mathcal{E}) \leq \frac{1}{2} \left[ \sum_{\alpha} \text{bias}(\alpha \cdot Y|\mathcal{E})^2 \right]^{\frac{1}{2}},$$

where the sum is over all  $\alpha$  in  $\{0, 1\}^n \setminus \{0^n\}$ . We split this sum into two parts, one for  $\alpha \in \text{span}(F)$  and one for  $\alpha$  not in  $\text{span}(F)$ , and analyze the two parts separately.

Since  $X$  is uniformly distributed, it follows that for any  $\alpha \notin \text{span}(F)$ , it holds that  $P_{\alpha \cdot X|F(X)}(\cdot|u) = \frac{1}{2}$  (for any  $u$ ). We conclude that

$$\begin{aligned} \frac{1}{2} &= P_{\alpha \cdot X|F(X)} = P_{\alpha \cdot X|F(X)W} = P_{\alpha \cdot X|F(X)\Theta WW'} \\ &= P_{\alpha \cdot Y|F(X)\Theta WW'} = P_{\alpha \cdot Y|\mathcal{E}} \quad \forall \alpha \notin \text{span}(F). \end{aligned}$$

The second equality follows since  $W$  is independent of  $X$ . The third equality holds by the fact that  $\Theta$  is computed from  $F(X) \oplus g(W)$  and  $W'$  is determined by  $\Theta$ . The fourth equality follows by the security of the one-time pad, i.e. recall that  $Y = X \oplus \Delta$ , where by Corollary 16 it holds that  $\Delta \in \{0, 1\}^n$  is independent of  $X$  when conditioned on fixed values for  $B = \mathfrak{c}(W)$  and  $\Theta$ . Hence, it follows that  $\text{bias}(\alpha \cdot Y|\mathcal{E}) = 0$  for  $\alpha \notin \text{span}(F)$ .

For any non-zero  $\alpha \in \text{span}(F)$ , we can write

$$\begin{aligned} \text{bias}(\alpha \cdot Y|\mathcal{E}) &= \text{bias}(\alpha \cdot (X \oplus \Delta)|\mathcal{E}) \\ &= \text{bias}(\alpha \cdot X \oplus \alpha \cdot \Delta|\mathcal{E}) && \text{(distributivity of dot product)} \\ &= \text{bias}(\alpha \cdot X|\mathcal{E})\text{bias}(\alpha \cdot \Delta|\mathcal{E}) && \text{(Corollary 16)} \\ &\leq \text{bias}(\alpha \cdot \Delta|\mathcal{E}) && \text{(bias}(\alpha \cdot X) \leq 1) \\ &= \prod_{i \in [n]} \text{bias}(\alpha_i \cdot \Delta_i|\mathcal{E}) && \text{(\Delta}_i \text{ independent)} \\ &= \prod_{i \in [n]: \alpha_i=1} \text{bias}(\Delta_i|\mathcal{E}) \\ &\leq \prod_{\substack{i \in [n]: \alpha_i=1 \\ \hat{\theta}_i = \mathfrak{c}(w)_i \oplus 1}} 2^{-\frac{1}{2}} && \text{(Theorem 18)} \\ &= 2^{-\frac{1}{2}|\alpha \odot (\mathfrak{c}(w) \oplus \hat{\theta})|} \leq 2^{-\frac{1}{4}(\frac{1}{4} - \beta)d} && \text{(by (8))} \end{aligned}$$

Combining the two parts, we get

$$\begin{aligned} d_{\text{unif}}(Y|\mathcal{E}) &\leq \frac{1}{2} \left[ \sum_{\alpha} \text{bias}(\alpha \cdot Y|\mathcal{E})^2 \right]^{\frac{1}{2}} \\ &= \frac{1}{2} \left[ \sum_{\alpha \in \text{span}(F) \setminus \{0^n\}} \text{bias}(\alpha \cdot Y|\mathcal{E})^2 + 0 \right]^{\frac{1}{2}} \leq \frac{1}{2} 2^{\frac{\ell}{2} - \frac{1}{4}(\frac{1}{4} - \beta)d}. \end{aligned}$$

Incorporating the error probability of having a “bad”  $F$  completes the proof.  $\square$

## 5.5 Attack against Q-ID with Operations on Pairs of Qubits

We present an attack with which the dishonest server  $S^*$  can discard two passwords in one execution of Q-ID using coherent operations on pairs of qubits.

Before discussing this attack, we first explain a straightforward strategy by which  $S^*$  can discard one password per execution:  $S^*$  chooses a candidate password  $\hat{w}$  and measures the state  $H^{c(W)}|X\rangle$  qubit-wise in the basis  $c(\hat{w})$  to obtain  $Y$ .  $S^*$  then computes  $F(Y) \oplus g(\hat{w})$  and compares this to  $Z = F(X) \oplus g(W)$ , which he received from the user. If indeed  $Z = F(Y) \oplus g(\hat{w})$ , then it is very likely that  $W = \hat{w}$ , i.e. that  $S^*$  guessed the password correctly.

Let us now explain the attack, which is obtained by modifying the above strategy. The attack is based on the following observation [DFSS05]: if  $S^*$  can perform Bell measurements on qubit pairs  $|x_1\rangle_a|x_2\rangle_a$ , for  $a \in \{0, 1\}$ , then he can learn the parity of  $x_1 \oplus x_2$  for both choices of  $a$  simultaneously. This strategy can also be adapted to determine both parities of a pair in which the first qubit is encoded in a basis that is opposite to that of the second qubit, i.e. by appropriately applying a Hadamard gate prior to applying the Bell measurement.

Let the first bit of  $Z$  be equal to  $f \cdot X \oplus g(W)_1$ ,<sup>11</sup> where  $f \in \text{span}(F) \setminus \{0^n\}$ . Let  $\hat{w}_1$  and  $\hat{w}_2$  be two candidate passwords. With the trick from above,  $S^*$  can measure the positions in the set

$$\mathcal{P} := \{i \in [n] : f_i = 1, c(\hat{w}_1)_i = 1 \oplus c(\hat{w}_2)_i\}$$

*pairwise* (assuming  $|\mathcal{P}|$  to be even) using Bell measurements, while measuring the positions where  $c(\hat{w}_1)$  and  $c(\hat{w}_2)$  coincide using ordinary single-qubit measurements. This allows him to compute both “check bits” corresponding to both passwords *simultaneously*, i.e. those check bits coincide with  $f \cdot Y_1 \oplus g(\hat{w}_1)_1$  and  $f \cdot Y_2 \oplus g(\hat{w}_2)_1$ , where  $Y_1$  and  $Y_2$  are the outcomes that  $S^*$  would have obtained if he had measured all qubits qubit-wise in either  $c(\hat{w}_1)$  or  $c(\hat{w}_2)$ , respectively. If both these check bits are different from the bit  $Z_1$ , then  $S^*$  can discard both  $w_1$  and  $w_2$ .

We have seen that in the *worst case*, the attack is capable of discarding two passwords in one execution, and hence clearly violates the security definition. On *average*, however, the attack seems to discard just one password per execution, i.e. a candidate password cannot be discarded if its check bit is consistent with  $Z_1$ , which essentially happens with probability  $1/2$ . This raises the question whether the security definition is unnecessarily strong, because it seems that not being able to discard more than one password on average would be sufficient. Apart from this, it might be possible to improve the attack, e.g. by selecting the positions where to measure pairwise in a more clever way, as to obtain multiple check bits (corresponding to multiple  $f$ s in the span of  $F$ ) per candidate password, thereby increasing the probability of discarding a wrong candidate password.

## 6 Conclusion

We view our work related to Q-ID as a first step in a promising line of research, aimed at achieving security in multiple models simultaneously. The main open problem in the context of the SQOM is to improve our results in a more general model in which the dishonest server  $S^*$  can choose his basis adaptively. Also, it would be interesting to see

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<sup>11</sup>By  $g(W)_1$  we mean the first bit of  $g(W)$ .

whether similar results can be obtained in a model where the adversary is restricted to performing quantum operations on blocks of several qubits.

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## A Proof of an Operator Norm Inequality (Proposition 26)

We first recall some basic properties of the operator norm  $\|A\| := \sup \|\!|A|\!\psi\rangle\|$ , where the supremum is over all norm-1 vectors  $|\psi\rangle \in \mathcal{H}$ . First of all, it is easy to see that

$$\left\| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\| = \max \{ \|A\|, \|B\| \}.$$

Also, from the fact that  $\|A\| = \sup |\langle \psi | A | \varphi \rangle|$ , where the supremum is over all norm-1  $|\psi\rangle, |\varphi\rangle \in \mathcal{H}$ , it follows that  $\|A^*\| = \|A\|$ , where  $A^*$  is the Hermitian transpose of  $A$ , and thus that for Hermitian matrices  $A$  and  $B$ :

$$\|AB\| = \|(AB)^*\| = \|B^*A^*\| = \|BA\|.$$

Furthermore, if  $A$  is Hermitian then  $\|A\| = \lambda_{\max}(A) := \max\{|\lambda_j| : \lambda_j \text{ an eigenvalue of } A\}$ . Finally, the operator norm is *unitarily invariant*, i.e.,  $\|A\| = \|UAV\|$  for all  $A$  and for all unitary  $U, V$ .

**Lemma 25.** *Any two  $n \times n$  matrices  $X$  and  $Y$  for which the products  $XY$  and  $YX$  are Hermitian satisfy*

$$\|XY\| = \|YX\|$$

*Proof.* For any two  $n \times n$  matrices  $X$  and  $Y$ ,  $XY$  and  $YX$  have the same eigenvalues, see e.g. [Bha97, Exercise I.3.7]. Therefore,  $\|XY\| = \lambda_{\max}(XY) = \lambda_{\max}(YX) = \|YX\|$ .  $\square$

We are now ready to state and prove the norm inequality. We recall that an orthogonal projector  $P$  satisfies  $P^2 = P$  and  $P^* = P$ .

**Proposition 26.** *For orthogonal projectors  $A_1, A_2, \dots, A_m$ , it holds that*

$$\|A_1 + \dots + A_m\| \leq 1 + (m - 1) \cdot \max_{1 \leq j < k \leq m} \|A_j A_k\|.$$

The case  $m = 2$  was proven in [DFSS05], adapting a technique by Kittaneh [Kit97]. We extend the proof to an arbitrary  $m$ .

*Proof.* Defining

$$X := \begin{pmatrix} A_1 & A_2 & \cdots & A_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad Y := \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A_m & 0 & \cdots & 0 \end{pmatrix}$$

yields

$$XY = \begin{pmatrix} A_1 + A_2 + \dots + A_m & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad YX = \begin{pmatrix} A_1 & A_1 A_2 & \cdots & A_1 A_m \\ A_2 A_1 & A_2 & \cdots & A_2 A_m \\ \vdots & \vdots & \ddots & \vdots \\ A_m A_1 & A_m A_2 & \cdots & A_m \end{pmatrix}$$

The matrix  $YX$  can be additively decomposed into  $m$  matrices according to the following pattern

$$YX = \begin{pmatrix} * & & & & \\ & * & & & \\ & & \ddots & & \\ & & & * & \\ & & & & * \end{pmatrix} + \begin{pmatrix} 0 & * & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & * \\ * & & & & 0 \end{pmatrix} + \dots + \begin{pmatrix} 0 & & & & * \\ * & 0 & & & \\ & & \ddots & & \\ & & & 0 & * \\ & & & * & 0 \end{pmatrix}$$

where the  $*$  stand for entries of  $YX$  and for  $i = 1, \dots, m$  the  $i$ th star-pattern after the diagonal pattern is obtained by  $i$  cyclic shifts of the columns of the diagonal pattern.

$XY$  and  $YX$  are Hermitian and thus we can apply Lemma 25. Then, by applying the triangle inequality, the unitary invariance of the operator norm and the facts that for all  $j \neq k : \|A_j\| = 1$ ,  $\|A_j A_k\| = \|A_k A_j\|$ , we obtain the desired statement.  $\square$

## B Proof of Lemma 5

To prove Lemma 5, we need to introduce some more tools.

The following proposition guarantees that the ‘‘averaging property’’ of the guessing probability (which holds by definition in the classical case) still holds when additionally conditioning on a quantum system.

**Proposition 27.** *For any state  $\rho_{XYE} \in \mathcal{D}(\mathcal{H}_X \otimes \mathcal{H}_Y \otimes \mathcal{H}_E)$  that is classical on  $X$  and  $Y$  it holds that*

$$p_{\text{guess}}(X|YE) = \sum_y P_Y(y) p_{\text{guess}}(X|E, Y = y).$$

*Proof.* First, note that for any matrix  $M_x$  acting on  $\mathcal{H}_Y \otimes \mathcal{H}_E$ , we can always write  $M_x = \sum_{y,y'} |y\rangle\langle y'| \otimes M_x^{y,y'}$ , where  $M_x^{y,y'}$  acts on  $\mathcal{H}_E$  for every  $x, y, y'$ . Now, we write

$$\begin{aligned}
p_{\text{guess}}(X|YE) &= \max_{\{M_x\}} \sum_x P_X(x) \text{tr}(M_x \rho_{YE}^x) \\
&= \max_{\{M_x\}} \sum_x P_X(x) \text{tr}(M_x \sum_y P_{Y|X}(y|x) |y\rangle\langle y| \otimes \rho_E^{x,y}) \\
&= \max_{\{M_x\}} \sum_{x,y} P_{XY}(x,y) \text{tr}((\sum_{v,w} |v\rangle\langle w| \otimes M_x^{v,w})(|y\rangle\langle y| \otimes \rho_E^{x,y})) \\
&= \max_{\{M_x\}} \sum_{x,y} P_{XY}(x,y) \sum_v \langle v|y\rangle \text{tr}(M_x^{v,y} \rho_E^{x,y}) \\
&= \max_{\{M_x\}} \sum_{x,y} P_{XY}(x,y) \text{tr}(M_x^{y,y} \rho_E^{x,y}) \\
&= \sum_y P_Y(y) \max_{\{M_x^{y,y}\}} \sum_x P_{X|Y}(x|y) \text{tr}(M_x^{y,y} \rho_E^{x,y}) \\
&= \sum_y P_Y(y) p_{\text{guess}}(X|E, Y=y).
\end{aligned}$$

□

The following proposition is known as the chain rule for min-entropy.

**Proposition 28** ([Ren05]). *The following holds for all  $\rho_{ABC} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ ,*

$$H_{\min}(A|BC) \geq H_{\min}(AB|C) - H_{\max}(B).$$

Finally, we need the following lemma.

**Lemma 29.** *For any state  $\rho_{XYE} \in \mathcal{D}(\mathcal{H}_X \otimes \mathcal{H}_Y \otimes \mathcal{H}_E)$  that is classical on  $X$  and  $Y$  it holds that*

$$H_{\min}(XE|Y=y) \geq H_{\min}(X|Y=y) \quad (10)$$

for every  $y \in \mathcal{Y}$ .

*Proof.* Note that it suffices to show that  $\lambda_{\max}(\rho_{XE}^y) \leq \lambda_{\max}(\rho_X^y)$  holds for every  $y \in \mathcal{Y}$ . Because  $\rho_{XE}^y$  is classical on  $X$ , there exists a unitary  $U$  acting on  $\mathcal{H}_X$  such that  $\tilde{\rho}_{XE}^y := (U \otimes \mathbb{I}_E) \rho_{XE}^y (U^\dagger \otimes \mathbb{I}_E)$  is classical with respect to the computational basis  $\{|x\rangle\}_{x \in \mathcal{X}}$  on  $\mathcal{H}_X$  with  $\mathcal{X} := [d]$ . In particular, this means that  $\tilde{\rho}_{XE}^y$  has block-diagonal structure:

$$\tilde{\rho}_{XE}^y = \sum_{x \in [d]} P_{X|Y}(x|y) |x\rangle\langle x| \otimes \rho_E^{x,y} = \begin{bmatrix} P_{X|Y}(1|y) \rho_E^{1,y} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & P_{X|Y}(d|y) \rho_E^{d,y} \end{bmatrix}.$$

Note that because  $U$  is unitary,  $\tilde{\rho}_{XE}^y$  has the same eigenvalues as  $\rho_{XE}^y$ , where these eigenvalues are given by the union of the eigenvalues of the blocks on the diagonal of  $\tilde{\rho}_{XE}^y$ . From this we see that the largest eigenvalue of  $\tilde{\rho}_{XE}^y$  (and thus of  $\rho_{XE}^y$ ) cannot be larger than the largest eigenvalue of  $\tilde{\rho}_X^y := \text{tr}_E(\tilde{\rho}_{XE}^y)$  (and thus of  $\rho_X^y$ ). □

*Proof of Lemma 5.* By (1) it is equivalent to show that

$$p_{\text{guess}}(X|YE) \leq p_{\text{guess}}(X|Y) 2^{H_{\max}(E)}.$$

Using Proposition 27, we write

$$\begin{aligned} p_{\text{guess}}(X|EY) &= \sum_y P_Y(y) p_{\text{guess}}(X|E, Y=y) = \sum_y P_Y(y) 2^{-H_{\min}(X|E, Y=y)} \\ &\leq \sum_y P_Y(y) 2^{-(H_{\min}(XE|Y=y) - H_{\max}(E))} \\ &\leq 2^{H_{\max}(E)} \sum_y P_Y(y) 2^{-H_{\min}(X|Y=y)} = 2^{H_{\max}(E)} p_{\text{guess}}(X|Y), \end{aligned}$$

where the first inequality is Proposition 28, and the second inequality follows by Lemma 29. Hence, the claim follows.  $\square$

## C Proof of Lemma 15

*Proof.* Let  $\alpha, \beta \in \mathbb{C}$  be such that  $\theta_i := \{\alpha|0\rangle + \beta|1\rangle, \beta|0\rangle - \alpha|1\rangle\}$ . (We can always find such  $\alpha$  and  $\beta$ .) Writing out the measurement explicitly gives

$$\begin{aligned} P_{Y_i|X_i B_i \Theta_i}(0|x_i, b_i, \theta_i) &= |(\alpha\langle 0| + \beta\langle 1|)H^{b_i}|x_i\rangle|^2 \quad \text{and} \\ P_{Y_i|X_i B_i \Theta_i}(1|x_i, b_i, \theta_i) &= |(\beta\langle 0| - \alpha\langle 1|)H^{b_i}|x_i\rangle|^2. \end{aligned}$$

Hence, it suffices to prove that

$$|(\alpha\langle 0| + \beta\langle 1|)H^{b_i}|x_i\rangle|^2 = |(\beta\langle 0| - \alpha\langle 1|)H^{b_i}|x_i \oplus 1\rangle|^2 \quad (11)$$

for every  $x_i, b_i \in \{0, 1\}$ .

We first show (11) for  $b_i = 0$ . Let  $\sigma_1$  be the first Pauli matrix defined by  $\sigma_1|a\rangle = |a \oplus 1\rangle$  for every  $a \in \{0, 1\}$ . It follows immediately from the definition that  $\sigma_1$  is a unitary matrix and it is easy to see that  $\sigma_1$  is Hermitian. Then,

$$\begin{aligned} |(\alpha\langle 0| + \beta\langle 1|)|x_i\rangle|^2 &= |(\alpha\langle 0| + \beta\langle 1|)\sigma_1\sigma_1|x_i\rangle|^2 = |(\alpha\langle 1| + \beta\langle 0|)|x_i \oplus 1\rangle|^2 \\ &= |(\beta\langle 0| - \alpha\langle 1|)|x_i \oplus 1\rangle|^2 \end{aligned}$$

The last equation follows because the expression equals either  $|\alpha|^2$  or  $|\beta|^2$  (depending on  $x_i \in \{0, 1\}$ ), hence we may freely change the sign of  $\alpha$ . For  $b_i = 1$ , we have

$$|(\alpha\langle 0| + \beta\langle 1|)H|x_i\rangle|^2 = |(\alpha\langle 0| + \beta\langle 1|)(|0\rangle + (-1)^{x_i}|1\rangle)|^2 = |\alpha + (-1)^{x_i}\beta|^2$$

and

$$|(\beta\langle 0| - \alpha\langle 1|)H|x_i \oplus 1\rangle|^2 = |(\beta\langle 0| - \alpha\langle 1|)(|0\rangle - (-1)^{x_i}|1\rangle)|^2 = |\beta + (-1)^{x_i}\alpha|^2.$$

We see that those expressions are equal for every  $x_i \in \{0, 1\}$ .  $\square$