Exact Weight Subgraphs and the k-Sum Conjecture

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Abstract

We consider the EXACT-WEIGHT-H problem of finding a (not necessarily induced) subgraph H of weight 0 in an edge-weighted graph G. We show that for every H, the complexity of this problem is strongly related to that of the infamous k-SUM problem. In particular, we show that under the k-SUM Conjecture, we can achieve tight upper and lower bounds for the EXACT-WEIGHT-H problem for various subgraphs H such as matching, star, path, and cycle.

One interesting consequence is that improving on the $O(n^3)$ upper bound for Exact-Weight-4-path or Exact-Weight-5-path will imply improved algorithms for 3-sum, 5-sum, All-Pairs Shortest Paths and other fundamental problems. This is in sharp contrast to the minimum-weight and (unweighted) detection versions, which can be solved easily in time $O(n^2)$. We also show that a faster algorithm for any of the following three problems would yield faster algorithms for the others: 3-sum, Exact-Weight-3-matching, and Exact-Weight-3-star.

1 Introduction

Two fundamental problems that have been extensively studied separately by different research communities for many years are the k-SUM problem and the problem of finding subgraphs of a certain form in a graph. We investigate the relationships between these problems and show tight connections between k-SUM and the "exact-weight" version of the subgraph finding problem.

The k-SUM problem is the parameterized version of the well known NP-complete problem SUBSET-SUM, and it asks if in a set of n integers, there is a subset of size k whose integers sum to 0. This problem can be solved easily in time $O(n^{\lceil k/2 \rceil})$, and Baran, Demaine, and Pătrașcu [3] show how the 3-SUM problem can be solved in time $O(n^2/\log^2 n)$ using certain hashing techniques. However, it has been a longstanding open problem to solve k-SUM for $some\ k$ in time $O(n^{\lceil k/2 \rceil - \epsilon})$ for some $\epsilon > 0$. In certain restricted models of computation, an $\Omega(n^{\lceil k/2 \rceil})$ lower bound has been established initially by Erickson [7] and later generalized by Ailon and Chazelle [1], and recently, Pătrașcu and Williams [16] show that $n^{o(k)}$ time algorithms for all k would refute the Exponential Time Hypothesis. The literature seems to suggest the following hypothesis, which we call the k-SUM Conjecture:

Conjecture 1 (The k-SUM Conjecture). There does not exist a $k \ge 2$, an $\varepsilon > 0$, and a randomized algorithm that succeeds (with high probability) in solving k-SUM in time $O(n^{\lceil \frac{k}{2} \rceil - \varepsilon})$.

The presumed difficulty of solving k-SUM in time $O(n^{\lceil k/2 \rceil - \varepsilon})$ for any $\varepsilon > 0$ has been the basis of many conditional lower bounds for problems in computational geometry. The k=3 case has re ceived even more attention, and proving 3-SUM-hardness has become common practice in the computational geometry

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literature. In a recent line of work, Pătrașcu [15], Vassilevska and Williams [17], and Jafargholi and Viola [11] show conditional hardness based on 3-SUM for problems in data structures and triangle problems in graphs.

The problem of determining whether a weighted or unweighted n-node graph has a subgraph that is isomorphic to a fixed k node graph H with some properties has been well-studied in the past [14, 12, 6]. There has been much work on detection and counting copies of H in graphs, the problem of listing all such copies of H, finding the minimum-weight copy of H, etc. [17, 13]. Considering these problems for restricted types of subgraphs H has received further attention, such as for subgraphs H with large indepedent sets, or with bounded treewidth, and various other structures [17, 18, 13, 8]. In this work, we focus on the following subgraph finding problem.

Definition 1 (The EXACT-WEIGHT-H Problem). Given an edge-weighted graph G, does there exist a (not necessarily induced) subgraph isomorphic to H such that the sum of its edge weights equals a given target value t?

No non-trivial algorithms were known for this problem. Theoretical evidence for the hardness of this problem was given in [17], where the authors prove that for any H of size k, an algorithm for the exact-weight problem can give an algorithm for the minimum-weight problem with an overhead that is only $O(2^k \cdot \log M)$, when the weights of the edges are integers in the range [-M,M]. They also show that improving on the trivial $O(n^3)$ upper bound for EXACT-WEIGHT-3-CLIQUE to $O(n^{3-\varepsilon})$ for any $\varepsilon>0$ would not only imply an $\tilde{O}(n^{3-\varepsilon})$ algorithm² for the minimum-weight 3-CLIQUE problem, which from [19] is in turn known to imply faster algorithms for the canonical ALL-PAIRS SHORTEST PATHS problem, but also an $O(n^{2-\varepsilon'})$ upper bound for the 3-SUM problem, for some $\varepsilon'>0$. They give additional evidence for the hardness of the exact-weight problem by proving that faster than trivial algorithms for the k-CLIQUE problem will break certain cryptographic assumptions.

Aside from the aforementioned reduction from 3-SUM to EXACT-WEIGHT-3-CLIQUE, few other connections between k-SUM and subgraph problems were known. The standard reduction from Downey and Fellows [5] gives a way to reduce the unweighted k-CLIQUE detection problem to $\binom{k}{2}$ -SUM on n^2 numbers. Also, in [15] and [11], strong connections were shown between the 3-SUM problem (or, the similar 3-XOR problem) and listing triangles in unweighted graphs.

1.1 Our Results

In this work, we study the exact-weight subgraph problem and its connections to k-SUM. We show three types of reductions: k-SUM to subgraph problems, subgraphs to other subgraphs, and subgraphs to k-SUM. These results give conditional lower bounds that can be viewed as showing hardness either for k-SUM or for the subgraph problems. We focus on showing implications of the k-SUM Conjecture and therefore view the first two kinds as a source for conditional lower bounds for EXACT-WEIGHT-H, while we view the last kind as algorithms for solving the problem. Our results are summarized in Table 1 and Figure 1, and are discussed below.

Hardness. By embedding the numbers of the k-SUM problem into the edge weights of the exact-weight subgraph problem, using different encodings depending on the structure of the subgraph, we prove four reductions that are summarized in Theorem 2:

 $^{^{1}}$ We can assume, without loss of generality, that the target value is always 0 and that H has no isolated vertices.

²In our bounds, k is treated as a constant. The notation $\tilde{O}(f(n))$ will hide polylog(n) factors.

Theorem 2. Let $k \geq 3$. If for all $\varepsilon > 0$, k-SUM cannot be solved in time $O(n^{\lceil k/2 \rceil - \varepsilon})$, then none of the following problems³ can be solved in time $O(n^{\lceil k/2 \rceil - \delta})$, for any $\delta > 0$:

- Exact-Weight-H on a graph on n nodes, for any subgraph H on k nodes.
- EXACT-WEIGHT-k-MATCHING on a graph on \sqrt{n} nodes.
- EXACT-WEIGHT-k-STAR on a graph on $n^{(1-1/k)}$ nodes.
- EXACT-WEIGHT-(k-1)-PATH on a graph on n nodes.

An immediate implication of Theorem 2 is that neither 3-STAR can be solved in time $O(n^{3-\varepsilon})$, nor can 3-MATCHING be solved in time $O(n^{4-\varepsilon})$ for some $\varepsilon>0$, unless 3-SUM can be solved in time $O(n^{2-\varepsilon'})$ for some $\varepsilon'>0$. We later show that an $O(n^{2-\varepsilon})$ algorithm for 3-SUM for some $\varepsilon>0$ will imply both an $\tilde{O}(n^{3-\varepsilon})$ algorithm for 3-STAR and an $\tilde{O}(n^{4-2\varepsilon})$ algorithm for 3-MATCHING. In other words, either all of the following three statements are true, or none of them are:

- 3-SUM can be solved in time $O(n^{2-\varepsilon})$ for some $\varepsilon > 0$.
- 3-STAR can be solved in time $O(n^{3-\varepsilon})$ for some $\varepsilon > 0$.
- 3-MATCHING can be solved in time $O(n^{4-\varepsilon})$ for some $\varepsilon > 0$.

From [17], we already know that solving 3-CLIQUE in time $O(n^{3-\varepsilon})$ for some $\varepsilon>0$ implies that 3-SUM can be solved in time $O(n^{2-\varepsilon})$ for some $\varepsilon>0$. By Theorem 2, this would imply faster algorithms for 3-STAR and 3-MATCHING as well.

Another corollary of Theorem 2 is the fact that 4-PATH cannot be solved in time $O(n^{3-\varepsilon})$ for some $\varepsilon>0$ unless 5-SUM can be solved in time $O(n^{3-\varepsilon'})$ for some $\varepsilon'>0$. This is in sharp contrast to the unweighted version (and the min-weight version) of 4-PATH, which can both be solved easily in time $O(n^2)$.

Theorem 2 shows that the k-SUM problem can be reduced to the EXACT-WEIGHT-H problem for various types of subgraphs H, and as we noted, this implies connections between the exact-weight problem for different subgraphs. It is natural to ask if for any other subgraphs the exact-weight problems can be related to one another. We will answer this question in the affirmative—in particular, we show a tight reduction from 3-CLIQUE to 4-PATH.

To get this result, we use the edge weights to encode information about the nodes in order to prove a reduction from EXACT-WEIGHT- H_1 to EXACT-WEIGHT- H_2 , where H_1 is what we refer to as a "vertex-minor" of H_2 . Informally, a vertex-minor of a graph is one that is obtained by edge deletions and node identifications (contractions) for arbitrary pairs of nodes of the original graph (see Section 4 for a formal definition). For example, the triangle subgraph is a vertex-minor of the path on four nodes, which is itself a vertex-minor of the cycle on four nodes.

Theorem 3. Let H_1, H_2 be subgraphs such that H_1 is a vertex-minor of H_2 . For any $\alpha \geq 2$, if Exact-Weight- H_2 can be solved in time $\tilde{O}(n^{\alpha})$, then Exact-Weight- H_1 can be solved in time $\tilde{O}(n^{\alpha})$.

Therefore, Theorem 3 allows us to conclude that 4-CYCLE cannot be solved in time $O(n^{3-\varepsilon})$ for some $\varepsilon>0$ unless 4-PATH can be solved in time $\tilde{O}(n^{3-\varepsilon})$, which cannot happen unless 3-CLIQUE can be solved in time $\tilde{O}(n^{3-\varepsilon})$.

To complete the picture of relations between 3-edge subgraphs, consider the subgraph composed of a 2-edge path along with another (disconnected) edge. We call this the "VI" subgraph and we define the

 $^{^3}k$ -MATCHING denotes the k-edge matching on 2k nodes. k-STAR denotes the k-edge star on k+1 nodes. k-PATH denotes the k-node path on k-1 edges.

EXACT-WEIGHT-VI problem appropriately. Since the path on four nodes is a vertex-minor of VI, we have that an $O(n^{3-\varepsilon})$ for some $\varepsilon>0$ algorithm for EXACT-WEIGHT-VI implies an $\tilde{O}(n^{3-\varepsilon})$ algorithm for 4-PATH. In Figure 1, we show this web of connections between the exact-weight 3-edge subgraph problems and its connection to 3-SUM, 5-SUM, and ALL-PAIRS SHORTEST PATHS. In fact, we will soon see that the conditional lower bounds we have established for these 3-edge subgraph problems are all tight. Note that the detection and minimum-weight versions of some of these 3-edge subgraph problems can all be solved much faster than $O(n^3)$ (in particular, $O(n^2)$), and yet such an algorithm for the exact-weight versions for *any* of these problems will refute the 3-SUM Conjecture, the 5-SUM Conjecture, and lead to breakthrough improvements in algorithms for solving ALL-PAIRS SHORTEST PATHS and other important graph and matrix optimization problems (cf. [19])!

Another $O(n^3)$ solvable problem is the EXACT-WEIGHT-5-PATH, and by noting that both 4-CYCLE and VI are vertex-minors of 5-PATH, we get that improved algorithms for 5-PATH will yield faster algorithms for all of the above problems. Moreover, from Theorem 2, 6-SUM reduces to 5-PATH. This established EXACT-WEIGHT-5-PATH as the "hardest" of the $O(n^3)$ time problems that we consider.

We also note that Theorem 3 yields some interesting consequences under the assumption that the k-CLIQUE problem cannot be solved in time $O(n^{k-\varepsilon})$ for some $\varepsilon>0$. Theoretical evidence for this assumption was provided in [17], where they show how an $O(n^{k-\varepsilon})$ for some $\varepsilon>0$ time algorithm for EXACT-WEIGHT-k-CLIQUE yields a sub-exponential time algorithm for the multivariate quadratic equations problem, a problem whose hardness is assumed in post-quantum cryptography.

We note that the 4-clique is a vertex-minor of the 8-node path, and so by Theorem 3, an $O(n^{4-\varepsilon})$ for some $\varepsilon>0$ algorithm for 8-PATH will yield a faster 4-CLIQUE algorithm. Note that an $O(n^{5-\varepsilon})$ algorithm for 8-PATH already refutes the 9-SUM Conjecture. However, this by itself is not known to imply faster clique algorithms⁴. Also, there are other subgraphs for which one can only rule out $O(n^{\lceil k/2 \rceil - \varepsilon})$ for $\varepsilon>0$ upper bounds from the k-SUM Conjecture, while assuming hardness for the k-CLIQUE problem and using Theorem 3, much stronger lower bounds can be achieved.

Algorithms. So far, our reductions only show one direction of the relationship between k-SUM and the exact-weight subgraph problems. We now show how to $use\ k$ -SUM to solve EXACT-WEIGHT-H, which will imply that many of our previous reductions are indeed tight. The technique for finding an H-subgraph is to enumerate over a set of d smaller subgraphs that partition H in a certain way. Then, in order to determine whether the weights of these d smaller subgraphs sum up to the target weight, we use d-SUM. We say that (S, H_1, \ldots, H_d) is a d-separator of H iff S, H_1, \ldots, H_d partition V(H) and there are no edges between a vertex in H_i and a vertex in H_j for any distinct $i, j \in [d]$.

Theorem 4. Let (S, H_1, \ldots, H_d) be a d-separator of H. Then, EXACT-WEIGHT-H can be reduced to $\tilde{O}(n^{|S|})$ instances of d-SUM each on $\max\{n^{|H_1|}, \ldots, n^{|H_d|}\}$ numbers.

By using the known d-SUM algorithms, Theorem 4 gives a non-trivial algorithm for exact-weight subgraph problems. The running time of the algorithm depends on the choice of the separator used for the reduction. We observe that the optimal running time can be achieved even when d=2, and can be identified (naively, in time $O(3^k)$) using the following expression. Let

$$\gamma(H) = \min_{(S,H_1,H_2) \text{ is a 2-separator}} \left\{ |S| + \max\left\{ |H_1|,|H_2|\right\} \right\}.$$

Corollary 5. Exact-Weight-H can be solved in time $\tilde{O}(n^{\gamma(H)})$.

⁴It is not known whether the assumption that k-CLIQUE cannot be solved in time $O(n^{k-\varepsilon})$ for any $\varepsilon > 0$ is stronger or weaker than the k-SUM Conjecture.

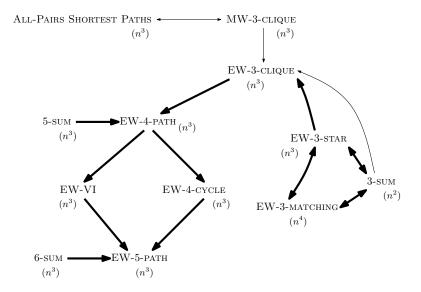


Figure 1: A diagram of the relationships between EXACT-WEIGHT-H (denoted EW, for small subgraphs H) and other important problems. The best known running times are given for each problem, and an arrow $A \to B$ denotes that A can be tightly reduced to B, in the sense that improving the stated running time for B will imply an improvement on the stated running time for A. The reductions established in this work are displayed in bold, the others are due to [19], [17].

Corollary 5 yields the upper bounds that we claim in Figure 1 and Table 1. For example, to achieve the $O(n^{\lceil (k+1)/2 \rceil})$ time complexity for k-PATH, observe that we can choose the set containing just the "middle" node of the path to be S, so that the graph $H \setminus S$ is split into two disconnected halves H_1 and H_2 , each of size at most $\lceil (k-1)/2 \rceil$. Note that this is the optimal choice of a separator, and so $\gamma(k$ -PATH) $= 1 + \lceil (k-1)/2 \rceil = \lceil (k+1)/2 \rceil$. It is interesting to note that this simple algorithm achieves running times that match many of our conditional lower bounds. This means that in many cases, improving on this algorithm will refute the k-SUM Conjecture, and in fact, we are not aware of any subgraph for which a better running time is known.

EXACT-WEIGHT-H is solved most efficiently by our algorithm when $\gamma(H)$ is small, that is, subgraphs with small "balanced" separators. Two such cases are when H has a large independent set and when H has bounded treewidth. We show that EXACT-WEIGHT-H can be solved in time $O(n^{k-\left\lfloor\frac{s}{2}\right\rfloor})$, if $\alpha(H)=s$, and in time $O(n^{\frac{2}{3}\cdot k+tw(H)})$. Also, we observe that our algorithm can be modified slightly to get an algorithm for the minimization problem.

Theorem 6. Let H be a subgraph on k nodes, with independent set of size s. Given a graph G on n nodes with node and edge weights, the minimum total weight of a (not necessarily induced) subgraph of G that is isomorphic to H can be found in time $\tilde{O}(n^{k-s+1})$.

This algorithm improves on the $O(n^{k-s+2})$ time algorithm of Vassilevska and Williams [17] for the MIN-WEIGHT-H problem.

Organization. We give formal definitions and preliminary reductions in Section 2. In Section 3 we present reductions from k-SUM to exact-weight subgraph problems that prove Theorem 2. In Section 4 we define vertex-minors and prove Theorem 3. In Section 5, we give reductions to k-SUM and prove Theorems 4 and 6.

Subgraph	Exact	Lower Bound	Condition	Detection	Min
3-CLIQUE	n^3	n^3	3-sum, APSP	n^{ω} [10]	n^3
4-path	n^3	n^3	3-clique, 5-sum	n^2	n^2
k-matching	$n^{2 \cdot \left\lceil \frac{k}{2} \right\rceil}$	$n^{2 \cdot \left \lceil \frac{k}{2} \right \rceil}$	k-sum	n^2	n^2
k-star	$n^{\left\lceil\frac{k}{2}\right\rceil+1}$	$n^{\left\lceil \frac{k+1}{2} \right\rceil}$	(k+1)-SUM	n	n^2
		$n^{\left\lceil \frac{k}{2} \right\rceil \cdot \frac{k}{k-1}}$	k-SUM		
k-PATH	$n^{\left\lceil \frac{k+1}{2} \right\rceil}$	$n^{\left\lceil \frac{k+1}{2} ight ceil}$	k-SUM	n^2	n^2
k-cycle	$n^{\left\lceil \frac{k}{2} \right\rceil + 1}$	$n^{\left\lceil rac{k+1}{2} ight ceil}$	k-PATH	n^2	n^3
Any	n^k	$n^{\lceil k/2 \rceil}$	k-sum	$n^{\omega k/3}$ [12]	n^k
		$n^{arepsilon k}$	(ETH)		
$\alpha(H) = s$	$n^{k-\left\lfloor \frac{s}{2} \right\rfloor}$	$n^{\left\lceil \frac{k}{2} ight ceil}$	k-sum	n^{k-s+1} [13]	n^{k-s+1} [Thm. 4]
tw(H) = w	$n^{\frac{2}{3}k+w}$	$n^{\lceil rac{k}{2} ceil}$	k-SUM	n^{w+1} [2]	n^{2w} [8]
k-sum	$n^{\lceil \frac{k}{2} \rceil}$	$n^{\lceil rac{k}{2} ceil}$	k-MATCHING, k -STAR	-	-

Table 1: The results shown in this work for EXACT-WEIGHT-H for various H. The second column has the upper bound achieved by our algorithm from Corollary 5. Improvements on the lower bound in the third column will imply improvements on the best known algorithms for the problems in the condition column. These lower bounds are obtained by our redctions, except for the first row which was proved in [17]. For comparison, we give the running times for the (unweighted) detection and minimum-weight versions of the subgraph problems. The last row shows our conditional lower bounds for k-SUM. $\alpha(H)$ represents the independence number of H, tw(H) is its treewidth. The results for the "Any" row hold for all subgraphs on k nodes. ETH stands for the Exponential Time Hypothesis.

2 Preliminaries and Basic Constructions

For a graph G, we will use V(G) to represent the set of vertices and E(G) to represent the set of edges. The notation N(v) will be used to represent the neighborhood of a vertex $v \in V(G)$.

2.1 Reducibility

2.2 The k-SUM Problem

Throughout the paper, it will be more convenient to work with a version of the k-SUM problem that is more structured than the basic formulation. This version is usually referred to as either TABLE-k-SUM or k-SUM', and is known to be equivalent to the basic formulation, up to k^k factors (by a simple extension of Theorem

3.1 in [9]). For convenience, and since f(k) factors are ignored in our running times, we will refer to this problem as k-SUM.

Definition 2 (k-SUM). Given k lists L_1, \ldots, L_k each with n numbers where $L_i = \{x_{i,j}\}_{j \in [n]} \subseteq \mathbb{Z}$, do there exist k numbers $x_{1,a_1}, \ldots, x_{k,a_k}$, one from each list, such that $\sum_{i=1}^k x_{i,a_i} = 0$?

In our proofs, we always denote an instance of k-SUM by L_1,\ldots,L_k , where $L_i=\{x_{i,j}\}_{j\in[n]}\subseteq\mathbb{Z}$, so that $x_{i,j}$ is the j^{th} number of the i^{th} list L_i . We define a k-solution to be a set of k numbers $\{x_{i,a_i}\}_{i\in[k]}$, one from each list. The sum of a k-solution $\{x_{i,a_i}\}_{i\in[k]}$ will be defined naturally as $\sum_{i=1}^k x_{i,a_i}$.

In [15], Pătrașcu defines the CONVOLUTION-3-SUM problem. We consider a natural extension of this problem.

Definition 3 (CONVOLUTION-k-SUM). Given k lists L_1, \ldots, L_k each with n numbers, where $L_i = \{x_{i,j}\}_{j \in [n]} \subseteq \mathbb{Z}$, does there exist a k-solution $\{x_{i,a_i}\}_{i \in [k]}$ such that $a_k = a_1 + \cdots + a_{k-1}$ and $\sum_{i=1}^k x_{i,a_i} = 0$?

Theorem 10 in [15] shows that 3-SUM $_2 \le _2$ CONVOLUTION-3-SUM. By generalizing the proof, we show the following useful lemma (see proof in Appendix C).

Lemma 7. For all $k \geq 2$, k-sum $\lceil k/2 \rceil \leq \lceil k/2 \rceil$ Convolution-k-sum.

2.3 *H*-Partite Graphs

Let H be a subgraph on k nodes with $V(H) = \{h_1, \dots, h_k\}$.

Definition 4 (H-partite graph). Let G be a graph such that V(G) can be partitioned into k sets P_{h_1}, \ldots, P_{h_k} , each containing n vertices. We will refer to these k sets as the super-nodes of G. A pair of super-nodes (P_{h_i}, P_{h_j}) will be called a super-edge if $(h_i, h_j) \in E(H)$. Then, we say that G is H-partite if every edge in E(G) lies in some super-edge of G.

We denote the set of vertices of an H-partite graph G by $V(G) = \{v_{i,j}\}_{i \in [k], j \in [n]}$, where $v_{i,j}$ is the j^{th} vertex in super-node P_{h_i} . We will say that G is the *complete* H-partite graph when $(v_{i,a}, v_{j,b}) \in E(G)$ if and only if (P_{h_i}, P_{h_j}) is a super-edge of G, for all $a, b \in [n]$.

An H-subgraph of an H-partite graph G, denoted by $\chi = \{v_{i,a_i}\}_{i \in [k]} \subseteq V(G)$, is a set of vertices for which there is exactly one vertex v_{i,a_i} from each super-node P_{h_i} , where a_i is an index in [n]. Given a weight function $w: (V(G) \cup E(G)) \to \mathbb{Z}$ for the nodes and edges of G, the total weight of the subgraph χ is defined naturally as

$$w(\chi) = \sum_{h_i \in V(H)} w(v_{i,a_i}) + \sum_{(h_i,h_j) \in E(H)} w(v_{i,a_i}, v_{j,a_j}).$$

Figure 2 illustrates our definitions and notations of H-partite graphs and H-subgraphs.

Now, we define a more structured version of the $\mathsf{EXACT} ext{-}\mathsf{WEIGHT} ext{-}H$ problem which is easier to work with.

Definition 5 (The EXACT-H Problem). Given a complete H-partite graph graph G with a weight function $w: (V(G) \cup E(G)) \to \mathbb{Z}$ for the nodes and edges, does there exist an H-subgraph of total weight 0?

In appendix B, we prove the following lemma, showing that the two versions of the EXACT-WEIGHT-H problem are reducible to one another in a tight manner. All of our proofs will use the formulation of EXACT-H, yet the results will also apply to EXACT-WEIGHT-H. Note that our definitions of H-partite graphs uses ideas similar to color-coding [2].

Lemma 8. Let $\alpha > 1$. Exact-Weight- $H_{\alpha} \equiv_{\alpha} \text{Exact-}H$.

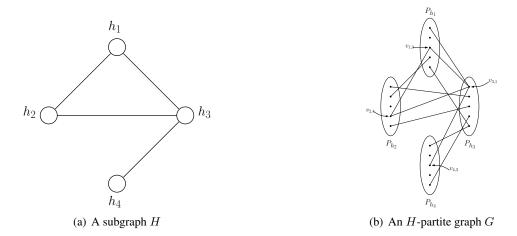


Figure 2: A subgraph H along with an H-partite graph G. We associate a partition $P_{h_i} \subseteq V(G)$ with each node $h_i \in V(H)$. The vertex $v_{i,j} \in V(G)$ represents the j^{th} node in partition P_{h_i} in G. Note that the set $\{v_{1,3}, v_{2,4}, v_{3,1}, v_{4,3}\}$ is an H-subgraph in G. Also, since there are no edges between a vertex in P_{h_3} and a vertex in P_{h_4} , G is a valid H-partite graph.

3 Reductions from k-SUM to Subgraph Problems

In this section we prove Theorem 2 by proving four reductions, each of these reductions uses a somewhat different way to encode k-SUM in the structure of the subgraph. First, we give a generic reduction from k-SUM to EXACT-H for an arbitrary H on k nodes. We set the node weights of the graph to be the numbers in the k-SUM instance, in a certain way.

Lemma 9 $(k\text{-SUM }_{\lceil k/2 \rceil} \leq_{\lceil k/2 \rceil} \text{EXACT-}H)$. Let H be a subgraph with k nodes. Then, k-SUM on n numbers can be reduced to a single instance of EXACT-H on kn vertices.

Proof. Let H be a subgraph with node set $\{h_1,\ldots,h_k\}$. Given a k-SUM instance of k lists L_1,\ldots,L_k , where each $L_i=\{x_{i,j}\}_{j\in[n]}$, we create a complete H-partite graph G on kn vertices where we associate each super-node P_{h_i} with a list L_i , and the j^{th} vertex in the super-node $v_{i,j}$ with the number $x_{i,j}\in L_i$. To do this, we set all edge weights to be 0, and for every $i\in[k], j\in[n]$, we set $w(v_{i,j})=x_{i,j}$. Now, for any H-subgraph $\chi=\{v_{i,a_i}\}_{i\in[k]}$ of G, the total weight of χ will be exactly $\sum_{i=1}^k x_{i,a_i}$, which is the sum of the k-solution $\{x_{i,a_i}\}_{i\in[k]}$. For the other direction, for any k-solution $\{x_{i,a_i}\}_{i\in[k]}$, the H-subgraph $\{v_{i,a_i}\}_{i\in[k]}$ of G has weight exactly $\sum_{i=1}^k x_{i,a_i}$. Therefore, there is a k-solution of sum 0 iff there is an H-subgraph in G of total weight 0.

We utilize the edge weights of the graph, rather than the node weights, to prove a tight reduction to k-MATCHING.

Lemma 10 (k-SUM $_{\lceil k/2 \rceil} \le_{2 \cdot \lceil k/2 \rceil}$ EXACT-k-MATCHING). Let H be the k-MATCHING subgraph. Then, k-SUM on n numbers can be reduced to a single instance of EXACT-H on $k\sqrt{n}$ vertices.

Proof. Given k lists L_1,\ldots,L_k each with n numbers, we will construct a complete H-partite graph G on $k\sqrt{n}$ vertices, where there will be super-edges $(P_{h_{2i-1}},P_{h_{2i}})$ for each $i\in[k]$, each with n edges over \sqrt{n} vertices. We place each number in L_i on an arbitrary edge within the i^{th} super-edge of G by setting $w(v_{2i-1,a},v_{2i,b})=x_{i,(a\cdot\sqrt{n}+b)}$ for all $a,b\in[\sqrt{n}]$. Now, note that the H-subgraph $\{v_{i,a_i}\}_{i\in[2k]}$ of G has weight $\sum_{i=1}^k x_{i,b_i}$, where $b_i=a_{2i-1}\cdot\sqrt{n}+a_{2i}$. This is precisely the sum of the k-solution $\{x_{i,b_i}\}_{i\in[k]}$. And,

for every k-solution $\{x_{i,b_i}\}_{i\in[k]}$, if we choose $a_{2i-1}=\lfloor b_i/\sqrt{n}\rfloor$ and $a_{2i}=b_i-a_{2i-1}\sqrt{n}$, the H-subgraph $\{v_{i,a_i}\}_{i\in[2k]}$ has weight $\sum_{i=1}^k x_{i,b_i}$. Therefore, there is a k-solution of sum 0 iff there is an H-subgraph in G of total weight 0.

Another special type of subgraph which can be shown to be tightly related to the k-SUM problem is the k-edge star subgraph. We define the k-STAR subgraph H to be such that $V(H) = \{h_1, \dots, h_k, h_{k+1}\}$ and $E(H) = \{(h_1, h_{k+1}), (h_2, h_{k+1}), \dots, (h_k, h_{k+1})\}$, so that h_{k+1} is the center node.

Lemma 11 $(k\text{-SUM}_{\left\lceil \frac{k}{2} \right\rceil} \leq_{\left\lceil \frac{k}{2} \right\rceil \cdot \frac{k}{k-1}} \text{EXACT-}k\text{-STAR})$. Let H be the k-STAR subgraph, and let $\alpha > 2$. If EXACT-H can be solved in $O(n^{\alpha})$ time, then k-SUM can be solved in $\tilde{O}(n^{(1-1/k)\cdot \alpha})$ time.

Proof. To prove the lemma we define the problem k-SUM n to be the following. Given a sequence of n k-SUM instances, each on n numbers, does there exist an instance in the sequence that has a solution of sum 0? Then, we prove two claims, one showing a reduction from k-SUM n to EXACT-k-STAR, and the other showing a self-reduction for k-SUM that relates it to k-SUM n .

Claim 12. Let H be the k-STAR subgraph. k-SUM n can be reduced to the EXACT-H problem on a graph of n nodes.

Proof. Given a k-SUM n instance, denote the i^{th} k-SUM instance in the sequence as $L_1^{(i)},\ldots,L_k^{(i)}$, where the j^{th} list of the i^{th} instance is $L_j^{(i)}=\{x_{j,\ell}^{(i)}\}_{\ell\in[n]}\subseteq\mathbb{Z}$. We create an H-partite graph G on kn nodes, where we associate the i^{th} vertex in super-node $P_{h_{k+1}}$ (vertex $v_{k+1,i}$), with the i^{th} instance of the sequence, and we assign the n numbers of list $L_j^{(i)}$ to the n edges incident to $v_{k+1,i}$ within the super-edge $(P_{h_j},P_{h_{k+1}})$. This can be done by setting, for every $i\in[n], j\in[k], \ell\in[n], w(v_{k+1,i},v_{j,\ell})=x_{j,\ell}^{(i)}$, and the weight of every vertex in G to 0.

Assume there is an H-subgraph $\chi = \{v_{j,a_j}\}_{j \in [k+1]}$ in G of total weight 0. Let $i = a_{k+1}$, and consider the k-solution for the i^{th} instance $\{x_{j,a_j}^{(i)}\}_{j \in [k]}$. Note that its sum is exactly the total weight of the H-subgraph χ , which is 0. For the other direction, assume the i^{th} instance has a k-solution $\{x_{j,a_j}^{(i)}\}_{j \in [k]}$ of sum 0, and define the H-subgraph $\chi = \{v_{j,a_j}\}_{j \in [k+1]}$, where $a_{k+1} = i$. Again, note that the total weight of χ is exactly $\sum_{j=1}^k x_{j,a_j}^{(i)} = 0$.

Claim 13. Let $k \geq 2$, and $\alpha \geq 2$. If k-SUMⁿ can be solved in $O(n^{\alpha})$ time, then k-SUM can be solved in $\tilde{O}(n^{\alpha \cdot \frac{k-1}{k}})$ time.

Proof. We will use the hashing scheme due to Dietzfelbinger [4] that we described and used in Appendix C, to hash the numbers into buckets. Given a k-SUM instance L_1, \ldots, L_k , our reduction is as follows:

- 1. Repeat the following $c \cdot k^k \cdot \log n$ times.
 - (a) Pick a hash function $h \in \mathcal{H}_{M,t}$, for t, and map each number $x_{i,j}$ to bucket $B_{i,h(x_{i,j})}$.
 - (b) Ignore all numbers mapped to "overloaded" buckets.
 - (c) Now each bucket has at most $N=kn/t=k\cdot n^{\frac{k-1}{k}}$ numbers. We will generate a sequence of $k\cdot t^{k-1}=k\cdot n^{\frac{k-1}{k}}=N$ instances of k-SUM, each on N numbers, such that one of them has a solution iff the original k-SUM input has a solution. This sequence will be the input to k-SUM N : Go over all t^{k-1} choices of k-1 buckets, $B_{1,a_1},\ldots,B_{k-1,a_{k-1}}$, and add k instances of k to the sequence, one for each of the k buckets, $B_{k,a^{(1)}},\ldots,B_{k,a^{(k)}}$, for which the last number in the solution might be in.

First note that if the k-SUM had a solution, it's numbers will be mapped into not "overloaded" buckets in one of the iterations, with probability $1 - O(n^{-c})$. Then, in such case, the reduction will succeed due to the "almost linearity" property of the hashing. Now, to conclude the proof of the claim, assume k-SUM N can be solved in time $O(N^{\alpha})$, and observe that using the reduction one gets an algorithm for n-SUM running in time $\tilde{O}(N^{\alpha}) = \tilde{O}(n^{\alpha \cdot \frac{k-1}{k}})$, as claimed.

Our final reduction between k-SUM and EXACT-H for a class of subgraphs H is as follows. First, define the k-PATH subgraph H to be such that $V(H) = \{h_1, \ldots, h_k\}$ and $E(H) = \{(h_1, h_2), (h_2, h_3), \ldots, (h_{k-1}, h_k)\}$.

Lemma 14 $(k\text{-SUM}_{\lceil \frac{k}{2} \rceil} \le \lceil \frac{k}{2} \rceil]$ EXACT-(k-1)-PATH). Let H be the k-PATH subgraph. If EXACT-H can be solved in time $O(n^{\lceil k/2 \rceil - \varepsilon})$ for some $\varepsilon > 0$, then k+1-SUM can be solved in time $O(n^{\lceil k/2 \rceil - \varepsilon'})$, for some $\varepsilon' > 0$.

Proof. We prove that an instance of CONVOLUTION-(k+1)-SUM on n numbers can be reduced to a single instance of EXACT-k-PATH, and by applying Lemma 7, this completes the proof. Given k+1 lists L_1,\ldots,L_{k+1} each with n numbers as the input to CONVOLUTION-(k+1)-SUM, we will construct a complete H-partite graph G on kn nodes. For every r and s such that $r-s\in [n]$, for all $i\in [k]$, define the edge weights of G in the following manner.

$$w(v_{i,r}, v_{i+1,s}) = \begin{cases} x_{1,r} + x_{2,s-r}, & \text{if } i = 1\\ x_{i+1,s-r}, & \text{if } 1 < i < k\\ x_{k,s-r} + x_{k+1,s}, & \text{if } i = k \end{cases}$$

Otherwise, if $r-s \not\in [n]$, we set $w(v_{i,r},v_{i+1,s}) = -\infty$ for all $i \in [k]$. Now to see the correctness of the reduction, take any H-subgraph $\{v_{i,a_i}\}_{i \in [k]}$ of G, and consider the (k+1)-solution $\{x_{i,b_i}\}_{i \in [k+1]}$, where $b_1 = a_1, b_{k+1} = a_k$, and for $2 \le i \le k, b_i = a_i - a_{i-1}$. First, note that the (k+1)-solution satisfies the property that $b_1 + \ldots + b_k = b_{k+1}$. Now, note that its total weight is $\sum_{i=1}^k w(v_{i,a_i}, v_{i+1,a_{i+1}}) = x_{1,a_1} + x_{2,a_2-a_1} + x_{3,a_3-a_2} + \ldots + x_{k-1,a_{k-1}-a_{k-2}} + x_{k,a_k-a_{k-1}} + x_{k+1,a_k} = \sum_{i=1}^{k+1} x_{i,b_i}$ which is exactly the sum of the (k+1)-solution. For the other direction, consider the (k+1)-solution $\{x_{i,b_i}\}_{i \in [k+1]}$ for which $b_{k+1} = b_1 + \ldots + b_{k+1}$. Then, the H-subgraph $\{v_{i,a_i}\}_{i \in [k]}$, where $a_i = b_1 + \ldots + b_i$, has total weight $\sum_{i=1}^{k+1} x_{i,b_i}$. Therefore, there is a k-solution of sum 0 iff there is an H-subgraph in G of total weight 0. \square

4 Relationships Between Subgraphs

In this section we prove Theorem 3 showing that EXACT- H_1 can be reduced to EXACT- H_2 if H_1 is a vertex-minor of H_2 . Then we give an additional observation that gives a reverse reduction. We start by defining vertex-minors.

Definition 6 (Vertex-Minor). A graph H_1 is called a vertex-minor of graph H_2 , and denoted $H_1 \leq_{\mathsf{vm}} H_2$, if there exists a sequence of subgraphs $H^{(1)}, \ldots, H^{(\ell)}$ such that $H_1 = H^{(1)}$, $H_2 = H^{(\ell)}$, and for every $i \in [\ell-1]$, $H^{(i)}$ can be obtained from $H^{(i+1)}$ by either

• Deleting a single edge $e \in E(H^{(i+1)})$, or

• Contracting two nodes⁵ $h_j, h_k \in V(H^{(i+1)})$ to one node $h_{jk} \in V(H^{(i)})$, such that $N(h_{jk}) = N(h_j) \cup N(h_k)$.

To prove Theorem 3 it suffices to show how to reduce $EXACT-H_1$ to $EXACT-H_2$ when H_1 is obtained by either a single edge deletion or a single vertex contraction. The edge deletion reduction is straightforward, and the major part of the proof will be showing the contraction reduction. The main observation is that we can make two copies of nodes and change their node weights in a way such that any H_2 -subgraph of total weight 0 that contains one of the copies will have to contain the other. This will allow us to claim that the subgraph obtained by replacing the two copies of a node with the original will be an H_1 -subgraph of total weight 0.

Lemma 15. Let H be a subgraph you get after deleting an edge from H'. EXACT-H on kn nodes can be reduced to a single instance of EXACT-H' on kn nodes.

Proof. Without loss of generality, denote $V(H) = V(H') = \{h_1, \dots, h_k\}, E(H') = \{e_1, \dots, e_m\}$, where $e_m = (h_{k-1}, h_k)$, and $E(H) = \{e_1, \dots, e_{m-1}\}$.

Given G, an H-partite graph as input to EXACT-H, we create an H'-partite graph G' which will have the same set of nodes as G, but will have an additional super-edge (h_{k-1}, h_k) where all of the edges within this super-edge will have weight 0. In other words, for all $a, b \in [n]$ define $w(v_{(k-1),a}, v_{k,b}) = 0$. Now, every H-subgraph in G is an H'-subgraph in G' with the same total weight, and vice versa, which proves the correctness of the reduction.

Lemma 16. Let H be a subgraph you get after contracting two nodes from H'. EXACT-H on (k+1)n nodes can be reduced to a single instance of EXACT-H' on kn nodes.

Proof. Without loss of generality, denote $V(H') = \{h_1, \dots, h_{k-1}\} \cup \{h_{k^{(1)}}, h_{k^{(2)}}\}, V(H) = \{h_1, \dots, h_{k-1}\} \cup \{h_k\}$, and assume you get H from H' by contracting the nodes $h_{k^{(1)}}, h_{k^{(2)}} \in V(H')$ into the node $h_k \in V(H)$. Given G, an H-partite graph, we create an H'-partite graph G' which will be almost the same as G, except that every vertex $v_{k,a}$ in the k^{th} partition of G will have two copies in G', one in $P_{h_{k^{(1)}}}$ and one in $P_{h_{k^{(2)}}}$, which we call $v_{k^{(1)},a}$ and $v_{k^{(2)},a}$, respectively. The weights in G' will be the same as in G, but we will add a unique integer u_a for $a \in [n]$ to the weight of $v_{k^{(1)},a}$ and subtract u_a from the weight of $v_{k^{(2)},a}$. This will ensure that in any H'-subgraph of total weight 0, if $v_{k^{(1)},a}$ is picked, then $v_{k^{(2)},a}$ must also be picked. This allows us to conclude that any H'-subgraph of total weight 0 in G' will directly correspond to an H-subgraph in G.

Let d=|E(H)|=|E(H')|, W the maximum weight of any edge or node in G, and $K=(d+k+1)\cdot W$. Create a complete H'-partite graph G', and define the edge weights $w':(E(G')\cup V(G'))\to \mathbb{Z}$ as follows. For every super-edge $(h_i,h_{\hat{k}})$ where $\hat{k}\in\{k^{(1)},k^{(2)}\}$, define $w'(v_{i,a},v_{\hat{k},b})=w(v_{i,a},v_{k,b})$. All other edges $(v_{i,a},v_{j,b})\in E(H')$ will have weight $w'(v_{i,a},v_{j,b})=w(v_{i,a},v_{j,b})$. For the vertices, we will set $w'(v_{k^{(1)},a})=a\cdot K$ and $w'(v_{k^{(2)},a})=-a\cdot K$ for all $a\in[n]$. All other vertices will have weight 0.

Let $\chi=\{v_{i,a_i}\}_{i\in[k-1]}\cup\{v_{k^{(1)},a_{k^{(1)}}},v_{k^{(2)},a_{k^{(2)}}}\}$ be an H'-subgraph of G' of total weight 0. First, we claim that $a_{k^{(1)}}=a_{k^{(2)}}$. This is true because the total weight of the subgraph χ is $(a_{k^{(1)}}-a_{k^{(2)}})\cdot K+X$, where X represents the sum of d edges and k nodes, each of weight at most W. Therefore, $X<(d+k+1)\cdot W=K$, which implies that $(a_{k^{(1)}}-a_{k^{(2)}})\cdot K+X=0$ can happen only if $a_{k^{(1)}}-a_{k^{(2)}}=0$. Second, note that the H-subgraph $\{v_{i,a_i}\}_{i\in[k]}$ of G, where $a_k=a_{k^{(1)}}=a_{k^{(2)}}$, will also have total weight 0. This is because the numbers added to the weights of the nodes $v_{k^{(1)},a_{k^{(1)}}}$ and $v_{k^{(1)},a_{k^{(1)}}}$ cancel out, and all of the other weights involved are defined to be the same as in G. Now for the other direction, note that

⁵The difference between our definition of vertex-minor and the usual definition of a graph minor is that we allow contracting two nodes that are not necessarily connected by an edge.

for any H-subgraph $\{v_{i,a_i}\}_{i\in[k]}$ in G, the H'-subgraph $\{v_{i,a_i}\}_{i\in[k-1]}\cup\{v_{k^{(1)},a_{k^{(1)}}},v_{k^{(2)},a_{k^{(2)}}}\}$ in G', where $a_{k^{(1)}}=a_{k^{(2)}}=a_k$, will have the same total weight. Therefore, there is an H'-subgraph of total weight 0 in G' if and only if there is an H-subgraph of total weight 0 in G.

Theorem 3 follows from these two lemmas, by the transitive property of our reducibility definition.

4.1 Reverse Direction

Next we give an observation which shows how $EXACT-H_2$ can be reduced to $EXACT-H_1$, where H_2 contains H_1 as an induced subgraph. This can be seen as a reversal of Theorem 3, since H_2 is a larger graph.

Proposition 17. Let H_2 be the subgraph you get by adding a node to H_1 that has edges to every node in H_1 , and let $\alpha \geq 2$. Then, $\mathsf{EXACT}\text{-}H_2$ $\alpha+1 \leq_{\alpha} \mathsf{EXACT}\text{-}H_1$.

Proof. Without loss of generality denote $V(H') = \{h_1, \dots, h_k\}, V(H) = \{h_1, \dots, h_k, h_{k+1}\}$ and $E(H) = E(H') \cup \{(h_i, h_{k+1})\}_{i=1}^k$. That is, h_{k+1} is the new node, and it's connected with edges to all the other nodes. To solve EXACT-H on a complete H-partite graph G on $(k+1) \cdot n$ nodes, we will create n instances of EXACT-H', one for every vertex $v_{k+1,a}$ in the super-node $P_{h_{k+1}}$. The instance will have a solution if and only if there is an H-subgraph in G of total weight 0 that has node $v_{k+1,a}$ in it:

For every $a \in [n]$, create an H'-partite graph G'_a that will be the same as G on all super-nodes and all super-edges that do not involve h_{k+1} , but will have the following additional weights: For every $i \in [k]$ and $b \in [n]$, we will add the weight of the edge $(v_{i,b}, v_{k+1,a})$ in G, to the weight of node $v_{i,b}$ in G'.

Now observe that every H'-subgraph in G'_a , $\{v_{i,a_i}\}_{i\in[k]}$ will have exactly the same weight as the H-subgraph of G which is $\{v_{i,a_i}\}_{i\in[k+1]}$, where $a_{k+1}=a$. And therefore, there is an H'-subgraph in G'_a of total weight 0, if and only if there is an H-subgraph of G which has the vertex $v_{k+1,a}$ and has weight 0. \square

5 Reductions to *k*-SUM (and Upper Bounds)

In this section we show how k-SUM can be used to solve exact-weight subgraph problems. First, we show how the standard reduction from clique detection to k-SUM can be generalized to relate k-SUM to exact-weight subgraph problems. However, this reduction does not give non-trivial implications for most subgraphs. Then, we show how to use a 2-SUM algorithm to solve EXACT-H for any subgraph H. This gives us a generic algorithm for solving EXACT-H, which we call the separator algorithm. Finally, we generalize this algorithm in a way that allows it to be phrased as a reduction from EXACT-H to d-SUM for any $d \le k - 1$, where k = |V(H)|. The bounds achieved by the algorithms depend on the structure of H.

In [5], a reduction that maps an unweighted k-CLIQUE detection instance to a $\binom{k}{2}$ -SUM instance on n^2 numbers is given in order to prove that k-SUM is W[1]-hard. The reduction maps each edge to a number that encodes the two vertices that are adjacent to the edge in a way such that the numbers encoded in the edges corresponding to a k-CLIQUE, when summed, cancel out to 0. In [11], the authors show how the same idea can be applied to show that triangle detection can be reduced to 3-XOR on n^2 vectors. We show that EXACT-H can be reduced to d-SUM on n^2 numbers, where d = |E(H)|.

Proposition 18. Let H be a subgraph on k nodes and d edges, and let $\alpha \geq 2$. Then, Exact-H $_{2\cdot \alpha} \leq_{\alpha} d$ -SUM. Moreover, the Exact-H problem on graphs with m edges can be reduced to d-SUM on m numbers.

Proof. We give a simple proof that uses out techniques. First note that any H with d edges is a vertex-minor of the d-MATCHING subgraph, and therefore by Theorem 3, EXACT-H $_{\alpha} \leq_{\alpha}$ EXACT-d-MATCHING, and note that in our reduction, the sparsity of the graph is preserved. Then use Theorem 4 to reduce EXACT-d-MATCHING to d-SUM, by choosing $S = \emptyset$ and H_i to be the two endpoints of the i^{th} edge, and observe that we get a d-SUM instance on m numbers.

The Separator Algorithm. We will say that (S, H_1, H_2) is a *separator* of a graph H iff S, H_1, H_2 partition V(H) and there are no edges between a vertex in H_1 and a vertex in H_2 . The set of all separators of H will be denoted as S(H). Consider the following algorithm to solve EXACT-H. We will call this algorithm the *separator algorithm*. First, find

$$(S, H_1, H_2) = \mathsf{argmin}_{(S', H_1', H_2') \in \mathcal{S}(H)}(|S'| + \max(|H_1'|, |H_2'|))$$

by naively brute-forcing over all 3^k possible choices for S, H_1 , and H_2 . Then, pick an S-subgraph $\chi_S = \{v_{i,a_i}\}_{h_i \in S}$. Construct a 2-SUM instance with target weight $w(\chi_S)$ and lists L_1 and L_2 constructed as follows: For every H_1 -subgraph $\chi_{H_1} = \{v_{i,a_i}\}_{h_i \in H_1}$, add $w(\chi_{H_1} \cup \chi_S)$ to L_1 . Similarly, for every H_2 -subgraph $\chi_{H_2} = \{v_{i,a_i}\}_{h_i \in H_2}$, add $w(\chi_{H_2} \cup \chi_S)$ to L_2 . We create an instance of 2-SUM for each possible S-subgraph χ_S . The algorithm outputs that there is an H-subgraph of total weight 0 iff some 2-SUM instance has a solution. The running time of this algorithm is $O(3^k + n^{|S|} \cdot (n^{|H_1|} + n^{|H_2|}))$. We will call $\gamma(H) = |S| + \max(|H_1|, |H_2|)$.

To see that the separator algorithm solves EXACT-H in time $O(n^{\gamma(H)})$, suppose there is an H-subgraph $\{v_{i,a_i}\}_{i\in[k]}=\{v_{i,a_i}\}_{h_i\in S}\cup\{v_{i,a_i}\}_{h_i\in H_1}\cup\{v_{i,a_i}\}_{h_i\in H_2}$ of weight 0. Then, the 2-SUM instance corresponding to $\chi_S=\{v_{i,a_i}\}_{h_i\in S}$ will contain an integer $w(\{v_{i,a_i}\}_{h_i\in H_1}\cup\{v_{i,a_i}\}_{h_i\in S})\in L_1$ and $w(\{v_{i,a_i}\}_{h_i\in H_2}\cup\{v_{i,a_i}\}_{h_i\in S})\in L_2$.

Since S, H_1, H_2 partition V(H), we can write the sum of these two integers as

$$w(\{v_{i,a_i}\}_{h_i \in S} \cup \{v_{i,a_i}\}_{h_i \in H_1} \cup \{v_{i,a_i}\}_{h_i \in H_2}) + w(\{v_{i,a_i}\}_{h_i \in S}).$$

Note that the term on the left is equal to $w(\{v_{i,a_i}\}_{h_i \in V(H)})$, which by our assumption is 0. Thus, the sum of these two integers is equal to the target, $w(\{v_{i,a_i}\}_{h_i \in S})$.

For the other direction, let $\chi_S = \{v_{i,a_i}\}_{h_i \in S}$ be the corresponding S-subgraph to the 2-SUM instance which has a solution of the form $w(\{v_{i,a_i}\}_{h_i \in H_1} \cup \{v_{i,a_i}\}_{h_i \in S}) \in L_1$ and $w(\{v_{i,a_i}\}_{h_i \in H_2} \cup \{v_{i,a_i}\}_{h_i \in S}) \in L_2$. Since S, H_1, H_2 partition V(H), we can write the sum of these two integers as:

$$w(\{v_{i,a_i}\}_{h_i \in H_1} \cup \{v_{i,a_i}\}_{h_i \in S} \cup \{v_{i,a_i}\}_{h_i \in H_2}) + w(\{v_{i,a_i}\}_{h_i \in S}).$$

Since the target of the 2-SUM instance is $w(\{v_{i,a_i}\}_{h_i \in S})$, we have that $w(\{v_{i,a_i}\}_{h_i \in H_1} \cup \{v_{i,a_i}\}_{h_i \in S} \cup \{v_{i,a_i}\}_{h_i \in H_2}) = w(\{v_{i,a_i}\}_{h_i \in V(H)}) = 0$.

Remark 1. The separator algorithm is quite simple, yet we are not aware of any subgraph H for which there is an algorithm that solves EXACT-H in time $O(n^{\gamma(H)-\varepsilon})$, for some $\varepsilon>0$. We have given examples of subgraphs for which improving on the separator algorithm is known to imply that the k-SUM Conjecture is false, and some for which this implication is not known.

Generalizing the Separator Algorithm. We can view the separator algorithm as an algorithm which finds the optimal way to "break" H into two subgraphs H_1 and H_2 , and enumerates all instances of H_1 and H_2 independently, and then solves 2-sum instances to combine the edge-disjoint subgraphs. One natural way to generalize this algorithm is to consider what happens when we divide H into d subgraphs H_1, \ldots, H_d . Then, by a similar algorithm, one can use d-SUM to solve the EXACT-H problem. This generalization is of interest due to the fact that it implies that faster d-SUM algorithms imply faster algorithms for EXACT-H.

We will say that $(S, H_1, ..., H_d)$ is a *d-separator* iff $S, H_1, ..., H_d$ partition V(H) and there are no edges between a vertex in H_i and a vertex in H_j for any distinct $i, j \in [1, d]$. The set of all *d*-separators of H will be denoted as $S^d(H)$.

Reminder of Theorem 4: Let (S, H_1, \ldots, H_d) be a d-separator of H. Then, EXACT-WEIGHT-H can be reduced to $\tilde{O}(n^{|S|})$ instances of d-SUM each on $\max\{n^{|H_1|}, \ldots, n^{|H_d|}\}$ numbers.

Proof. [of Theorem 4] We can generalize the separator algorithm to hold for arbitrary d-separators. Pick an S-subgraph $\chi_S = \{v_{i,a_i}\}_{h_i \in S}$. Construct a d-SUM instance with target weight $(d-1) \cdot w(\chi_S)$ and lists L_1, \ldots, L_d constructed as follows: For all $j \in [d]$, for every H_j -subgraph $\chi_{H_j} = \{v_{i,a_i}\}_{h_i \in H_j}$, add $w(\chi_{H_j} \cup \chi_S)$ to L_j . We create an instance of d-SUM for each possible S-subgraph χ_S . The algorithm outputs that there is an H-subgraph of total weight 0 iff some d-SUM instance has a solution. The proof of correctness for this reduction follows similarly to the proof of correctness for the separator algorithm. The $\tilde{O}(\cdot)$ in the number of instances comes from the EXACT-WEIGHT-H < EXACT-H reduction.

Corollary 19. Let H be a graph on k nodes and let I be an independent set of H where |I| = s. Then, EXACT-H can be reduced to $O(n^{k-s})$ instances of s-SUM on n integers.

Proof. Consider the separator (S, H_1, \dots, H_s) , where $S = V(H) \setminus I$, and H_i is a singleton containing the ith vertex in I.

Corollary 20. Let H be a graph on k nodes with treewidth bounded by tw. Then, EXACT-H can be solved in time $O(n^{\frac{2}{3}k+tw})$.

Proof. Observe that there will be a d-separator of size tw, for some d>1, where each of the d disconnected components has at most k/2 nodes, and therefore can be separated into two disconnected components with at most 2k/3 nodes each. Thus, $\gamma(H) \leq 2k/3 + tw$.

Remark 2. Note that under the current best known running times for k-SUM, the separator algorithm (of Corollary 5) will always be at least as good as the algorithm one can get from Theorem 4. This is implied by the fact that the fastest known way to solve k-SUM is by a reduction to 2-SUM. However, if it turns out that there exists a k_0 for which k_0 -SUM can be solved fast enough, the algorithm of Theorem 4 can be faster than the separator algorithm. As an example, assume 3-SUM can be solved in linear time, and H is a subgraph composed of 3 disconnected k/3-node cliques.

Now notice that if one wanted to find the minimum total weight of an H-subgraph in the input graph G, the same procedure can be applied, with a slight modification that makes it more efficient. When going over an S-subgraphs χ_S of G, instead of solving d-SUM on the d lists L_1,\ldots,L_d , it is enough to find the minimum number in each list. Observe that the sum of these numbers, minus $(d-1)\cdot w(\chi_S)$, equals the minimum total weight of an H-subgraph in G that uses the nodes in χ_S . Therefore, by going over all S-subgraphs, and taking the minimum of these numbers, one gets the minimum total weight of an H-subgraph in G. The running time of this modified procedure is $O(n^{|S|}\cdot (n^{|H_1|}+\cdots+n^{|H_d|}))$.

Reminder of Theorem 6: Let H be a subgraph on k nodes, with independent set of size s. Given a graph G on n nodes with node and edge weights, the minimum total weight of a (not necessarily induced) subgraph of G that is isomorphic to H can be found in time $\tilde{O}(n^{k-s+1})$.

Proof. First, observe that by our proof of the reduction EXACT-WEIGHT- $H \leq \text{EXACT-}H$ in Appendix B, an algorithm for the minimization problem that assumes the graph is H-partite yields an algorithm for the original problem with the same running time, up to $k^k \cdot \text{poly} \log n$ factors. Then, use the procedure mentioned above where (S, H_1, \ldots, H_d) are as in the proof of Corollary 19, to solve the structured version of the problem in time $O(n^{k-s} \cdot n)$.

6 Conclusions

We conclude with two interesting open questions:

- 1. Perhaps the simplest subgraph for which we cannot give tight lower and upper bounds is the 5-CYCLE subgraph. Can we achieve $O(n^{4-\varepsilon})$ for some $\varepsilon>0$ without breaking the k-SUM Conjecture, or can we prove that it is not possible?
- 2. Can we prove that EXACT-WEIGHT-4-PATH $_3 \le _3$ EXACT-WEIGHT-3-STAR? This would show that breaking the 3-SUM Conjecture will imply an $O(n^{3-\varepsilon})$ for some $\varepsilon>0$ algorithm for ALL-PAIRS SHORTEST PATHS.

Acknowledgements. The authors would like to thank Ryan and Virginia Williams for many helpful discussions and for sharing their insights, and Hart Montgomery for initiating the conversation that led up to this work. We would also like to thank the anonymous reviewers for their comments and suggestions.

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A Reducibility

Our definition of reducibility is a mild extension of the definition of sub cubic reducibility in [19](Definition C.1). In weighted graph problems where the weights are integers in [-M, M], n will refer to the number of nodes times $\log M$. For k-SUM problems where the input integers are in [-M, M], n will refer to the number of integers times $\log M$.

Definition 7. Let A and B be two decision problems. We say that $A_a \leq_b B$, if there is an algorithm A with oracle access to B, such that for every $\varepsilon > 0$ there is a $\delta > 0$ satisfying three properties:

- For every instance x of A, A solves the problem A on x probability 1 o(1).
- A runs in time $O(n^{a-\delta})$ time on instances of size n.
- For every instance x of A of size n, let n_i be the size of the i^{th} oracle access to B in A(x). Then $\sum_i n_i^{b-\varepsilon} \leq n^{a-\delta}$.

The proofs of Propositions 1 and 2 in [19], prove that this definition has the following two properties that we will use:

- Let A, B, C be problems so that $A \underset{a \leq_b}{\leq_c} B$ and $B \underset{b \leq_c}{\leq_c} C$, then $A \underset{a \leq_c}{\leq_c} C$.
- If $A \ _a \leq_b B$ then an $O(n^{b-\varepsilon})$ algorithm for B for some $\varepsilon > 0$, implies an $O(n^{a-\delta})$ algorithm for A for some $\delta > 0$, that succeeds with probability 1 o(1).

B Proof of Lemma 8

Claim 21. Exact-H $_{\alpha}\leq_{\alpha}$ Exact-Weight-H.

Proof. Let $G = \{v_{i,a_i}\}_{i \in [k]}$ be an H-partite graph with weight function $w: V(G) \cup E(G) \to \mathbb{Z}$ and target weight 0. In this proof, we will construct a new weight function $w^*: E(G) \to \mathbb{Z}$ and a target $t \in \mathbb{Z}$ such that G, H, w^* , and t make up an instance of EXACT-WEIGHT-H. We will build w^* in the following manner. Let W be the maximum weight of a node or edge in the graph, and let $d = W \cdot |E(H)|$. First, initialize $w^*(v_{i,a}, v_{j,b}) = w(v_{i,a}, v_{j,b})$ for all edges in G. Then, let (P_{h_a}, P_{h_b}) be the z^{th} super-edge of H. For each edge $(v_{i,a}, v_{j,b})$, we add the integer d^z to $w^*(v_{i,a}, v_{j,b})$. Now, for each $i \in [n]$, pick an arbitrary $j \in [n]$ such that $(h_i, h_j) \in E(H)$, and add $w(v_{i,a})$ to $w^*(v_{i,a}, v_{j,b})$ for all $a, b \in [n]$. We set the target $t = \sum_{i=1}^d d^i$.

To prove correctness, let $\chi = \{v_{i,a_i}\}_{i \in [k]}$ be an H-subgraph of G of total weight 0. Since each super-edge of G is used exactly once by χ , it follows that the sum of the edges of χ will have total weight t under weight function w^* . For the reverse direction, let $S = \{v_i\}_{i \in [k]}$ be a subgraph isomorphic to H whose edge weights sum to t under weight function w^* . Then, each v_i must lie in a distinct super-node of G, for otherwise, if the j^{th} super-node is unoccupied, then the total weight of S cannot possibly sum to t. Now, relabel the vertices of S as $\{v_{i,a_i}\}_{i \in [k]}$. Then, the total weight of S under w^* can be expressed as $\sum_{h_i \in V(H)} w(v_{i,a_i}) + \sum_{(h_i,h_j) \in E(H)} w(v_{i,a_i},v_{j,a_j}) + t$. Therefore, we conclude that the sum of the weights of the nodes and edges of S is S0, as desired.

Claim 22. Exact-Weight-H $_{\alpha} \leq_{\alpha}$ Exact-H.

Proof. We will use a simple color coding trick to ensure that the reduction succeeds with probability $1/k^k$. This procedure can be derandomized using standard techniques.

Let G^* be the graph of an instance of EXACT-WEIGHT-H. We construct an H-partite graph G in the following manner. For each vertex $v_i \in V(G)$, we will pick a random $j \in [k]$ and put v_i in super-node P_{h_j} . In other words, we maintain the structure of the graph while partitioning the vertices into k parts. This can also be seen analogously as color-coding the vertices using k colors. The graph G (along with the original weight function) is now an instance of EXACT-H.

For correctness, note that if G contains an H-subgraph χ , then χ is an isomorphic copy of H in G^* with probability 1. For the other direction, we will show that with probability at least $1/k^k$, a set of vertices $\chi = \{v_i\}_{i \in [k]}$ from G^* will form an H-subgraph in G. For each vertex v_i , there is a 1/k probability that it is assigned to partition P_{h_i} . Thus, with probability $1/k^k$, this event holds for all v_i for $i \in [k]$, and so χ is an H-subgraph of G. To translate this into a reduction, we simply repeat this randomized procedure $O(k^k)$ times.

C Proof of Lemma 7

Proof. Assume CONVOLUTION-k-SUM can be solved in time $O(n^{\lceil \frac{k}{2} \rceil - \varepsilon})$, for some $\varepsilon > 0$. We follow the outline of the proof of Theorem 10 in [15] to give an $O(n^{\lceil \frac{k}{2} \rceil - \varepsilon'})$ time algorithm for k-SUM.

We use a hashing scheme due to Dietzfelbinger [4] to hash the numbers of the n-SUM instance to t buckets. In [4], a simple hash family $\mathcal{H}_{M,t}$ is given, such that if one picks a function $h:[M] \to [t]$ at random from $\mathcal{H}_{M,t}$, and maps each number $x_{i,j} \in L_i$ to bucket $B_{i,h(x_{i,j})}$, the following will hold:

- (Good load balancing) W.h.p. only O(kt) numbers will be mapped to "overloaded" buckets, that is, buckets with more than kn/t numbers. Moreover, each number will be hashed to an "overloaded" bucket with o(1) probability.
- (Almost linearity) For any k-1 buckets $B_{1,a_1},\ldots,B_{k-1,a_{k-1}}$, and any k-1 numbers $y_1\in B_{1,a_1},\ldots,y_{k-1}\in B_{k-1,a_{k-1}}$, the number $z=-(y_1+\cdots+y_{k-1})$ can only be mapped to one of certain k buckets (w.p. 1): $B_{k,a^{(1)}},\ldots,B_{k,a^{(k)}}$, where w.l.o.g. we can assume that $a^{(1)}=\sum_{i=1}^{k-1}a_i$, and for $1< i\leq k$, $a^{(i)}=a^{(i-1)}+1$.

Given a k-SUM instance L_1, \ldots, L_k , our reduction is as follows:

- 1. Repeat the following $c \cdot k^k \cdot \log n$ times.
 - (a) Pick a hash function $h \in \mathcal{H}_{M,t}$, for t to be set later, and map each number $x_{i,j}$ to bucket $B_{i,h(x_{i,j})}$.
 - (b) Ignore all numbers mapped to "overloaded" buckets.
 - (c) Now each bucket has at most R = kn/t numbers. We create $k \cdot R^k$ instances of CONVOLUTION-k-SUM, one for every choice of numbers $(i_1, \ldots, i_k) \in [R]^k$ and a number $0 \le y < k$, where in each instance, the lists will contain only t numbers. These instances will test all k-solutions that might lead to a solution. For a fixed $(i_1, \ldots, i_k) \in [R]^k$ and $0 \le y < k$, we create k lists L'_1, \ldots, L'_k as input for CONVOLUTION-k-SUM, where for every $j \in [k-1]$, $x'_{j,a} \in L'_j$ will be set to the i_j -th number of bucket $B_{j,a}$ of L_j , while $x'_{k,a} \in L'_k$ will be set to the i_k -th number of bucket $B_{k,a+y}$ of L_k .

To see the correctness of the reduction, assume there was a solution to the k-SUM problem, $\{x_{j,a_j}\}_{j\in[k]}$, and note that with probability $1-O(n^{-c})$, there will be an iteration for which these numbers are not mapped to "overloaded" buckets. Now let h be the hash function in a good iteration, $a=\sum_{j=1}^{k-1}h(x_{j,a_j})$, and y be such that $h(x_{k,a_k})=a+y$. Note that by the "almost linearity" property, such $y\in[k]$ must exist. Now let $(i_1,\ldots,i_k)\in[R]^k$ be such that for every $j\in[k-1]$, x_{j,a_j} is the i_j -th element in bucket $B_{j,h(x_{j,a_j})}$ of L_j , while x_{k,a_k} is the i_k -th element in bucket $B_{k,a+y}$. Now consider the CONVOLUTION-k-SUM instance that we get for these (i_1,\ldots,i_k) and y, and consider the k-solution $\{x'_{j,h(x_{j,a_j})}\}_{j\in[k-1]}\cup\{x'_{k,h(x_{k,a_k})-y}\}$. Its sum will be exactly $\sum_{j\in[k]}x_{j,a_j}$, since $x'_{j,h(x_{j,a_j})}$ will be set to x_{i,a_j} , for every $j\in[k]$. And it will satisfy the convolution property, since $\sum_{j=1}^{k-1}h(x_{j,a_j})=a=h(x_{k,a_k})-y$. For the other direction, any k-solution in any convolution problem is a legitimate k-solution in the original k-SUM problem with the same sum. Therefore, with probability $1-o(n^c)$, there is a solution iff one of the CONVOLUTION-k-SUM instances has a solution.

The total running time of the reduction is $\tilde{O}(t \cdot n^{\left\lceil \frac{k-1}{2} \right\rceil} + (n/t)^{\left\lceil \frac{k}{2} \right\rceil - \varepsilon})$. Now set $t = n^{\varepsilon}$, and note that when k is odd, the first term is insignificant, to get a running time of $\tilde{O}(n^{(1-\varepsilon)\cdot(\left\lceil \frac{k}{2} \right\rceil - \varepsilon)}) = \tilde{O}(n^{\left\lceil \frac{k}{2} \right\rceil - \varepsilon'})$, for some $\varepsilon' > 0$.