# Universal Witnesses for State Complexity of Boolean Operations and Concatenation Combined with Star* 

Janusz Brzozowski and David Liu<br>David R. Cheriton School of Computer Science, University of Waterloo, Waterloo, ON, Canada N2L 3G1<br>\{brzozo, dyliu\}@uwaterloo.ca


#### Abstract

We study the state complexity of boolean operations and product (concatenation, catenation) combined with star. We derive tight upper bounds for the symmetric differences and differences of two languages, one or both of which are starred, and for the product of two starred languages. We prove that the previously discovered bounds for the union and the intersection of languages with one or two starred arguments, for the product of two languages one of which is starred, and for the star of the product of two languages can all be met by the recently introduced universal witnesses and their variants.


Keywords: boolean operation, combined operation, concatenation, regular language, product, star, state complexity, universal witness

## 1 Introduction

The state complexity of a regular language is the number of states in the minimal deterministic finite automaton (DFA) recognizing the language. The state complexity of an operation on regular languages is the worst-case state complexity of the result of the operation as a function of the state complexities of the arguments. For more information on this topic see 11211 .

Let $K$ and $L$ be two regular languages over alphabet $\Sigma$, and let their state complexities be $m$ and $n$, respectively. In 2007 A. Salomaa, K. Salomaa, and Yu 10 showed using ternary witnesses that the complexity of $(K \cup L)^{*}$ is $2^{m+n-1}-\left(2^{m-1}+2^{n-1}-1\right)$. They also established a lower bound for $(K \cap L)^{*}$ using an alphabet of 8 letters. These results were improved by Jirásková and Okhotin [9] who showed that binary witnesses suffice for $(K \cup L)^{*}$, and that $3 \cdot 2^{m n-2}$ is a tight upper bound for $(K \cap L)^{*}$; they used an alphabet of 6 letters. In 2012, Gao and Yu [8] showed with ternary witnesses that the complexity of $K \cup L^{*}$ is $m\left(2^{n-1}+2^{n-2}-1\right)+1$, and that the same upper bound applies to $K \cap L^{*}$. Moreover, it was shown in [6] by Gao, Kari and Yu that quaternary witnesses meet the bound $\left(2^{m-1}+2^{m-2}-1\right)\left(2^{n-1}+2^{n-2}-1\right)+1$ for $K^{*} \cup L^{*}$

[^0]and $K^{*} \cap L^{*}$. In 2008, Gao, K. Salomaa, and Yu [7] demonstrated using quaternary witnesses that $2^{m+n-1}+2^{m+n-4}-\left(2^{m-1}+2^{n-1}-m-1\right)$ is a tight upper bound for $(K L)^{*}$. The complexity of $K L^{*}$ was studied by Cui, Gao, Kari and $\mathrm{Yu}[5$ in 2012. They proved with ternary witnesses that the tight bound is $m\left(2^{n-1}+2^{n-2}\right)-2^{n-2}$. The same authors also showed in 4] using quaternary witnesses that the complexity of $K^{*} L$ is $5 \cdot 2^{m+n-3}-2^{m-1}-2^{n}+1$. In summary, nine operations using union, intersection, and product (also called concatenation or catenation) combined with star have been studied.

To establish the state complexity of an operation one finds an upper bound and languages to act as witnesses to show that the bound is tight. A witness is usually a sequence ( $L_{n} \mid n \geqslant k$ ) of languages, where $k$ is some small positive integer; we will call such a sequence a stream of languages. The languages in a stream normally differ only in the parameter $n$. In the past, two different streams have been used for most binary operations.

Recently, Brzozowski [2] proposed the DFA $\mathcal{U}_{n}(a, b, c)=(Q, \Sigma, \delta, 0,\{n-1\})$ of Fig. 1 and its language $U_{n}(a, b, c)$ as the "universal witness" DFA and language, respectively, for $n \geqslant 3$. The restrictions of the DFA and the language to alphabet $\{a, b\}$ are denoted by $\mathcal{U}_{n}(a, b, \emptyset)$ and $U_{n}(a, b, \emptyset)$. It was proved in [2] that the bound $2^{n-1}+2^{n-2}$ for star is met by $U_{n}(a, b, \emptyset)$, and the bound $2^{n}$ for reversal, by $U_{n}(a, b, c)$. The bound $(m-1) 2^{n}+2^{n-1}$ for product is met by $U_{m}(a, b, c)$ and $U_{n}(a, b, c)$. The bound $m n$ for union, intersection, difference $(K \backslash L)$ and symmetric difference $(K \oplus L)$ is met by the streams $U_{m}(a, b, c)$ and $U_{n}(a, b, c)$ if $m \neq n$, as was conjectured in [2] and proved in [3]. If $m=n$, it is necessary to use two different streams; however, it is possible to use streams that are almost the same, in the following sense. Two languages $K$ and $L$ over $\Sigma$ are permutationally equivalent if one can be obtained from the other by permuting the letters of the alphabet, and a similar definition applies to DFA's. It was proved in [2] that two permutationally equivalent streams $U_{m}(a, b, c)$ and $U_{n}(b, a, c)$ are witnesses to the bound for the boolean operations: union $(K \cup L)$, intersection $(K \cap L)$, difference $(K \backslash L)$, and symmetric difference $(K \oplus L)$. Thus $U_{n}(a, b, c)$ is indeed a universal witness for the basic operations.


Fig. 1. DFA $\mathcal{U}_{n}(a, b, c)$ of language $U_{n}(a, b, c)$.

It turns out that the witness $U_{n}(a, b, c)$ cannot meet the bound for some combined operations. However, the notion of universal witness can be broadened to include "dialects" of $U_{n}(a, b, c)$. Some terminology is required, before we define this concept.

The inputs of DFA $\mathcal{U}_{n}$ perform the following transformations on the set $Q=\{0, \ldots, n-1\}$ of states. Input $a$ is a cycle of all $n$ states, and this is denoted by $a:(0, \ldots, n-1)$. Input $b$ is a transposition of 0 and 1 , and does not affect any other states; this is denoted by $b:(0,1)$, and by $b:(i, j)$, if $i$ and $j$ are transposed. Input $c$ is a singular transformation sending state $n-1$ to state 0 , and not affecting any other states; this is denoted by $c:\binom{n-1}{0}$, and by $c:\binom{i}{j}$, in general. The constant transformation sending all states to state $i$ is denoted by $\binom{Q}{i}$. The identity transformation on $Q$ is denoted by $\mathbf{1}_{Q}$.

It is known [2] that the inputs of $\mathcal{U}_{n}(a, b, c)$ of Fig. 11 perform all $n^{n}$ transformations of states.

A dialect of $U_{n}(a, b, c)$ is the language of any DFA with three inputs $a, b$, and $c$, where $a$ is a cycle of length $n$ as above, $b$ is the transposition of any two states $(i, j)$, and $c$ is a singular transformation $c:\binom{i}{j}$ sending any state $i$ to any state $j$. The initial state is always 0 , but the set of final states is arbitrary, as long as the resulting DFA is minimal.

Since there are operations for which ternary witnesses do not meet the worstcase bounds, the notions of universal witness and dialect have been extended to quaternary alphabets [2], by adding a fourth input $d$ which performs the identity permutation, denoted by $d: \mathbf{1}_{Q}$. The concepts of permutational equivalence and dialects were extended in the obvious way to quaternary languages and DFA's. The following dialects are used in this paper:

1. $\mathcal{U}_{\{0\}, n}(a, b, c)$, which is $\mathcal{U}_{n}(a, b, c)$ with $\{0\}$ as the set of final states.
2. $\mathcal{T}_{n}(a, b, c)=\left(Q, \Sigma, \delta_{T}, 0,\{n-1\}\right)$, where $a:(0, \ldots, n-1), b:(0,1)$, and $c:\binom{1}{0}$.
3. $\mathcal{W}_{n}(a, b, c, d)=\left(Q, \Sigma, \delta_{\mathcal{W}}, 0,\{n-1\}\right)$, where $a:(0, \ldots, n-1), b:(n-2, n-1)$, $c:\binom{1}{0}$, and $d: \mathbf{1}_{Q}$.
4. $\mathcal{W}_{\{0\}, n}(a, b, c, d)$, which is $\mathcal{W}_{n}(a, b, c, d)$ with $\{0\}$ as the set of final states.

We use the convention that $\mathcal{X}$ is a DFA if and only if $X$ is its language. The operation $K \circ L$ represents any one of the four boolean operations union, intersection, difference and symmetric difference.

In this paper, we consider the following 13 operations that use boolean operations and product combined with star :

$$
\begin{aligned}
& K \cup L^{*}, K \cap L^{*}, K \oplus L^{*}, K \backslash L^{*}, L^{*} \backslash K, \\
& K^{*} \cup L^{*}, K^{*} \cap L^{*}, K^{*} \oplus L^{*}, K^{*} \backslash L^{*}, \\
& K L^{*}, K^{*} L, K^{*} L^{*},(K L)^{*} .
\end{aligned}
$$

Our contributions are as follows:

1. We derive the bound $m\left(2^{n-1}+2^{n-2}-1\right)+1$ for $K_{m} \backslash L_{n}^{*}, L_{n}^{*} \backslash K_{m}$ and $K_{m} \oplus L_{n}^{*}$. We show that the known bounds for $K_{m} \cup L_{n}^{*}, K_{m} \oplus L_{n}^{*}$ and $L_{n}^{*} \backslash K_{m}$ are met by the streams $U_{m}(a, b, c)$ and $U_{n}(b, a, c)$, and that, for $K_{m} \cup L_{n}^{*}$ and $K_{m} \backslash L_{n}^{*}$, the dialect $U_{\{0\}, m}(a, b, c)$ and the language $U_{n}(b, a, c)$ act as witnesses. This corrects an error in [8], where it is claimed that the witnesses that serve for union also work for intersection.
2. We derive the bound $\left(2^{m-1}+2^{m-2}-1\right)\left(2^{n-1}+2^{n-2}-1\right)+1$ for $K_{m}^{*} \backslash L_{n}^{*}$, and $K_{m}^{*} \oplus L_{n}^{*}$. We show that the known bounds for $K_{m}^{*} \cup L_{n}^{*}$ and $K_{m}^{*} \cap L_{n}^{*}$ are met
by the dialects $W_{m}(a, b, c, d)$ and $W_{n}(d, c, b, a)$, and that, for $K_{m}^{*} \backslash L_{n}^{*}$ and $K_{m}^{*} \oplus L_{n}^{*}$, the dialects $W_{\{0\}, m}(a, b, c, d)$ and $W_{n}(d, c, b, a)$ act as witnesses.
3. We prove that the known bound $m\left(2^{n-1}+2^{n-2}\right)-2^{n-2}$ for $K_{m} L_{n}^{*}$ is met by the dialects $T_{m}(a, b, c)$ and $T_{n}(b, a, c)$.
4. We show that the known bound $5 \cdot 2^{m+n-3}-2^{m-1}-2^{n}+1$ for $K_{m}^{*} L_{n}$ is met by $U_{m}(a, b, c, d)$ and $U_{n}(d, c, b, a)$.
5. We derive the bound $2^{m+n-1}-2^{m-1}-3 \cdot 2^{n-2}+2$ for $K_{m}^{*} L_{n}^{*}$ and show that it is met by $U_{m}(a, b, c, d)$ and $U_{n}(d, c, b, a)$.
6. We prove that the known bound $2^{m+n-1}+2^{m+n-4}-\left(2^{m-1}+2^{n-1}-m-1\right)$ for $\left(K_{m} L_{n}\right)^{*}$ is met by $W_{m}(a, b, c, d)$ and $W_{n}(d, c, b, a)$.
7. In obtaining these results, we prove Conjectures 7, 9, 10, 12, 15 and 17 of [2].

Sections 2 and 3 study boolean operations with one and two starred arguments, respectively. Products with one or two starred arguments are examined in Section 4. In Section 5 we consider stars of product, intersection, and difference, and Section 6 concludes the paper.

## 2 Boolean Operations with One Starred Argument

Recall that the complexity of $L_{n}^{*}$ is $2^{n-1}+2^{n-2}$. Gao and Yu [8] showed that the complexity of $K_{m} \cup L_{n}^{*}$ is $m\left(2^{n-1}+2^{n-2}-1\right)+1$. They used the following DFA's over alphabet $\Sigma=\{a, b, c\}$ : For $K$, let $\mathcal{D}_{K}=\left(Q_{K}, \Sigma, \delta_{K}, 0,\{m-1\}\right)$, with $Q_{K}=\{0, \ldots, m-1\}, a, b: \mathbf{1}_{Q_{K}}$, and $c:(0, \ldots n-1)$. For $L$, let $\mathcal{D}_{L}=$ $\left(Q_{L}, \Sigma, \delta_{L}, 0,\{n-1\}\right)$, with $Q_{L}=\{0, \ldots, n-1\}, a:(0, \ldots, n-1), b$ defined by $\delta_{L}(0, b)=0, \delta_{L}(i, b)=i+1(\bmod n)$, for $i=1, \ldots, n-1$, and $c: \mathbf{1}_{Q_{L}}$. They showed that the same bound also holds for $K_{m} \cap L_{n}^{*}$, and claimed that the same witnesses work. That claim is incorrect, however, as is shown below.

The results of [8] for union are extended here to $K_{m} \cup L_{n}^{*}, K_{m} \oplus L_{n}^{*}$ and $L_{n}^{*} \backslash K_{m}$ with witnesses $U_{m}(a, b, c)$ and $U_{n}(b, a, c)$, and to $K_{m} \cap L_{n}^{*}$ and $K_{m} \backslash L_{n}^{*}$ with witnesses $U_{\{0\}, m}(a, b, c)$ and $U_{n}(b, a, c)$.

Proposition 1. Let $K_{m}$ and $L_{n}$ be two regular languages with complexities $m$ and $n$. Then the complexities of $K_{m} \circ L_{n}^{*}$ and $L_{n}^{*} \backslash K_{m}$ are at most $m\left(2^{n-1}+\right.$ $\left.2^{n-2}-1\right)+1$, for $n \geqslant 3$.

Proof. Let $\mathcal{D}_{1}=\left(Q_{1}, \Sigma, \delta_{1}, 0, F_{1}\right)$ with $Q_{1}=\{0, \ldots, m-1\}$ be the DFA of $K_{m}$, and let $\mathcal{D}_{2}=\left(Q_{2}, \Sigma, \delta_{2}, 0, F_{2}\right)$ with $Q_{2}=\{0, \ldots, n-1\}$ be the DFA of $L_{n}$. Construct $\mathcal{N}_{2}$, an NFA accepting $L_{n}^{*}$, by adding a new final state $s$ to $\mathcal{D}_{2}$, with the same outgoing transitions as state 0 , and $\varepsilon$-transitions from each final state in $F_{2}$ to 0 . Now $\mathcal{N}_{2}$ has initial state $\{s\}$ instead of $\{0\}$. See Fig. 2 for an illlustration. Let $\mathcal{S}_{2}$ be the minimal DFA obtained from $\mathcal{N}_{2}$ by the subset construction and minimization, and let $\mathcal{P}$ be the direct product of $\mathcal{D}_{1}$ and $\mathcal{S}_{2}$.

For all five boolean operations, the states of $\mathcal{P}$ are ordered pairs, where the first element is a state $i \in Q_{1}$ and the second is either $\{s\}$ or a subset of $Q_{2}$. Because of the $\varepsilon$-transitions, the allowable states are $(0,\{s\})$, all states of the form $(i, S)$ where $S$ is non-empty and $S \cap F_{2}=\emptyset$, and all states of the form $(i, S)$


Fig. 2. DFA $\mathcal{D}_{1}$ of $U_{4}(a, b, c)$ and NFA $\mathcal{N}_{2}$ of $\left(U_{5}(b, a, c)\right)^{*}$.
where $S$ contains at least one final state together with 0 . The total number of possible states is largest if there is only one final state, say $n-1$. Hence the number of states in $\mathcal{P}$ cannot exceed 1 plus $m\left(2^{n-1}-1\right)$ for states of the form $(i, S)$ where $S$ is non-empty and $n-1 \notin S$, and $m 2^{n-2}$ for states of the form $(i, S)$ where $0, n-1 \in S$. Therefore the complexity of $K_{m} \circ L_{n}^{*}$ and $L_{n}^{*} \backslash K_{m}$ cannot exceed $1+m\left(2^{n-1}+2^{n-2}-1\right)$.

Theorem $1\left(K \circ L^{*}\right)$. Let $K_{m}=U_{m}(a, b, c)$ and $L_{n}=U_{n}(b, a, c)$. For $m, n \geqslant 3$, the complexities of $K_{m} \cup L_{n}^{*}, K_{m} \oplus L_{n}^{*}$, and $L_{n}^{*} \backslash K_{m}$ are all $m\left(2^{n-1}+2^{n-2}-1\right)+1$. Let $K_{m}^{\prime}$ be the language $U_{\{0\}, m}(a, b, c)$. Then the complexities of $K_{m}^{\prime} \cap L_{n}^{*}$ and $K_{m}^{\prime} \backslash L_{n}^{*}$ are also $m\left(2^{n-1}+2^{n-2}-1\right)+1$.

Proof. Let the various automata be defined as in the proof of Proposition 1 , but this time with $K_{m}=U_{m}(a, b, c)$ and $L_{n}=U_{n}(b, a, c)$. We show that all $m\left(2^{n-1}+2^{n-2}-1\right)+1$ allowable states of $\mathcal{P}$ are reachable. We use the notation $(i, S) \xrightarrow{w}(j, T)$ to denote that state $(j, T)$ is reached from $(i, S)$ by word $w$. We have $(0,\{s\}) \xrightarrow{c}(0,\{0\}) \xrightarrow{(b a)^{i-1}}(i,\{0\})$ for $2 \leqslant i \leqslant m-1$. If $m$ is odd, $(0,\{0\}) \xrightarrow{a^{m+1}}(1,\{0\})$; if $m$ is even, $(0,\{0\}) \xrightarrow{a^{m-1} c a}(1,\{0\})$.

Brzozowski showed in [2] that all allowable states of $\mathcal{N}_{2}$ are reachable from $\{0\}$ by words in $\{a, b\}^{*}$. These words act as permutations on $\mathcal{D}_{1}$. To reach state $(i, S)$ apply the word $w$ that takes $\{0\}$ to $S$ in $\mathcal{N}_{2}$ to state $(j,\{0\})$, where $j$ is such that $j \xrightarrow{w} i$. Therefore all the allowable states are reachable.

For distinguishability, first consider two states $(i, S)$ and $(j, T)$, where $S \neq T$. Then there is a $k$ either in $S \backslash T$ or in $T \backslash S$; without loss of generality, assume $k \in S \backslash T$. By applying $b^{n-1-k}$, we reach states $\left(i^{\prime}, S^{\prime}\right)$ and $\left(j^{\prime}, T^{\prime}\right)$, where $n-1 \in$ $S^{\prime} \backslash T^{\prime}$. Note that applying some cyclic shift $a^{l}$ to $\mathcal{D}_{1}$, we reach states $\left(i^{\prime \prime}, S^{\prime \prime}\right)$ and ( $j^{\prime \prime}, T^{\prime \prime}$ ), where $n-1 \in S^{\prime \prime} \backslash T^{\prime \prime}$. These states are distinguishable for the boolean operations as follows:
$-K_{m} \cup L_{n}^{*}, K_{m} \oplus L_{n}^{*}, L_{n}^{*} \backslash K_{m}$ : apply a cyclic shift so $i^{\prime}, j^{\prime}$ are non-final in $\mathcal{D}_{1}$. This is possible since as $\mathcal{D}_{1}$ has a single final state and $m \geqslant 3$.
$-K_{m}^{\prime} \cap L_{n}^{*}$ : map $i$ to the final state of $\mathcal{D}_{1}$.
$-K_{m}^{\prime} \backslash L_{n}^{*}$ : map $j$ to the final state of $\mathcal{D}_{1}$.
Now consider two states $(i, S)$ and $(j, S), i<j$. We may assume $j<m-1$ because, since $m \geqslant 3$, we can apply a cyclic shift of $a$ 's so that neither $i$ nor $j$
is equal to $m-1$. Doing so might change $S$ to $S^{\prime}$, but $S^{\prime}$ is the same in both states and $S^{\prime}$ remains non-empty. The states are distinguishable as follows:
$-K_{m} \cup L_{n}^{*}, K_{m} \oplus L_{n}^{*}, K_{m}^{\prime} \backslash L_{n}^{*}$ : apply $c$ so that $n-1 \notin S$, then $a^{k}$ for some $k$ to map $j$ to a final state.

- $K_{m}^{\prime} \cap L_{n}^{*}, L_{n}^{*} \backslash K_{m}$ : since $S$ is non-empty, apply a cyclic shift so $n-1 \in S$, then another shift so $j$ is final, and hence $i$ is non-final.
Finally, note that only states $(0,\{s\})$ and $(0,\{0\})$ reach $(1,\{1\})$ on applying $a$; therefore by the previous argument, $(0,\{s\})$ is distinguishable from all other states except possibly $(0,\{0\})$. Note now that states $(0,\{s\})$ and $(0,\{0\})$ are distinguishable in $K_{m} \cup L_{n}^{*}, K_{m} \oplus L_{n}^{*}$ and $L_{n}^{*} \backslash K_{m}$, but equivalent in $K_{m} \cap L_{n}^{*}$ and $K_{m} \backslash L_{n}^{*}$. Hence we cannot have the same witnesses for both intersection and union. However, the choice of final states distinguishes $(0,\{s\})$ from $(0,\{0\})$ for $K_{m}^{\prime} \cap L_{n}^{*}$ and $K_{m}^{\prime} \backslash L_{n}^{*}$. Therefore all reachable states are distinguishable.


## 3 Boolean Operations with Two Starred Arguments

Gao, Kari and Yu [6] showed that the bounds for $K_{m}^{*} \cup L_{n}^{*}$ and $K_{m}^{*} \cap L_{n}^{*}$ are both $\left(2^{m-1}+2^{m-2}-1\right)\left(2^{n-1}+2^{n-2}-1\right)+1$. They used the following DFA's over alphabet $\Sigma=\{a, b, c, d\}$ : For $K$, let $\mathcal{D}_{K}=\left(Q_{K}, \Sigma, \delta_{K}, 0,\{m-1\}\right)$, with $Q_{K}=\{0, \ldots, m-1\}, a:(0, \ldots, m-1), b$ defined by $\delta_{K}(0, b)=0, \delta_{K}(i, b)=i+1$ $(\bmod m)$, for $i=1, \ldots, m-1$, and $c, d: \mathbf{1}_{Q_{K}}$. For $L$, let $\mathcal{D}_{L}=\left(Q_{L}, \Sigma, \delta_{L}, 0,\{n-\right.$ $1\}$ ), with $Q_{L}=\{0, \ldots, n-1\}, a, b: \mathbf{1}_{Q_{L}}, c:(0, \ldots n-1)$, and $d$ defined by $\delta_{K}(0, d)=0, \delta_{K}(i, d)=i+1(\bmod n)$, for $i=1, \ldots, n-1$.

We extend these results to $K_{m} \oplus L_{n}^{*}$ and $K_{n}^{*} \backslash L_{m}^{*}$, for which we now derive upper bounds.

Proposition 2. Let $K_{m}$ and $L_{n}$ be two regular languages with complexities $m$ and $n$. Then the complexities of $K_{m}^{*} \circ L_{n}^{*}$ are at most $\left(2^{m-1}+2^{m-2}-1\right)\left(2^{n-1}+\right.$ $\left.2^{n-2}-1\right)+1$ for $m, n \geqslant 3$.

Proof. Let $\mathcal{D}_{1}=\left(Q_{1}, \Sigma, \delta_{1}, 0, F_{1}\right)$ be the DFA of $K_{m}$, and $\mathcal{D}_{2}=\left(Q_{2}, \Sigma, \delta_{2}, 0, F_{2}\right)$, the DFA of $L_{n}$. Let $\mathcal{N}_{1}\left(\mathcal{N}_{2}\right)$ be the NFA for $K_{m}^{*}\left(L_{n}^{*}\right)$ obtained by adding a new initial and final state $s_{1}\left(s_{2}\right)$, transitions from state $s_{1}\left(s_{2}\right)$ the same as from 0 in $\mathcal{D}_{1}\left(\mathcal{D}_{2}\right)$, and an $\varepsilon$-transition from each final state of $\mathcal{D}_{1}\left(\mathcal{D}_{2}\right)$ to the initial state 0 of $\mathcal{D}_{1}\left(\mathcal{D}_{2}\right)$. See Fig. 3 for an example of this construction. Let $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$ be the minimal DFA's obtained from $\mathcal{N}_{1}$ and $\mathcal{N}_{2}$ by the subset construction and minimization. Finally, let $\mathcal{P}$ be the direct product of $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$.

The states of $\mathcal{P}$ are ordered pairs, where the first element is a subset of $\left\{s_{1}\right\} \cup Q_{1}$ and the second is a subset of $\left\{s_{2}\right\} \cup Q_{2}$. Note that $s_{1}$ and $s_{2}$ can only appear in the initial state $\left(\left\{s_{1}\right\},\left\{s_{2}\right\}\right)$ of $\mathcal{P}$. After any input is applied to $\mathcal{P}$, the state has the form $(S, T)$, where $S$ is a state of $\mathcal{S}_{1}$ other than $\left\{s_{1}\right\}$ (there are at most $2^{m-1}+2^{m-2}-1$ such states), and $T$ is a state of $\mathcal{S}_{2}$ other than $\left\{s_{2}\right\}$ (there are at most $2^{n-1}+2^{n-2}-1$ such states), and this is independent of the witnesses used. Thus $\left(2^{m-1}+2^{m-2}-1\right)\left(2^{n-1}+2^{n-2}-1\right)+1$ is an upper bound for the number of states of the DFA for $K^{*} \circ L^{*}$.


Fig. 3. NFA's $\mathcal{N}_{1}$ and $\mathcal{N}_{2}$ of $\left(W_{4}(a, b, c, d)\right)^{*}$ and $\left(W_{5}(d, c, b, a)\right)^{*}$.

Theorem $2\left(K^{*} \circ L^{*}\right)$. Let $K_{m}=W_{m}(a, b, c, d)$ and $L_{n}=W_{n}(d, c, b, a)$. For $m, n \geqslant 3$, the complexities of $K_{m}^{*} \cup L_{n}^{*}$ and $K_{m}^{*} \cap L_{n}^{*}$ are $\left(2^{m-1}+2^{m-2}-\right.$ 1) $\left(2^{n-1}+2^{n-2}-1\right)+1$. If $K_{m}^{\prime}$ is the language of $\mathcal{W}_{\{0\}, m}$, then the complexities of $\left(K_{m}^{\prime}\right)^{*} \backslash L_{n}^{*}$ and $\left(K_{m}^{\prime}\right)^{*} \oplus L_{n}^{*}$ are also $\left(2^{m-1}+2^{m-2}-1\right)\left(2^{n-1}+2^{n-2}-1\right)+1$.

Proof. Let the various automata be defined as in the proof of Proposition 2, but this time with $K_{m}=W_{m}(a, b, c, d)$ and $L_{n}=W_{n}(d, c, b, a)$. We now show that all $\left(2^{m-1}+2^{m-2}-1\right)\left(2^{n-1}+2^{n-2}-1\right)+1$ allowable states discussed in Proposition 2 are reachable.

We first show that all allowable subsets of $Q_{1}$ are reachable in $\mathcal{D}_{1}$, ignoring $\mathcal{D}_{2}$. First, $\left\{s_{1}\right\} \xrightarrow{c}\{0\} \xrightarrow{a^{m-1}}\{0, m-1\}$. Suppose all states $S$ with $\{0, m-1\} \subseteq$ $S \subseteq Q_{1},|S|=k, k \geqslant 2$ are reachable. All states $S$ with $\{0,1\} \subseteq S \subseteq Q_{1}$ of size $k$ are now reachable by applying $a$. If $S=\left\{i_{1}, \ldots, i_{k}\right\}$ with $i_{1}<\cdots<i_{k}<m-1$, let $j=i_{2}-i_{1}-1$; then $\left\{0,1, i_{3}-j-i_{1}, \ldots, i_{k}-j-i_{1}\right\} \xrightarrow{(a c)^{j} a^{i_{1}}} S$.

Now states $\{0, m-1\} \subseteq S$ of size $k+1$ can now be reached as follows: $\left\{i_{1}-1, \ldots, i_{k-1}-1, m-2\right\} \xrightarrow{a}\left\{0, i_{1}, \ldots, i_{k-1}, m-1\right\}$.

Therefore all allowable states of $\mathcal{D}_{1}$ are reachable by words in $\{a, c\}^{*}$.
In $\mathcal{N}_{2}, a$ and $c$ map states $s_{2}$ and 0 to 0 . Therefore all allowable states of $\mathcal{P}$ of the form $(S,\{0\})$ are reachable. A symmetric argument shows that all states $T$ of $\mathcal{D}_{2}$ are reachable by words in $\left\{b^{2}, d\right\}^{*}$ (as $b^{2}$ and $b$ are the same transformation on $\mathcal{D}_{2}$ ). All of these words map states $S \subseteq Q_{1}$ to themselves, except in the case $0, m-1 \notin S, m-2 \in S$. Let $S=\left\{i_{1}, \ldots, i_{k}\right\}$ be such a state; then for all allowable $T,\left(\left\{i_{1}-1, \ldots, i_{k}-1\right\}, T\right)$ is reachable, and reaches $(S, T)$ when $a$ is applied. Therefore all allowable states are reachable.

Next we show that all the states of $\mathcal{P}$ are distinguishable. Recall that for $K_{m}^{*} \cup L_{n}^{*}$ and $K_{m}^{*} \cap L_{n}^{*}$, we use $\{m-1\}$ as the final state of $\mathcal{N}_{1}$, and for $\left(K_{m}^{\prime}\right)^{*} \oplus L_{n}^{*}$ and $\left(K_{m}^{\prime}\right)^{*} \backslash L_{n}^{*}$, we use $\{0\}$.

Suppose we have states $\left(S_{1}, T_{1}\right),\left(S_{2}, T_{2}\right)$ with $T_{1} \neq T_{2}$. Then there is a $k$ either in $T_{1} \backslash T_{2}$ or in $T_{2} \backslash T_{1}$; without loss of generality, assume $k \in T_{1} \backslash T_{2}$. By applying $d^{n-1-k}$, we reach states $\left(S_{1}, T_{1}^{\prime}\right)$ and $\left(S_{2}, T_{2}^{\prime}\right)$, where $n-1 \in T_{1}^{\prime} \backslash T_{2}^{\prime}$. Apply $c^{2} a c^{2}$ so that $T_{1}^{\prime}$ and $T_{2}^{\prime}$ are unchanged, but now $1,2 \notin S_{1}^{\prime} \cup S_{2}^{\prime}$. Then apply $a^{m-2}$ so $0, m-1 \notin S_{1}^{\prime \prime} \cup S_{2}^{\prime \prime}$. This distinguishes the two states for $K_{m}^{*} \cup L_{n}^{*}$ and $\left(K_{m}^{\prime}\right)^{*} \oplus L_{n}^{*}$. For $K_{m}^{*} \cap L_{n}^{*}$, since $S_{1} \neq \emptyset$, we may apply a cyclic shift to $\mathcal{D}_{1}$ so that $m-1 \in S_{1}^{\prime}$ to distinguish the states. For $\left(K_{m}^{\prime}\right)^{*} \backslash L_{n}^{*}$, we can assume that $h \in S_{2}^{\prime \prime}$, and use $a^{m-1-h}$ to map $S_{2}^{\prime \prime}$ to $S_{2}^{\prime \prime \prime}$, where $\{0, m-1\} \subseteq S_{2}^{\prime \prime \prime}$. This also
maps $S_{1}^{\prime \prime}$ to $S_{1}^{\prime \prime \prime}$, and keeps $T_{1}^{\prime}$ and $T_{2}^{\prime}$ unchanged. Since $n-1 \in T_{1}^{\prime} \backslash T_{2}^{\prime}$, we have ( $S_{1}^{\prime \prime \prime}, T_{1}^{\prime}$ ) is non-final and ( $S_{2}^{\prime \prime \prime}, T_{2}^{\prime}$ ) is final for $\left(K_{m}^{\prime}\right)^{*} \backslash L_{n}^{*}$.

Now suppose $S_{1} \neq S_{2}$. For $K_{m}^{*} \cup L_{n}^{*}$ and $K_{m}^{*} \cap L_{n}^{*}$ the above argument is symmetric. For the other two operations, apply a cyclic shift so that $m-1 \in$ $S_{1}^{\prime} \backslash S_{2}^{\prime}$. Now apply $(c b a)^{m-3}$ so that $m-1 \in S_{1}^{\prime \prime} \backslash S_{2}^{\prime \prime}$, and $2, \ldots, m-2 \notin S_{1}^{\prime \prime} \cup S_{2}^{\prime \prime}$. Apply $a$ so that $0 \in S_{1}^{\prime \prime \prime} \backslash S_{2}^{\prime \prime \prime}$. Then as above, apply $b^{2} d^{n-2}$ so that $n-1 \notin T_{1}^{\prime} \cup T_{2}^{\prime}$, while leaving $S_{1}^{\prime \prime \prime}$ and $S_{2}^{\prime \prime \prime}$ unchanged. This distinguishes the states for $\left(K_{m}^{\prime}\right)^{*} \backslash L_{n}^{*}$ and $\left(K_{m}^{\prime}\right)^{*} \oplus L_{n}^{*}$.

Therefore all $\left(2^{m-1}+2^{m-2}-1\right)\left(2^{n-1}+2^{n-2}-1\right)$ states of the form $(S, T)$ are distinguishable. It remains to distinguish $\left(\left\{s_{1}\right\},\left\{s_{2}\right\}\right)$ from the other states. As in Theorem $]^{1}\left(\left\{s_{1}\right\},\left\{s_{2}\right\}\right)$ is distinguished from all states except ( $\{0\},\{0\}$ ) by $a$. It is distinguishable from $(\{0\},\{0\})$ by the choice of final state of $\mathcal{D}_{1}$.

## 4 Products with Starred Arguments

### 4.1 The Language $K L^{*}$

The complexity of $K L^{*}$ was studied by Cui, Gao, Kari, and Yu [5]. They showed that $m\left(2^{n-1}+2^{n-2}\right)-2^{n-2}$ is a tight bound using the following witnesses over alphabet $\Sigma=\{a, b, c\}$ : For $K$, let $\mathcal{D}_{K}=\left(Q_{K}, \Sigma, \delta_{K}, q_{0},\{m-1\}\right)$, with $Q_{K}=\left\{q_{0}, \ldots, q_{m-1}\right\}, a:\left(q_{0}, \ldots, q_{m-1}\right), \delta_{K}\left(q_{i}, b\right)=q_{i+1}$ for $i=0, \ldots, m-3$, $\delta_{K}\left(q_{m-2}, b\right)=q_{0}, \delta_{K}\left(q_{m-1}, b\right)=q_{m-2}$, and $\delta_{K}\left(q_{i}, c\right)=q_{i+1}$ for $i=0, \ldots, m-3$, $\delta_{K}\left(q_{m-2}, c\right)=q_{0}, \delta_{K}\left(q_{m-1}, c\right)=q_{m-1}$. For $L$, let $\mathcal{D}_{L}=\left(Q_{L}, \Sigma, \delta_{L}, 0,\{n-1\}\right)$, with $Q_{L}=\{0, \ldots, n-1\}, a:(0, \ldots n-1), \delta_{L}(0, b)=0, \delta_{L}(i, b)=i+1$ for $i=1, \ldots, n-2, \delta(n-1, b)=1 ; c:\binom{n-1}{1}$. We prove that two permutationally equivalent dialects of $U_{n}(a, b, c)$ also meet the bound.

Theorem 3 ( $K L^{*}$ ). Let $K_{m}=T_{m}(a, b, c)$, and $L_{n}=T_{n}(b, a, c)$. For $m, n \geqslant 3$, the complexity of $K_{m} L_{n}^{*}$ is $m\left(2^{n-1}+2^{n-2}\right)-2^{n-2}$.

Proof. Let $\mathcal{D}_{1}=\left(Q_{1}, \Sigma, \delta_{1}, q_{0},\left\{q_{m-1}\right\}\right)$ with $Q_{1}=\left\{q_{0}, \ldots, q_{m-1}\right\}$ be the DFA of $K_{m}$, and let $\mathcal{D}_{2}=\left(Q_{2}, \Sigma, \delta_{2}, 0,\{n-1\}\right)$ with $Q_{2}=\{0, \ldots, n-1\}$ be the DFA of $L_{n}$. Let $\mathcal{N}_{2}$ be the NFA for $L_{n}^{*}$, and let $\mathcal{N}$ be the NFA for the product $K_{m} L_{n}^{*}$. Figure $\mathbb{Z}_{\text {shows our witnesses }} \mathcal{T}_{4}(a, b, c)$ and $\mathcal{T}_{5}(b, a, c)$ and the NFA $\mathcal{N}$ for $K L^{*}$. We perform the subset construction and minimization of $\mathcal{N}$ to obtain the DFA $\mathcal{P}$ for the product $K L^{*}$.


Fig. 4. Witness $\mathcal{N}$ for $\mathcal{T}_{4}(a, b, c)\left(\mathcal{T}_{5}(b, a, c)\right)^{*}$.

The states of $\mathcal{P}$ are subsets of $Q_{1} \cup Q_{2} \cup\{s\}$. Note that $q_{m-1}$ cannot appear in a state of $\mathcal{P}$ without $s$, and vice versa. Also, $n-1$ cannot appear without 0 , but 0 can appear without $n-1$. Each state of $\mathcal{D}$ must contain exactly one of $\left\{q_{0}\right\}, \ldots,\left\{q_{m-2}\right\}$ or $\left\{q_{m-1}, s\right\}$, and either a (possibly empty) subset of $Q_{2}$ not containing $n-1$, or subset of $Q_{2}$ containing both $n-1$ and 0 . Hence there are at most $m\left(2^{n-1}+2^{n-2}\right)$ reachable subsets; we now show that all these subsets can be reached.

Set $\left\{q_{0}\right\}$ is the initial state of $\mathcal{P}$, set $\left\{q_{i}\right\}$ for $i \leqslant m-2$ is reached by $a^{i}$, and $\left\{q_{m-1}, s\right\}$, by $a^{m-1}$.

Suppose all allowable states of the form $\left\{q_{m-1}, s\right\} \cup S,|S| \leqslant k, k \geqslant 0$, are reachable. Let $S \subseteq Q_{2},|S|=k+1$. If $1 \in S$ and $0 \notin S$, then we have $\left\{q_{m-1}, s\right\} \cup(S \backslash\{1\}) \xrightarrow{a}\left\{q_{0}\right\} \cup S$. If $0,1 \in S$, then $\left\{q_{m-1}, s\right\} \cup(S \backslash\{0\}) \xrightarrow{a}\left\{q_{0}\right\} \cup S$. If $0 \in S$ and $1 \notin S$, then $\left\{q_{m-1}, s\right\} \cup(S \backslash\{0\}) \xrightarrow{a c}\left\{q_{0}\right\} \cup S$. Therefore all states $\left\{q_{0}\right\} \cup S,|S|=k+1$, and either $0 \in S$ or $1 \in S$, are reachable. Every state $\left\{q_{0}\right\} \cup S$, where $n-1 \notin S$, is reachable by an even number of $b$ 's from a state containing either 0 or 1 . Every $S=\left\{0, i_{1}, \ldots, i_{k-1}, n-1\right\}$ is also reachable in this way (by mapping either 0 or 1 to $i_{1}$ ). So all states $\left\{q_{0}\right\} \cup S,|S|=k+1$, are reachable. By applying cyclic shifts $a^{i}$, all states $\left\{q_{i}\right\} \cup S, i<m-1$ and $\left\{q_{m-1}, s\right\} \cup S$ are reachable.

Any state of the form $\left\{q_{m-1}, s\right\} \cup T$, where $T \subset Q_{L} \backslash\{0, n-1\}$, is equivalent to $\left\{q_{m-1}, s, 0\right\} \cup T$, as they are both final and are mapped to the same state under any input. So the number of distinguishable states of $\mathcal{D}$ is at most $m\left(2^{n-1}+\right.$ $\left.2^{n-2}\right)-2^{n-2}$. We prove that there are precisely that many distinguishable states.

Consider two states of the form $\left\{q_{i}\right\} \cup S,\left\{q_{m-1}, s\right\} \cup T$, where $i<m-1$. These states are distinguished by $c b^{n-2}$. Any pair $\left\{q_{i}\right\} \cup S,\left\{q_{j}\right\} \cup T, i \neq j$ can by transformed into states of this form by applying a cyclic shift. Now consider $\left\{q_{i}\right\} \cup S,\left\{q_{i}\right\} \cup T, S \neq T, i<m-1$. There exists a cyclic shift $b^{k}$ which transforms the states so that $n-1 \in S \oplus T$, and this distinguishes the states.

Then the only remaining case is $\left\{q_{m-1}, s\right\} \cup S,\left\{q_{m-1}, s\right\} \cup T$, and $S \neq T$. As we stated earlier, if $S \oplus T=\{0\}$ then the states are indistinguishable. Otherwise, let $k \in S \oplus T, k>0$. Apply $b^{n-1-k}$ so that $n-1 \in S \oplus T$. Then applying $a$ to map $\left\{q_{m-1}, s\right\}$ to $\left\{q_{0}, 1\right\}$ distinguishes the states.

### 4.2 The Language $K^{*} L$

Cui, Gao, Kari and Yu 4 proved using quaternary witnesses that the complexity of $K^{*} L$ is $5 \cdot 2^{m+n-3}-2^{m-1}-2^{n}+1$. Let $\Sigma=\{a, b, c, d\}$. For $K$ they used $\mathcal{D}_{K}=\left(Q_{K}, \Sigma, \delta_{K}, q_{0},\{m-1\}\right)$, with $Q_{K}=\left\{q_{0}, \ldots, q_{m-1}\right\}, a:\left(q_{0}, \ldots, q_{m-1}\right)$, $\delta_{K}\left(q_{0}, b\right)=q_{0}, \delta_{K}\left(q_{i}, b\right)=i+1 \bmod m$ for $i=1, \ldots, m-1$, and $c, d: \mathbf{1}_{Q_{K}}$. For $L$, their witness was $\mathcal{D}_{L}=\left(Q_{L}, \Sigma, \delta_{L}, 0,\{n-1\}\right)$, with $Q_{L}=\{0, \ldots, n-$ $1\}, a, b: \mathbf{1}_{Q_{L}}, c:(0, \ldots n-1), d:\binom{Q_{L}}{0}$. We show here that two quaternary permutationally equivalent languages also work.

Theorem $4\left(K^{*} L\right)$. Let $K_{m}=U_{m}(a, b, c, d)$ and $L_{n}=U_{n}(d, c, b, a)$. For $m, n \geqslant$ 3 , the complexity of $K_{m}^{*} L_{n}$ is $5 \cdot 2^{m+n-3}-2^{m-1}-2^{n}+1$.


Fig. 5. NFA $\mathcal{N}$ for $\left(U_{4}(a, b, c, d)\right)^{*} U_{5}(d, c, b, a)$.

Proof. Let $\mathcal{D}_{1}=\left(Q_{1}, \Sigma, \delta_{1}, q_{0},\left\{q_{m-1}\right\}\right)$ with $Q_{1}=\left\{q_{0}, \ldots, q_{m-1}\right\}$ be the DFA of $K_{m}$, and let $\mathcal{D}_{2}=\left(Q_{2}, \Sigma, \delta_{2}, 0,\{n-1\}\right)$ with $Q_{2}=\{0, \ldots, n-1\}$ be the DFA of $L_{n}$. Let $\mathcal{N}_{1}$ be the NFA for $K_{m}^{*}$, and let $\mathcal{N}$ be the NFA for the product $K_{m}^{*} L_{n}$. We perform the subset construction and minimization of $\mathcal{N}$ to obtain the DFA $\mathcal{P}$ for the product $K^{*} L$. The construction is illustrated in Fig. [5]

Owing to the $\varepsilon$-transitions, the allowable states of the DFA are $\{s, 0\}$, all $\left(2^{m-1}-1\right)\left(2^{n}-1\right)$ subsets of the form $S \cup T$ where $\emptyset \subsetneq S \subseteq Q_{1}, q_{m-1} \notin S$, $\emptyset \subsetneq T \subseteq Q_{2}$, and all $\left(2^{m-2}-1\right)\left(2^{n-1}-1\right)$ subsets of the form $S \cup T$, where $q_{0}, q_{m-1} \in S \subseteq Q_{1}$ and $0 \in T \subseteq Q_{2}$. There are $5 \cdot 2^{m+n-3}-2^{m-1}-2^{n}+2$ such subsets and we will now show that they are all reachable.

The initial state of $\mathcal{P}$ is $\{s, 0\}$. It is known from [2] that all allowable subsets of $\mathcal{N}_{1}$ are reachable by words in $\{a, b\}^{*}$. These inputs all map 0 to itself, and hence all allowable states of the form $S \cup\{0\}$ are reachable.

If $q_{m-1} \notin S$ and $T=\left\{t_{1}, \ldots t_{k}\right\}$, then $S \cup\left\{0, t_{2}-t_{1}, \ldots, t_{k}-t_{1}\right\} \xrightarrow{d^{t_{1}}} S \cup T$. Let $T=\left\{0, t_{1}, \ldots, t_{k}\right\}, 0<t_{1}<\cdots<t_{k}$, and $S=\left\{q_{i_{1}}, \ldots, q_{i_{l}}\right\}, i_{1}<\cdots<i_{l}<$ $m-1$. Also, let $S^{\prime}=\left\{q_{i_{2}-i_{1}-1}, \ldots, q_{i_{l}-i_{1}-1}, q_{m-2}\right\}$ and $T^{\prime}=\left\{t_{1}, \ldots, t_{k}\right\}$. Then

$$
S^{\prime} \cup T^{\prime} \xrightarrow{a c^{2}}\left\{0, q_{i_{2}-i_{1}}, \ldots, q_{i_{l}-i_{1}}\right\} \cup T \xrightarrow{a^{i_{1}}} S \cup T .
$$

Moreover, $S \cup\left\{t_{0}, t_{1}+t_{0}, \ldots, t_{k}+t_{0}\right\}$ can be reached from $S \cup T$ by $d^{t_{0}}$. Combining these results shows that all allowable states $S \cup T$ with $q_{m-1} \notin S$ are reachable. Finally, if $S=\left\{q_{0}, q_{i_{1}}, \ldots, q_{i_{k}}, q_{m-1}\right\}$, and $0 \in T$, then $\left\{q_{i_{1}-1}, \ldots, q_{i_{l}-1}, q_{m-2}\right\} \cup$ $T \xrightarrow{a} S \cup T$. Therefore all allowable states are reachable.

For distinguishability, first consider states $S_{1} \cup T_{1}, S_{2} \cup T_{2}$. If $T_{1} \neq T_{2}$, then applying a cyclic shift $d^{k}$ transforms the states so that $n-1 \in T_{1} \oplus T_{2}$, distinguishing the states. If $S_{1} \neq S_{2}$, apply a cyclic shift $a^{k}$ so that $q_{m-1} \in$ $S_{1} \oplus S_{2}$. Then apply $b d$ so that $0 \in T_{1} \oplus T_{2}$, and the states are distinguishable by the previous case.

Finally, the initial state $\{s\} \cup\{0\}$ is indistinguishable from $\left\{q_{0}\right\} \cup\{0\}$, as any non-empty input transforms these two states into the same state. So then there are $5 \cdot 2^{m+n-3}-2^{m-1}-2^{n}+1$ distinguishable states.

### 4.3 The Language $K^{*} L^{*}$

The combined operation $K^{*} L^{*}$ appears not to have been studied before.
Proposition 3. The complexity of the operation $K_{m}^{*} L_{n}^{*}$ is at most $2^{m+n-1}-$ $2^{m-1}-3 \cdot 2^{n-2}+2$ for $m, n \geqslant 3$.

Proof. Let $\mathcal{D}_{1}=\left(Q_{1}, \Sigma, \delta_{1}, q_{0}, F_{1}\right)$ with $Q_{1}=\left\{q_{0}, \ldots, q_{m-1}\right\}$ be the DFA of $K_{m}$, and let $\mathcal{D}_{2}=\left(Q_{2}, \Sigma, \delta_{2}, 0, F_{2}\right)$ with $Q_{2}=\{0, \ldots, n-1\}$ be the DFA of $L_{n}$. Construct NFA's $\mathcal{N}_{1}$ and $\mathcal{N}_{2}$ accepting $K_{m}^{*}$ and $L_{n}^{*}$ by adding new initial states $s_{1}$ and $s_{2}$, which are also final. Let $\mathcal{N}$ be the NFA for $K_{m}^{*} L_{n}^{*}$, and let $\mathcal{P}$ be the DFA obtained by the subset construction and minimization of $\mathcal{N}$. These constructions are illustrated in Fig. 6.


Fig. 6. NFA $\mathcal{N}$ of $\left(U_{4}(a, b, c, d)\right)^{*}\left(U_{5}(d, c, b, a)\right)^{*}$.

The initial state of $\mathcal{P}$ is $\left\{s_{1}, s_{2}\right\}$. Note that any state $R$ of $\mathcal{P}$ containing $s_{2}$ but not 0 , is equivalent to $R \cup\{0\}$, since both states are final because of $s_{2}$, and $s_{2}$ and 0 have identical outgoing transitions. Hence we can ignore states like $R$ in our counting, and assume that every state containing $s_{2}$ also contains 0 . Due to the $\varepsilon$-transitions, the allowable states of the DFA are $\left\{s_{1}, s_{2}\right\}$, and all subsets of the form $S \cup T$, where $\emptyset \subsetneq S \subseteq Q_{1}, \emptyset \subsetneq T \subseteq\left\{s_{2}\right\} \cup Q_{2}$, and fall into one of the following cases:
$-S \cap F_{1}=\emptyset, T \cap F_{2}=\emptyset ;$
$-S \cap F_{1}=\emptyset, T$ contains at least one state of $F_{2}$ and 0 ;

- $S$ contains at least one state of $F_{1}$ and $s_{2}, 0 \in T$.

One verifies that the possible number of states is greatest when there is only one final state, say $q_{m-1}$, in $F_{1}$ and only one final state, say $n-1$, in $F_{2}$. Hence we have the cases:

$$
\begin{aligned}
& -q_{m-1} \notin S, n-1 \notin T:\left(2^{m-1}-1\right)\left(2^{n-1}-1\right) \text { states; } \\
& -q_{m-1} \notin S, 0, n-1 \in T:\left(2^{m-1}-1\right) 2^{n-2} \text { states; } \\
& -q_{0}, q_{m-1} \in S, s_{2}, 0 \in T: 2^{m+n-3} \text { states. }
\end{aligned}
$$

Therefore there are a total of $2^{m+n-1}-2^{m-1}-3 \cdot 2^{n-2}+2$ allowable states. Hence the complexity of $K_{m}^{*} L_{n}^{*}$ is at most $2^{m+n-1}-2^{m-1}-3 \cdot 2^{n-2}+2$.

Theorem $5\left(K^{*} L^{*}\right)$. Let $K_{m}=U_{m}(a, b, c, d)$ and $L_{n}=U_{n}(d, c, b, a)$. For $m, n \geqslant 3$, the complexity of $K_{m}^{*} L_{n}^{*}$ is $2^{m+n-1}-2^{m-1}-3 \cdot 2^{n-2}+2$.

Proof. Let the various automata be defined as in the proof of Proposition 3, but this time with $K_{m}=U_{m}(a, b, c, d)$ and $L_{n}=U_{n}(d, c, b, a)$. The reachability of all of the states of $\mathcal{P}$ follows the proof in Theorem 4 for all states $S \cup T$ where $n-1 \notin T$. Let $T=\left\{0, t_{1}, \ldots, t_{k}, n-1\right\}$. If $q_{m-1} \notin S$, then $S \cup\left\{0, t_{2}-\right.$ $\left.t_{1}, \ldots, t_{k}-t_{1}, n-1-t_{1}\right\} \xrightarrow{d^{t_{1}}} S \cup T$. If $q_{m-1} \in S$, say $S=\left\{q_{0}, q_{i_{1}}, \ldots, q_{i_{l}}, q_{m-1}\right\}$, then $\left\{q_{i_{1}}-1, \ldots, q_{i_{l}-1}, q_{m-2}\right\} \cup T \xrightarrow{a} S \cup T$. Therefore all allowable states are reachable.

For distinguishability, first consider states $S_{1} \cup T_{1}, S_{2} \cup T_{2}$, where $S_{1}, S_{2} \subseteq Q_{1}$ and $T_{1}, T_{2} \subseteq\left\{s_{2}\right\} \cup Q_{2}$. The set of final states of the NFA is $\left\{s_{2}, n-1\right\}$; however, any set containing $s_{1}$ or $q_{m-1}$ also contains $s_{2}$, and hence is a final state of $\mathcal{P}$. Note that applying $c$ always results in a state $S \cup T$, where $q_{m-1}, s_{2} \notin S$, and applying $b$ causes $n-1 \notin T$. If $T_{1} \neq T_{2}$, then applying a cyclic shift $d^{k}$ transforms the states so that $n-1 \in T_{1} \oplus T_{2}$, and then applying $c$ distinguishing the states. If $S_{1} \neq S_{2}$, apply a cyclic shift $a^{k}$ so that $q_{m-1} \in S_{1} \oplus S_{2}$, then apply $b$ to distinguish the states.

Finally, consider the initial state $\left\{s_{1}, s_{2}\right\}$, and any state $R$ not contain $s_{1}$, since the initial state is the only one containing $s_{1}$. There are three cases:

1. $q_{0} \notin R$ : Applying $a$, from $\left\{s_{1}, s_{2}\right\}$ we reach $\left\{q_{1}, 0\right\}$, and from $R$ we reach $R^{\prime}$, where $q_{1} \notin R^{\prime}$. By the argument in the second paragraph of the proof, $\left\{s_{1}, s_{2}\right\}$ is distinguished from $R$.
2. $q_{0} \in R$, and $R \neq\left\{q_{0}, 0\right\}$ : If $a d$ is applied, then $\left\{s_{1}, s_{2}\right\}$ goes to $\left\{q_{1}, 1\right\}$, and $R$ goes to $R^{\prime}$ such that there exists $x \in R^{\prime}, x \notin\left\{q_{1}, 1\right\}$. Then these two states are distinguishable by the previous argument.
3. $R=\left\{q_{0}, 0\right\}$ : State $\left\{s_{1}, s_{2}\right\}$ is final, but $\left\{q_{0}, 0\right\}$ is not.

Hence all the allowable states are distinguishable and the theorem holds.

## 5 Stars of Binary Operations

### 5.1 The Language ( $K L$ )*

In 2008 Gao, K. Salomaa, and Yu [7] proved that $2^{m+n-1}+2^{m+n-4}-\left(2^{m-1}+\right.$ $\left.2^{n-1}-m-1\right)$ is a tight upper bound for $(K L)^{*}$. They used the following DFA's over alphabet $\Sigma=\{a, b, c, d\}$ : For $K$, let $\mathcal{D}_{K}=\left(Q_{1}, \Sigma, \delta_{K}, q_{0},\left\{q_{m-1}\right\}\right)$ with $a:\left(q_{0}, \ldots, q_{m-1}\right), b: \mathbf{1}_{Q_{K}}, c$ defined by $\delta_{K}\left(q_{0}, c\right)=\delta_{K}\left(q_{m-1}, c\right)=q_{0}, \delta_{K}\left(q_{i}, c\right)=$ $q_{i+1}$, for $i=1, \ldots, m-2$, and $d: \mathbf{1}_{Q_{K}}$. For $L$, let $\mathcal{D}_{L}=\left(Q_{L}, \Sigma, \delta_{L}, 0,\{n-1\}\right)$ with $a: \mathbf{1}_{Q_{L}}, b:(0, \ldots, n-1), c: \mathbf{1}_{Q_{L}}$, and $d$ defined by $\delta_{L}(0, d)=\delta_{L}(n-1, d)=0$, $\delta_{L}(i, d)=i+1$, for $i=1, \ldots, n-2$. We show that two permutationally equivalent dialects $W_{m}(a, b, c, d)$ and $W_{n}(d, c, b, a)$ of $U_{n}(a, b, c, d)$ also meet the bound.

Theorem $6\left((K L)^{*}\right)$. Let $K_{m}=W_{m}(a, b, c, d)$ and $L_{n}=W_{n}(d, c, b, a)$. For $m, n \geqslant 3$, the complexity of $\left(K_{m} L_{n}\right)^{*}$ is $2^{m+n-1}+2^{m+n-4}-\left(2^{m-1}+2^{n-1}-m-1\right)$.


Fig. 7. NFA for $\left(\left(W_{4}(a, b, c, d) W_{5}(d, c, b, a)\right)^{*}\right.$.

Proof. Let $\mathcal{D}_{1}=\left(Q_{1}, \Sigma, \delta_{1}, q_{0},\left\{q_{m-1}\right\}\right)$ with $Q_{1}=\left\{q_{0}, \ldots, q_{m-1}\right\}$ be the DFA of $K_{m}$, and let $\mathcal{D}_{2}=\left(Q_{2}, \Sigma, \delta_{2}, 0,\{n-1\}\right)$ with $Q_{2}=\{0, \ldots, n-1\}$ be the DFA of $L_{n}$. Let $\mathcal{N}$ be the NFA for $(K L)^{*}$. This NFA is shown in Fig. 7 for $m=4$ and $n=5$. Let $\mathcal{D}$ be the DFA obtained from $\mathcal{N}$ by the subset construction and minimization.

The states of $\mathcal{D}$ are the initial state $\{s\}$ and states of the form $S \cup T$ where $\emptyset \subsetneq S \subseteq Q_{1}$ and $T \subseteq Q_{2}$. Because of the $\varepsilon$-transitions, the allowable states $S \cup T$ must have either $q_{m-1} \notin S$ or $q_{m-1} \in S$, and $0 \in T$. Moreover, if $|S|>1$, then $T \neq \emptyset$, as at least one $\varepsilon$-transition from $n-1$ to $q_{0}$ must have been used. The number of allowable states is counted as follows:

1. First, we have the initial state $\{s\}$.
2. If $T=\emptyset$, then $|S|=1$, and $q_{m-1} \notin S$. There are $m-1$ such states.
3. If $T \neq \emptyset$, then $|S| \geqslant 1$.
(a) $n-1 \notin T$ : If $q_{m-1} \notin S$, then there are $\left(2^{m-1}-1\right)\left(2^{n-1}-1\right)$ such states. Otherwise, $q_{m-1} \in S$ and $0 \in T$, and there are $2^{m+n-3}$ such states.
(b) $n-1 \in T$ : Then $q_{0} \in S$. If $q_{m-1} \notin S$, there are $2^{m+n-3}$ such states. Otherwise, $q_{m-1} \in S$ and $0 \in T$, and there are $2^{m+n-4}$ such states.

Altogether we have $2^{m+n-1}+2^{m+n-4}-\left(2^{m-1}+2^{n-1}-m-1\right)$ states. We will now show they are all reachable.

The initial state is $\{s\}$. We have $\{s\} \xrightarrow{b}\left\{q_{0}\right\} \xrightarrow{a^{i}}\left\{q_{i}\right\}$ for $i<m-1$.
For $i<m-1$ and $T=\left\{t_{1}, \ldots, t_{k}\right\} \subseteq Q_{2} \backslash\{n-1\}$ with $t_{1}<\cdots<t_{k}$, the state $\left\{q_{i}\right\} \cup T$ is reachable by $\left\{q_{i}\right\} \cup\left\{t_{2}-t_{1}, \ldots, t_{k}-t_{1}\right\} \xrightarrow{a^{m} d^{t_{1}}}\left\{q_{i}\right\} \cup T$. Suppose $n-1 \in T$, say $T=\left\{t_{1}, \ldots, t_{k}, n-1\right\}$. If $T \neq Q_{2}$, then the state $\left\{q_{0}\right\} \cup T$ is reachable by a applying a cyclic shift $d^{l}$ to some $\left\{q_{0}\right\} \cup T^{\prime}$, where $n-1 \notin T^{\prime}$. Moreover, $\left\{q_{m-2}\right\} \cup\left(Q_{2} \backslash\{n-1\}\right) \xrightarrow{d a}\left\{q_{0}, q_{1}, q_{m-1}\right\} \cup Q_{2} \xrightarrow{c a c}\left\{q_{0}\right\} \cup Q_{2}$. Finally, if $0 \in T$ then $\left\{q_{m-2}\right\} \cup T \xrightarrow{a}\left\{q_{m-1}\right\} \cup T$. So all allowable states of the form $S \cup T,|S|=1$ are reachable.

Let $S=\left\{q_{i_{1}}, \ldots, q_{i_{k}}\right\}, 0<i_{1}<\cdots<i_{k}$. Since $n-1 \notin T$, we have $\left\{q_{i_{2}-i_{1}}, \ldots, q_{i_{k}-i_{1}}\right\} \cup T \xrightarrow{d^{n} a^{i_{1}}} S \cup T$. Now suppose $S=\left\{q_{0}, q_{i_{2}}, \ldots, q_{i_{k}}\right\}$. If $n-1 \in$ $T$, then $\left\{q_{0}, q_{i_{3}-i_{2}}, \ldots, q_{i_{k}-i_{2}}\right\} \cup T \xrightarrow{a\left(a c^{2}\right)^{i_{2}-1}} S \cup T$. If $n-1 \notin T$ and $q_{m-1} \in S$, then $T=\left\{0, t_{2}, \ldots, t_{l}\right\}$ and $t_{l}<n-1$. Let $T^{\prime}=\left\{0, t_{2}-1, \ldots, t_{l}-1, n-1\right\}$.

Then $S \cup T^{\prime}$ is reachable, and $S \cup T^{\prime} \xrightarrow{d} S \cup T \cup\{1\}$; if $1 \notin T$, apply $b^{2}$ to get $S \cup T$.

Finally, suppose $q_{0} \in S, q_{m-1} \notin S$, and $n-1 \notin T$. Suppose $T=\left\{t_{1}, \ldots, t_{l}\right\}$, $t_{1}<\cdots<t_{l}$, and let $T^{\prime \prime}=\left\{t_{2}-t_{1}-1, \ldots, t_{l}-t_{1}-1, n-1\right\}$. Since $q_{0} \in S$ and $n-1 \in T$, state $S \cup T^{\prime \prime}$ is reachable. Then we reach $S \cup T$ from $S \cup T^{\prime \prime}$ by applying $d^{t_{1}+1}$.

Therefore all the allowable states are reachable.
We now show all states are disintinguishable. Let $S_{1} \cup T_{1}, S_{2} \cup T_{2}$ be two distinct states. If $T_{1} \neq T_{2}$, then the states are distinguishable by a cyclic shift $d^{k}$. If $S_{1} \neq S_{2}$, without loss of generality we may assume $q_{m-1} \in S_{1} \oplus S_{2}$. Then applying $b^{2} d^{n-1}$ results in states $S_{1}^{\prime} \cup T_{1}^{\prime}, S_{2}^{\prime} \cup T_{2}^{\prime}$, where $0 \in T_{1}^{\prime} \oplus T_{2}^{\prime}$, so the states are distinguishable. Finally, the initial state $\{s\}$ is distinguished from every state other than $\left\{q_{0}\right\}$ by $a$; it is distinguishable from $\left\{q_{0}\right\}$ because it is final.

### 5.2 The Languages $(K \cup L)^{*}$

In 2007 A. Salomaa, K. Salomaa, and S. Yu 10 showed that the complexity of $(K \cup L)^{*}$ is $2^{m+n-1}-\left(2^{m-1}+2^{n-1}-1\right)$ with ternary witnesses. Jirásková and Okhotin [9] used binary witnesses: For $K$, let $\mathcal{D}_{K}=\left(Q_{1}, \Sigma, \delta_{K}, 0,\{0\}\right)$ with $a:(0, \ldots, m-1)$, and $b$ defined by $\delta_{K}(i, b)=i+1$, for $i=0, \ldots, m-2$, $\delta_{K}(m-1, b)=1$. For $L$, let $\mathcal{D}_{L}=\left(Q_{L}, \Sigma, \delta_{L}, 0,\{0\}\right)$ with $a:\binom{0}{1}$ and $b:$ $(0, \ldots, n-1)$. Permutationally equivalent binary dialects of $U_{n}(a, b, c)$ can also be used. Let $\mathcal{S}_{n}=\mathcal{S}_{n}(a, b)=\left(Q, \Sigma, \delta_{S}, 0,\{0\}\right)$, where $a:(0, \ldots, n-1)$, and $b:\binom{0}{1}$. The following theorem was proved in [2]:

Theorem $7\left(\left(K_{m} \cup L_{n}\right)^{*}\right)$. For $m, n \geqslant 3$, the complexity of $\left(S_{m}(a, b) \cup S_{n}(b, a)\right)^{*}$ is $2^{m+n-1}-\left(2^{m-1}+2^{n-1}-1\right)$.

### 5.3 The Language $(K \cap L)^{*}$

It was also proved in [9] that the complexity of $(K \cap L)^{*}$ is $2^{m n-1}+2^{m n-2}$, which is the composition of the complexities of intersection and star. Their witnesses $K$ and $L$ were over an alphabet of six letters, $\Sigma=\{a, b, c, d, e, f\}$ : For $K$, let $\mathcal{D}_{K}=\left(Q_{K}, \Sigma, \delta_{K}, 0,\{m-1\}\right)$, with $Q_{K}=\{0, \ldots, m-1\}$. For $L$, let $\mathcal{D}_{L}=$ $\left(Q_{L}, \Sigma, \delta_{L}, 0,\{n-1\}\right)$, with $Q_{L}=\{0, \ldots, n-1\}$. The transitions were as follows:

| $\mathcal{D}_{K}$ | $\mathcal{D}_{L}$ |
| :---: | :---: |
| $a:(0, \ldots, m-1)$ | $a:(0, \ldots, n-1)$ |
| $b: 1_{Q_{K}}$ | $b:(0, \ldots, n-1)$ |
| $c:(1, \ldots, m-1)$ | $c: 1_{Q_{L}}$ |
| $d: 1_{Q_{K}}$ | $d:(1, \ldots, n-1)$ |
| $e:\binom{1}{0}$ | $e: 1_{Q_{L}}$ |
| $f: 1_{Q_{K}}$ | $f:\binom{1}{0}$ |

We conjecture that quinary witnesses can also be used. Let $\Sigma=\{a, b, c, d, e\}$ and $\mathcal{U}_{n}(a, b, c, d, e)=\left(Q_{K}, \Sigma, \delta_{\mathcal{U}}, 0,\{n-1\}\right)$, where $Q_{K}=(0, \ldots, n-1\}, a$ :
$(0, \ldots, n-1), b:(0,1), c:\binom{n-1}{0}, d=1_{Q_{K}}$, and $e:(1, \ldots n-1)$. Let $\pi$ be the permutation that sends $\{a, b, c, d, e\}$ to $\{e, c, b, a, d\}$, let $\mathcal{D}_{1}=\mathcal{U}_{n}(a, b, c, d, e)$, and $\mathcal{D}_{2}=\mathcal{U}_{n}(e, c, b, a, d)$. The transitions in $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ are:

| $\mathcal{D}_{1}$ | $\mathcal{D}_{2}$ |
| :---: | :---: |
| $a:(0, \ldots, m-1)$ | $\pi(a): 1_{Q_{2}}$ |
| $b:(0,1)$ | $\pi(b):\binom{n-1}{0}$ |
| $c:\binom{m-1}{0}$ | $\pi(c):(0,1)$ |
| $d: 1_{Q_{1}}$ | $\pi(d):(1, \ldots, n-1)$ |
| $e:(1, \ldots, m-1)$ | $\pi(e):(0, \ldots, n-1)$ |

Note that $U_{n}(a, b, c, d, e)$ is an extension of $U_{n}(a, b, c, d)$ to 5 letters.
Conjecture $1\left(\left(K_{m} \cap L_{n}\right)^{*}\right)$. Let $K_{m}=U_{m}(a, b, c, d, e)$ and $L_{n}=U_{n}(e, c, b, a, d)$. Then the complexity of $\left(K_{m} \cap L_{n}\right)^{*}$ is $2^{m n-1}+2^{m n-2}$ for $m, n \geqslant 3$.

This has been verified for $m=3$ and $n=3,4,5,6$ and for $m=4$ and $n=4,5$.

### 5.4 The Language $(K \backslash L)^{*}$

Theorem $8\left(\left(K_{m} \backslash L_{n}\right)^{*}\right)$. The complexity of the operation $\left(K_{m} \backslash L_{n}\right)^{*}$ is $2^{m n-1}+2^{m n-2}$ for $m, n \geqslant 3$, and it is met by the witnesses $K_{m}$ and $\overline{L_{n}}$, where $K_{m}$ and $L_{n}$ are the witnesses of Jirásková and Okhotin for intersection.

Proof. This follows since $(K \backslash \bar{L})^{*}=(K \cap L)^{*}$.
If Conjecture 1 holds, then we also have
Conjecture $2\left(\left(K_{m} \backslash L_{n}\right)^{*}\right)$. Let $K_{m}=U_{m}(a, b, c, d, e)$ and $L_{n}=\overline{U_{n}(e, c, b, a, d)}$. Then the complexity of $\left(K_{m} \cap L_{n}\right)^{*}$ is $2^{m n-1}+2^{m n-2}$ for $m, n \geqslant 3$.

### 5.5 The Language $(K \oplus L)^{*}$

The complexity of this combined operation remains open.

## 6 Conclusions

We have proved that the universal witnesses $U_{n}(a, b, c)$ and $U_{n}(a, b, c, d)$, along with their permutational equivalents $U_{n}(b, a, c)$ and $U_{n}(d, c, b, a)$, and dialects $U_{\{0\}, n}(a, b, c), T_{n}(a, b, c), T_{n}(b, a, c), W_{n}(a, b, c, d), W_{\{0\}, n}(a, b, c, d), W_{n}(d, c, b, a)$ suffice to act as witnesses for all state complexity bounds involving binary boolean operations and product combined with star. In the case of one or two starred arguments, we have shown that it is efficient to consider all four boolean operations together. The use of universal witnesses and their dialects simplified several proofs, and allowed us to utilize the similarities in the witnesses.

Acknowledgment We thank Baiyu Li for careful proofreading and correcting several flaws in an earlier version of the paper.

## References

1. Brzozowski, J.: Quotient complexity of regular languages. J. Autom. Lang. Comb. 15(1/2) (2010) 71-89
2. Brzozowski, J.: In search of the most complex regular languages. In Moreira, N., Reis, R., eds.: Proceedings of the 17th International Conference on Implementation and Application of Automata (CIAA). Volume 7381 of LNCS, Springer (2012) 5-24
3. Brzozowski, J., Liu, D.: Universal witnesses for state complexity of basic operations combined with reversal. http://arxiv.org/abs/1207.0535 (July 2012)
4. Cui, B., Gao, Y., Kari, L., Yu, S.: State complexity of combined operations with two basic operations. Theoret. Comput. Sci. 437 (2012) 82-102
5. Cui, B., Gao, Y., Kari, L., Yu, S.: State complexity of two combined operations: catenation-star and catenation-reversal. Int. J. Found. Comput. Sc. 23(1) (2012) 51-66
6. Gao, Y., Kari, L., Yu, S.: State complexity of union and intersection of star on $k$ regular languages. Theoret. Comput. Sci. 429 (2012) 98-107
7. Gao, Y., Salomaa, K., Yu, S.: The state complexity of two combined operations: star of catenation and star of reversal. Fund. Inform. 83(1-2) (2008) 75-89
8. Gao, Y., Yu, S.: State complexity of combined operations with union, intersection, star, and reversal. Fund. Inform. 116 (2012) 1-14
9. Jirásková, G., Okhotin, A.: On the state complexity of star of union and star of intersection. Fund. Inform. 109 (2011) 1-18
10. Salomaa, A., Salomaa, K., Yu, S.: State complexity of combined operations. Theoret. Comput. Sci. 383 (2007) 140-152
11. Yu, S.: State complexity of regular languages. J. Autom. Lang. Comb. 6 (2001) 221-234

[^0]:    * This work was supported by the Natural Sciences and Engineering Research Council of Canada under grant No. OGP0000871.

