# Model Counting for Formulas of Bounded Clique-Width\*

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Abstract. We show that #SAT is polynomial-time tractable for classes of CNF formulas whose incidence graphs have bounded symmetric clique-width (or bounded clique-width, or bounded rank-width). This result strictly generalizes polynomial-time tractability results for classes of formulas with signed incidence graphs of bounded clique-width and classes of formulas with incidence graphs of bounded modular treewidth, which were the most general results of this kind known so far.

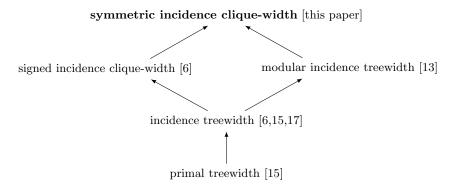
## 1 Introduction

Propositional model counting (#SAT) is the problem of computing the number of satisfying truth assignments for a given CNF formula. It is a well-studied problem with applications in Artificial Intelligence, such as probabilistic inference [1,16]. It is also a notoriously hard problem: #SAT is #P-complete in general [18] and remains #P-hard even for monotone 2CNF formulas and Horn 2CNF formulas [14]. It is NP-hard to approximate the number of satisfying truth assignments of a formula with n variables to within  $2^{n^{1-\varepsilon}}$  for any  $\varepsilon > 0$ . As in the exact case, this hardness result even holds for monotone 2CNF formulas and Horn 2CNF formulas [14]. While these syntactic restrictions do not make the problem easier, #SAT becomes tractable under certain structural restrictions [6,8,9,11,12,13,15,17]. Structural restriction are obtained by bounding parameters of (hyper)graphs associated with formulas. We extend this line of research and study #SAT for classes of formulas whose incidence graphs (that is, the bipartite graph whose vertex classes consist of variables and clauses, with variables adjacent to clauses they occur in) have bounded symmetric clique-width [4]. Symmetric clique-width is a parameter that is closely related to clique-width, rank-width, and Boolean-width: a class of graphs has bounded symmetric clique-width iff it has bounded clique-width iff it has bounded rank-width iff it has bounded Boolean-width. For a graph class  $\mathcal{C}$ , let  $\#SAT(\mathcal{C})$  be the restriction of #SAT to instances F with incidence graph  $I(F) \in \mathcal{C}$ . We prove:

**Theorem 1.** #SAT(C) is polynomial-time tractable for any graph class C of bounded symmetric clique-width.

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This result generalizes polynomial-time tractability results for classes of formulas with signed incidence graphs of bounded clique-width [6] and classes of formulas with incidence graphs of bounded modular treewidth [13]. The situation is illustrated in Figure 1 (for a survey of results for width-based parameters, see [12,13]). Our result is obtained through a combination of dynamic program-



**Fig. 1.** A hierarchy of structural parameters. An arc from a parameter p to a parameter q reads as "for any class of formulas, q is bounded whenever p is bounded." Bold type is used to indicate parameters that render #SAT polynomial-time tractable when bounded by a constant.

ming on a decomposition tree with the representation of truth assignments by projections (i.e., sets of clauses satisfied by these assignments). This extends the techniques used to prove polynomial-tractability of #SAT for classes of formulas with incidence graphs of bounded modular treewidth [13]; there, partial assignments are partitioned into equivalence classes by an equivalence relation roughly defined as follows: two assignments are equivalent whenever they satisfy the same set of clauses of a certain formula induced by a subtree of the decomposition. To make bottom-up dynamic programming work, it is enough to record the number of assignments in each equivalence class. This approach does not carry over to the case of bounded symmetric clique-width for principal reasons: the number of equivalence classes of such a relation can be exponential in the size of the (sub)formula.

To deal with this, our algorithm uses the technique of taking into account an "expectation from the outside" [2,7,8]. The underlying idea is that the information one has to record for any particular partial solution can be reduced significantly if one includes an "expectation" about what this partial solution will be combined with to form a complete solution. This trick allows us to bound the number of records required for dynamic programming by a polynomial in the number of clauses of the input formula.

For all parameters considered in Figure 1, propositional model counting is polynomial-time tractable if the parameter is bounded by a constant, but some of them even admit so-called FPT algorithms. The runtime of an FPT algorithm is bounded by a function of the form f(k)p(l), where f is an arbitrary computable function and p is a polynomial with order independent of the parameter k. As we will see, the order of the polynomial bounding the runtime in Theorem 1 is dependent on the parameter. One may wonder whether this can be avoided, that is, whether the problem admits an FPT algorithm. The following result shows that this is not possible, subject to an assumption from parameterized complexity.

**Theorem 2** ([12]). SAT, parameterized by the symmetric clique-width of the incidence graph of the input formula, is W[1]-hard.

To be precise, the result proven in [12] is stated in terms of clique-width. However, since the clique-width of a graph is at most twice its symmetric clique-width (see [4]), the result carries over to symmetric clique-width.

## 2 Preliminaries

Let  $f: X \to Y$  be a function and  $X' \subseteq X$ . We let  $f(X') = \{f(x) \in Y : x \in X'\}$ . Let  $X^*$  and  $Y^*$  be sets, and let  $g: X^* \to Y^*$  be a function with g(x) = f(x) for all  $x \in X \cap X^*$ . Then the function  $f \cup g: X \cup X^* \to Y \cup Y^*$  is defined as  $(f \cup g)(x) = f(x)$  if  $x \in X$  and  $(f \cup g)(x) = g(x)$  if  $x \in X^* \setminus X$ .

Graphs. The graphs considered in this paper are loopless, simple, and undirected. If G is a graph and v is a vertex of G, we let N(v) denote the set of all neighbors of v in G. For a tree T we write L(T) to denote the set of leaves of T. Let C be a class of graphs and let f be a mapping (invariant under isomorphisms) that associates each graph G with a non-negative real number. We say C has bounded f if there is a C such that  $f(G) \leq C$  for every  $G \in C$ .

Formulas. We assume an infinite supply of propositional variables. A literal is a variable x or a negated variable  $\overline{x}$ ; we put  $var(x) = var(\overline{x}) = x$ ; if  $y = \overline{x}$  is a literal, then we write  $\overline{y} = x$ . For a set S of literals we write  $\overline{S} = \{\overline{x} : x \in S\}$ ; S is tautological if  $S \cap \overline{S} \neq \emptyset$ . A clause is a finite non-tautological set of literals. A finite set of clauses is a CNF formula (or formula, for short). The length of a formula F is given by  $\sum_{C \in F} |C|$ . A variable x occurs in a clause C if  $x \in C \cup \overline{C}$ . We let var(C) denote the set of variables that occur in C. A variable x occurs in a formula F if it occurs in at least one of its clauses, and we let  $var(F) = \bigcup_{C \in F} var(C)$ . If F is a formula and X a set of variables, we let  $F|_X = \{C \in F : X \subseteq var(C)\}$ . The incidence graph of a formula F is the bipartite graph I(F) with vertex set  $var(F) \cup F$  and edge set  $\{Cx : C \in F \text{ and } x \in var(C)\}$ .

Let F be a formula. A truth assignment is a mapping  $\tau: X \to \{0,1\}$  defined on some set of variables  $X \subseteq var(F)$ . We call  $\tau$  total if X = var(F) and partial

otherwise. For  $x \in X$ , we define  $\tau(\overline{x}) = 1 - \tau(x)$ . A truth assignment  $\tau$  satisfies a clause C if C contains some literal  $\ell$  with  $\tau(\ell) = 1$ . If  $\tau$  satisfies all clauses of F, then  $\tau$  satisfies F; in that case we call F satisfiable. The Satisfiability (SAT) problem is that of testing whether a given formula is satisfiable. The propositional model counting (#SAT) problem is a generalization of SAT that asks for the number of satisfying total truth assignments of a given formula. For a graph class C, we let #SAT(C) be the restriction of #SAT to instances F with  $I(F) \in C$ .

Decomposition Trees. We review decomposition trees following the presentation in [3]. Let G = (V, E) be a graph. A decomposition tree for G is a pair  $(T, \delta)$ , where T is a rooted binary tree and  $\delta : L(T) \to V$  is a bijection. For a subset  $X \subseteq V$  let  $\overline{X} = V \setminus X$ . We associate every edge  $e \in E(T)$  with a bipartition  $P_e$  of V obtained as follows. If  $T_1$  and  $T_2$  are the components obtained by removing e from T, we let  $P_e = (L(T_1), L(T_2))$ . Note that  $L(T_2) = \overline{X}$  for  $X = L(T_1)$ . A function  $f : 2^V \to \mathbb{R}$  is symmetric if  $f(X) = f(\overline{X})$  for all  $X \subseteq V$ . Let  $f : 2^V \to \mathbb{R}$  be a symmetric function. The f-width of  $(T, \delta)$  is the maximum of  $f(X) = f(\overline{X})$  taken over the bipartitions  $P_e = (X, \overline{X})$  for all  $e \in E(T)$ . The f-width of G is the minimum of the f-widths of the decomposition trees of G.

Let A(G) stand for the adjacency matrix of G, that is, the  $V \times V$  matrix  $A(G) = (a_{vw})_{v \in V, w \in V}$  such that  $a_{vw} = 1$  if  $vw \in E$  and  $a_{vw} = 0$  otherwise. For  $X, Y \subseteq V$ , let A(G)[X, Y] denote the  $X \times Y$  submatrix  $(a_{vw})_{v \in X, w \in Y}$ . The cut-rank function  $\rho_G : 2^V \to \mathbb{R}$  of G is defined as

$$\rho_G(X) = rank(A(G)[X, V \setminus X]),$$

where rank is the rank function of matrices over  $\mathbb{Z}_2$ . The row and column ranks of any matrix are equivalent, so this function is symmetric. The rank-width of a decomposition tree  $(T, \delta)$  of G, denoted  $rankw(T, \delta)$ , is the  $\rho_G$ -width of  $(T, \delta)$ , and the rank-width of G, denoted rankw(G), is the  $\rho_G$ -width of G.

Let X be a proper nonempty subset of V. We define an equivalence relation  $\equiv_X$  on X as

$$x \equiv_X y$$
 iff, for every  $z \in V \setminus X$ ,  $xz \in E \Leftrightarrow yz \in E$ .

The index of X in G is the cardinality of  $X/\equiv_X$ , that is, the number of equivalence classes of  $\equiv_X$ . We let  $index_G: 2^V \to \mathbb{R}$  be the function that maps each proper nonempty subset X of V to its index in G. We now define the function  $\iota_G: 2^V \to \mathbb{R}$  as

$$\iota_G(X) = \max(index_G(X), index_G(V \setminus X)).$$

This function is trivially symmetric. The *index* of a decomposition tree  $(T, \delta)$  of G, denoted  $index(T, \delta)$ , is the  $\iota_G$ -width of  $(T, \delta)$ . The *symmetric clique-width* [4] of G, denoted scw(G), is the  $\iota_G$ -width of G.

Symmetric clique-width and rank-width are closely related graph parameters. In fact, the index of a decomposition tree can be bounded in terms of its rank-width.

**Lemma 3.** For every graph G and decomposition tree  $(T, \delta)$  of G, rankw $(T, \delta) \leq index(T, \delta) \leq 2^{rankw(T, \delta)}$ .

Proof. Let G = (V, E) be a graph and X be a nonempty proper subset of V. For every pair of vertices  $x, y \in X$  the rows of  $A(G)[X, V \setminus X]$  with indices x and y are identical if and only if  $x \equiv_X y$ . So  $index_G(X)$  is precisely the number of distinct rows of  $A(G)[X, V \setminus X]$ , which is an upper bound on the rank of  $A(G)[X, V \setminus X]$  over  $\mathbb{Z}_2$ . Symmetrically,  $index_G(V \setminus X)$  is the number of distinct columns of  $A(G)[X, V \setminus X]$ , which is also an upper bound on the rank. So  $\rho_G(X) \leq \iota_G(X)$ , which proves the left inequality. The rank of  $A(G)[X, V \setminus X]$  is the cardinality of a basis for the matrix's row (column) space. That is, each of its row (column) vectors can be represented as a linear combination of  $\rho_G(X)$  row (column) vectors. Over  $\mathbb{Z}_2$ , any linear combination can be obtained using only 0 and 1 as coefficients. Accordingly, there can be at most  $2^{\rho(X)}$  distinct rows (columns) in  $A(G)[X, V \setminus X]$ . So  $\iota_G(X) \leq 2^{\rho_G(X)}$ , and the right inequality follows.

Corollary 4. For every graph G,  $rankw(G) \leq scw(G) \leq 2^{rankw(G)}$ .

Runtime bounds for the dynamic programming algorithm presented below are more naturally stated in terms the index of the underlying decomposition tree than in terms of its rank-width. However, to the best of our knowledge, there is no polynomial-time algorithm for computing decomposition trees of minimum index directly – instead, we will use the following result to compute decomposition trees of minimum rank-width.

**Theorem 5** ([5]). Let  $k \in \mathbb{N}$  be a constant and  $n \geq 2$ . For an n-vertex graph G, we can output a decomposition tree of rank-width at most k or confirm that the rank-width of G is larger than k in time  $O(n^3)$ .

Projections. Let F be a set of clauses and X a set of variables. For an assignment  $\sigma \in 2^X$  we write  $F(\sigma)$  to denote the set of clauses of F satisfied by  $\sigma$ , and call  $F(\sigma)$  a projection of F. We write  $\operatorname{proj}(F,X) = \{F(\sigma) : \sigma \in 2^X\}$  for the set of projections of F with respect to a set X of variables.

**Proposition 6.** Let F be a formula with m clauses and let  $X \subseteq var(F)$  be a set of variables. We have  $|\mathsf{proj}(F|_X, X)| \leq m+1$ . Moreover, the set  $\mathsf{proj}(F|_X, X)$  can be computed in time polynomial in l, where l is the length of F.

Proof. Let  $\sim_X$  be the relation on clauses defined as  $C \sim_X C'$  if  $\{\ell \in C : var(\ell) \in X\} = \{\ell \in C' : var(\ell) \in X\}$ . Clearly  $\sim_X$  is an equivalence relation. Let  $\mathcal{C}_1, \ldots, \mathcal{C}_l$  be the equivalence classes of  $\sim_X$  on  $F|_X$ . Recall that every clause C in  $F|_X$  contains all variables in X. As a consequence, an assignment  $\tau \in 2^X$  either satisfies all clauses in  $F|_X$  or it satisfies all clauses in  $F|_X$  except those in a unique class  $\mathcal{C}_i$  for  $i \in \{1, \ldots, l\}$ , in which case  $F|_X(\tau) = F|_X \setminus \mathcal{C}_i$ . Since  $F|_X \subseteq F$  we get  $l \leq m$ , and thus  $|\operatorname{proj}(F|_X, X)| \leq m+1$ . Computing  $\operatorname{proj}(F|_X, X)$  boils down to computing  $\mathcal{C}_1, \ldots, \mathcal{C}_l$  and in turn  $F|_X \setminus \mathcal{C}_i$  for each  $i \in \{1, \ldots, l\}$ , which can be done in time polynomial in the length of F. The set  $F|_X$  is contained in  $\operatorname{proj}(F|_X, X)$  if and only if  $l < 2^{|X|}$ , which can be checked in polynomial time as well.

## 3 An Algorithm for #SAT

In this section, we will describe an algorithm for #SAT via dynamic programming on a decomposition tree. To simplify the statements of intermediate results, we fix a formula F with |F| = m clauses and a decomposition tree  $(T, \delta)$  of I(F) with  $index(T, \delta) = k$ . For a node  $z \in V(T)$ , let  $T_z$  denote the maximal subtree of T rooted at z. We write  $var_z$  for the set of variables  $var(F) \cap \delta(L(T_z))$  and  $F_z$  for the set of clauses  $F \cap \delta(L(T_z))$ . Moreover, we let  $\overline{F_z} = F \setminus F_z$  and  $\overline{var_z} = var(F) \setminus var_z$ .

Our algorithm combines techniques from [13] with dynamic programming using "expectations" [2,7,8]. We briefly describe the information maintained for each node  $z \in V(T)$  of the decomposition. Classes of truth assignments  $\sigma \in 2^{var_z}$  will be represented by two sets of clauses. The first set (typically denoted out) corresponds to the projection  $\overline{F_z}(\sigma)$ , that is, the set of clauses outside the current subtree that is satisfied by  $\sigma$ . The second set is a projection  $F_z(\tau)$  for some  $\tau \in 2^{\overline{var_z}}$  so that the combined assignment  $\sigma \cup \tau$  satisfies  $F_z$ . This set of clauses (typically denoted in) is "expected" to be satisfied from outside the current subtree by an "incoming" assignment. Adopting the terminology of [8], we call these pairs of sets shapes.

**Definition 7 (Shape).** Let  $z \in V(T)$ , let  $out_z \subseteq \overline{F_z}$ , and let  $in_z \subseteq F_z$ . We call the pair  $(out_z, in_z)$  a shape (for z), and say an assignment  $\tau \in 2^{var_z}$  is of shape  $(out_z, in_z)$  if it satisfies the following conditions.

- (i)  $\overline{F_z}(\tau) = out_z$ .
- (ii) For each clause  $C \in F_z$ , the assignment  $\tau$  satisfies C or  $C \in in_z$ .

If  $out_z \in \operatorname{proj}(\overline{F_z}, var_z)$  and  $in_z \in \operatorname{proj}(F_z, \overline{var_z})$  then the shape  $(out_z, in_z)$  is proper. We denote the set of shapes for  $z \in V(T)$  by  $\operatorname{shapes}(z)$  and write  $N_z(s)$  to denote the set of assignments in  $2^{var_z}$  of shape  $s \in \operatorname{shapes}(z)$ . Moreover, we let  $n_z(s) = |N_z(s)|$ .

Note that an assignment can have multiple shapes, so shapes do not partition assignments into equivalence classes.

**Lemma 8.** A truth assignment  $\tau \in 2^{var(F)}$  satisfies F if and only if it has shape  $(\emptyset, \emptyset)$ . Moreover, the shape  $(\emptyset, \emptyset)$  is proper.

Proof. Observe that  $var_r = var(F)$ , and let  $\tau \in 2^{var_r}$ . Suppose  $\tau$  satisfies F. Since  $\overline{F_r}$  is empty, we immediately get  $\overline{F_r}(\tau) = \emptyset$ , so  $\tau$  satisfies condition (i). Moreover  $\tau$  satisfies every clause of  $F = F_r$ , so condition (ii) is satisfied as well. For the right to left direction, suppose  $\tau$  has shape  $(\emptyset, \emptyset)$ . It follows from condition (ii) that  $\tau$  must satisfy  $F_r = F$ . To see that  $(\emptyset, \emptyset)$  is proper note that  $\overline{F_r}(\sigma) = \emptyset$  for any  $\sigma \in 2^{var_r}$ , and that  $2^{\overline{var_r}}$  contains only the empty function  $\epsilon : \emptyset \to \{0,1\}$  with  $F_r(\epsilon) = \emptyset$ .

This tells us that  $n_r((\emptyset, \emptyset))$  is equal to the number of satisfying truth assignments of F. Let  $x, y, z \in V(T)$  such that x and y are the children of z, and let  $s_x, s_y, s_z$ 

be shapes for x, y, z, respectively. The assignments in  $N_x(s_x)$  and  $N_y(s_y)$  contribute to  $N_z(s_z)$  if certain conditions are met. These are captured by the following definition.

**Definition 9.** Let  $x, y, z \in V(T)$  such that x and y are the children of z. We say two shapes  $(out_x, in_x) \in \operatorname{shapes}(x)$  and  $(out_y, in_y) \in \operatorname{shapes}(y)$  generate the shape  $(out_z, in_z) \in \operatorname{shapes}(z)$  whenever the following conditions are satisfied.

- (1)  $out_z = (out_x \cup out_y) \cap \overline{F_z}$
- (2)  $in_x = (in_z \cup out_y) \cap F_x$
- (3)  $in_y = (in_z \cup out_x) \cap F_y$

We write generators<sub>z</sub>(s) for the set of pairs in shapes(x) × shapes(y) that generate  $s \in \text{shapes}(z)$ .

**Lemma 10.** Let  $x, y, z \in V(T)$  such that x and y are the children of z, and let  $\tau_x \in 2^{var_x}$  be of shape  $(out_x, in_x) \in \operatorname{shapes}(x)$  and  $\tau_y \in 2^{var_y}$  be of shape  $(out_y, in_y) \in \operatorname{shapes}(y)$ . If  $(out_x, in_x)$  and  $(out_y, in_y)$  generate the shape  $(out_z, in_z) \in \operatorname{shapes}(z)$ , then  $\tau = \tau_x \cup \tau_y$  is of shape  $(out_z, in_z)$ . Moreover, if  $(out_z, in_z)$  is proper then  $(out_x, in_x)$  and  $(out_y, in_y)$  are proper.

Proof. Suppose  $(out_x, in_x)$  and  $(out_y, in_y)$  generate  $(out_z, in_z)$ . To see that  $\tau$  satisfies condition (i), note that a clause is satisfied by  $\tau$  if and only if it is satisfied by  $\tau_x$  or  $\tau_y$ , so  $\overline{F_z}(\tau_z) = \overline{F_z}(\tau_x) \cup \overline{F_z}(\tau_y) = (out_x \cap \overline{F_z}) \cup (out_y \cap \overline{F_z}) = out_z$ . For condition (ii), let  $C \in F_z = F_x \cup F_y$ . Without loss of generality assume that  $C \in F_x$ . Suppose  $\tau$  does not satisfy C. Then  $\tau_x$  does not satisfy C, so we must have  $C \in in_x$  because  $\tau_x$  is of shape  $(out_x, in_x)$ . But  $\tau_y$  does not satisfy C either, so  $C \notin out_y$ . Combining these statements, we get  $C \in in_x \setminus out_y$ . Because  $(out_x, in_x)$  and  $(out_y, in_y)$  generate  $(out_z, in_z)$  we have  $in_x = (in_z \cup out_y) \cap F_x$  by condition (2). It follows that  $C \in in_z$ .

The assignments  $\tau_x$  and  $\tau_y$  are of shapes  $(out_x, in_x)$  and  $(out_y, in_y)$  so  $out_x \in \operatorname{proj}(\overline{F_x}, var_x)$  and  $out_y \in \operatorname{proj}(\overline{F_y}, var_y)$  by condition (i). Suppose  $(out_z, in_z)$  is proper. Then there is an assignment  $\rho \in 2^{\overline{var_z}}$  such that  $in_z = F_z(\rho)$ . The shapes  $(out_x, in_x)$  and  $(out_y, in_y)$  generate  $(out_z, in_z)$ , so  $in_x = (in_z \cup out_y) \cap F_x$ . Thus  $in_x = (F_z(\rho) \cup \overline{F_y}(\tau_y)) \cap F_x$ . Equivalently,  $in_x = (F_z(\rho) \cap F_x) \cup (\overline{F_y}(\tau_y) \cap F_x)$ . Since  $F_x \subseteq F_z$  and  $F_x \subseteq \overline{F_y}$  this can be rewritten once more as  $in_x = F_x(\rho) \cup F_x(\tau_y)$ . The domains  $\overline{var_z}$  of  $\rho$  and  $var_y$  of  $\tau_y$  are disjoint, so  $F_x(\rho) \cup F_x(\tau_y) = F_x(\rho \cup \tau_y)$ . Because  $\overline{var_z} \cup var_y = \overline{var_x}$  it follows that  $in_x \in \operatorname{proj}(F_x, \overline{var_x})$  and so  $(out_x, in_x)$  is proper. A symmetric argument shows that  $(out_y, in_y)$  is proper.

**Corollary 11.** Let  $x, y, z \in V(T)$  such that x and y are the children of z in T, and let  $s \in \mathsf{shapes}(z)$  be proper. Suppose  $s_x \in \mathsf{shapes}(x)$  and  $s_y \in \mathsf{shapes}(y)$  generate s and both  $N_x(s_x)$  and  $N_y(s_y)$  are nonempty. Then  $s_x$  and  $s_y$  are proper.

**Lemma 12.** Let  $x, y, z \in V(T)$  such that x and y are the children of z, and let  $\tau \in 2^{var_z}$  be a truth assignment of shape  $(out_z, in_z) \in shapes(z)$ . Let

 $au_x = au|_{var_x} \ and \ au_y = au|_{var_y}.$  There are unique shapes  $(out_x, in_x) \in \operatorname{shapes}(x)$  and  $(out_y, in_y) \in \operatorname{shapes}(y)$  generating  $(out_z, in_z)$  such that  $au_x$  has shape  $(out_x, in_x)$  and  $au_y$  has shape  $(out_y, in_y)$ .

Proof. We define  $out_x = \overline{F_x}(\tau_x)$ ,  $out_y = \overline{F_y}(\tau_y)$  and let  $in_x = (in_z \cap F_x) \cup F_x(\tau_y)$ ,  $in_y = (in_z \cap F_y) \cup F_y(\tau_x)$ . We prove that  $(out_x, in_x)$  and  $(out_y, in_y)$  generate  $(out_z, in_z)$ . Since  $\tau$  has shape  $(out_z, in_z)$  by condition (i) we have  $out_z = \overline{F_z}(\tau)$ . We further have  $\overline{F_z}(\tau) = \overline{F_z}(\tau_x) \cup \overline{F_z}(\tau_y)$  by choice of  $\tau_x$  and  $\tau_y$ . Because  $\overline{F_z} \subseteq \overline{F_x}$  and  $\overline{F_z} \subseteq \overline{F_y}$  we get  $\overline{F_z}(\tau) = (F_x(\tau_x) \cap \overline{F_z}) \cup (F_y(\tau_y) \cap \overline{F_z})$  and thus  $\overline{F_z}(\tau) = (out_x \cup out_y) \cap \overline{F_z}$ . That is, condition (1) is satisfied. From  $F_x \subseteq \overline{F_y}$  and  $F_y \subseteq \overline{F_x}$  it follows that  $F_x(\tau_y) = \overline{F_y}(\tau_y) \cap F_x$  and  $F_y(\tau_x) = \overline{F_x}(\tau_x) \cap F_y$ . Thus  $F_x(\tau_y) = out_y \cap F_x$  and  $F_y(\tau_x) = out_x \cap F_y$  by construction of  $out_x$  and  $out_y$ . By inserting in the definitions of  $in_x$  and  $in_y$  we get  $in_x = (in_z \cap F_x) \cup (out_y \cap F_x)$  and  $in_y = (in_z \cap F_y) \cup (out_x \cap F_y)$ , so conditions (2) and (3) are satisfied. We conclude that  $(out_x, in_x)$  and  $(out_y, in_y)$  generate  $(out_z, in_z)$ .

We proceed to showing that  $\tau_x$  is of shape  $(out_x, in_x)$ . Condition (i) is satisfied by construction. To see that condition (ii) holds, pick any  $C \in F_x$  not satisfied by  $\tau_x$ . If  $\tau_y$  satisfies C, then  $C \in F_x(\tau_y) \subseteq in_x$ . Otherwise,  $\tau = \tau_x \cup \tau_y$  does not satisfy C. Since  $\tau$  of shape  $(out_z, in_z)$  this implies  $C \in in_z$ . Again we get  $C \in in_x$  as  $in_z \cap F_x \subseteq in_x$ . The proof that  $\tau_y$  has shape  $(out_y, in_y)$  is symmetric.

To show uniqueness, let  $(out_x', in_x') \in \operatorname{shapes}(x)$  and  $(out_y', in_y') \in \operatorname{shapes}(y)$  generate  $(out_z, in_z)$ , and suppose  $\tau_x$  has shape  $(out_x', in_x')$  and  $\tau_y$  has shape  $(out_y', in_y')$ . From condition (i) we immediately get  $out_x' = \overline{F_x}(\tau_x) = out_x$  and  $out_y' = \overline{F_y}(\tau_y) = out_y$ . Since the pairs  $(out_x', in_x'), (out_y', in_y')$  and  $(out_x, in_x), (out_y, in_y)$  both generate  $(out_z, in_z)$ , it follows from condition (2) that  $in_x' = in_x$  and  $in_y' = in_y$ .

**Lemma 13.** Let  $x, y, z \in V(T)$  such that x and y are the children of z in T, and let  $s \in \mathsf{shapes}(z)$ . The following equality holds.

$$n_z(s) = \sum_{(s_x, s_y) \in \mathsf{generators}_z(s)} n_x(s_x) \, n_y(s_y) \tag{1}$$

Proof. Let  $M(s) = \bigcup_{(s_x,s_y) \in \mathsf{generators}_z(s)} N_x(s_x) \times N_y(s_y)$ . We first show that the function  $f: \tau \mapsto (\tau|_{var_x}, \tau|_{var_y})$  is a bijection from  $N_z(s)$  to M(s). By Lemma 12 for every  $\tau \in N_z(s)$  there is a pair  $(s_x, s_y) \in \mathsf{generators}_z(s)$  such that  $\tau|_{var_x} \in N_x(s_x)$  and  $\tau|_{var_y} \in N_y(s_y)$ . So f is into. By Lemma 10, for every pair of assignments  $\tau_x \in N_x(s_x), \tau_y \in N_y(s_y)$  with  $(s_x, s_y) \in \mathsf{generators}_z(s)$  the assignment  $\tau_x \cup \tau_y$  is in  $N_z(s)$ . Hence f is surjective. It is easy to see that f is surjective, so f is indeed a bijection.

We prove that |M(s)| is equivalent to the right hand side of Equality 1. Since  $|N_x(s_x) \times N_y(s_y)| = n_x(s_x) \, n_y(s_y)$  for every pair  $(s_x, s_y) \in \mathsf{generators}_z(s)$ , we only have to show that the sets  $N_x(s_x) \times N_y(s_y)$  and  $N_x(s_x') \times N_y(s_y')$  are disjoint for distinct pairs  $(s_x, s_y), (s_x', s_y') \in \mathsf{generators}_z(s)$ . Let  $(s_x, s_y), (s_x', s_y') \in \mathsf{generators}_z(s)$  and suppose  $(N_x(s_x) \times N_y(s_y)) \cap (N_x(s_x') \times N_y(s_y'))$  is nonempty.

Pick any  $(\tau_x, \tau_y) \in (N_x(s_x) \times N_y(s_y)) \cap (N_x(s_x') \times N_y(s_y'))$ . The function f is a bijection, so  $\tau_x \cup \tau_y \in N_z(s)$ . By Lemma 12 there is at most one pair  $(s_x'', s_y'') \in \text{generators}_z(s)$  of shapes such that  $\tau_x \in N_x(s_x'')$  and  $\tau_y \in N_y(s_y'')$ , so  $(s_x, s_y) = (s_x'', s_y'') = (s_x', s_y')$ .

**Corollary 14.** Let  $x, y, z \in V(T)$  such that x and y are the children of z in T, and let  $s \in \text{shapes}(z)$  be proper. Let  $P = \{ (s_x, s_y) \in \text{generators}_z(s) : s_x \text{ and } s_y \text{ are proper} \}$ . The following equality holds.

$$n_z(s) = \sum_{(s_x, s_y) \in P} n_x(s_x) \, n_y(s_y) \tag{2}$$

*Proof.* By Corollary 11 the product  $n_x(s_x)n_y(s_y)$  is nonzero only if  $s_x$  and  $s_y$  are proper, for any pair  $(s_x, s_y) \in \mathsf{generators}_z(s)$ . In combination with (1) this implies (2).

Corollary 14 in combination with Lemma 8 implies that, for each  $z \in V(T)$ , it is enough to compute the values  $n_z(s)$  for proper shapes  $s \in \mathsf{shapes}(z)$ . To turn this insight into a polynomial time dynamic programming algorithm, we still have to show that the number of proper shapes in  $\mathsf{shapes}(z)$  can be polynomially bounded, and that the set of such shapes can be computed in polynomial time. We will achieve this by specifying a subset of  $\mathsf{shapes}(z)$  for each  $z \in V(T)$  that contains all proper shapes and can be computed in polynomial time.

We define families  $\mathscr{X}_z$  and  $\overline{\mathscr{X}_z}$  of sets of variables for each node  $z \in V(T)$ , as follows.

$$\mathcal{X}_z = \{ X \subseteq var_z : \exists C \in \overline{F_z} \text{ such that } X = var_z \cap var(C) \}$$

$$\overline{\mathcal{X}_z} = \{ X \subseteq \overline{var_z} : \exists C \in F_z \text{ such that } X = \overline{var_z} \cap var(C) \}$$

The next lemma follows from the definition of a decomposition tree's index.

**Lemma 15.** For every node  $z \in V(T)$ ,  $\max(|\mathscr{X}_z|, |\overline{\mathscr{X}_z}|) \leq k$ .

Let  $z \in V(T)$  and let f be a function with domain  $\mathscr{X}_z$  that maps every set X to some projection  $f(X) \in \operatorname{proj}(\overline{F_z}|_X, X)$ . We denote the set of such functions by outfunctions(z). Symmetrically, we let infunctions(z) denote the set of functions g that map every set  $Y \in \overline{\mathscr{X}_z}$  to some projection  $g(Y) \in \operatorname{proj}(F_z|_Y, Y)$ .

**Lemma 16.** For every  $z \in V(T)$ ,  $|\mathsf{outfunctions}(z)| \leq (m+1)^k$  as well as  $|\mathsf{infunctions}(z)| \leq (m+1)^k$ .

*Proof.* By Proposition 6 that the cardinality of  $\operatorname{proj}(\overline{F_z}|_X, X)$  is bounded by m+1 for every  $X \in \mathscr{X}_z$ . In combination with Lemma 15 this yields  $|\operatorname{outfunctions}(z)| \leq (m+1)^k$ . The proof of  $|\operatorname{infunctions}(z)| \leq (m+1)^k$  is symmetric.

Let  $\mathsf{union}(f)$  denote  $\bigcup_{X \in \mathsf{dom}(f)} f(X)$ , where  $\mathsf{dom}(f)$  is the domain of f. We define the set of restricted shapes for  $z \in V(T)$  as follows.

$$\mathsf{rshapes}(z) = \{ \, (\mathit{out}, \mathit{in}) \in \mathsf{shapes}(z) : \exists f \in \mathsf{outfunctions}(z) \, \mathit{s.t.} \, \, \mathit{out} = \mathsf{union}(f) \\ \land \exists g \in \mathsf{infunctions}(z) \, \mathit{s.t.} \, \, \mathit{in} = \mathsf{union}(g) \, \}$$

Every pair  $(f,g) \in \text{outfunctions}(z) \times \text{infunctions}(z)$  uniquely determines a shape in rshapes(z). Accordingly, Lemma 16 allows us to bound the cardinality of rshapes(z) as follows.

Corollary 17. For any  $z \in V(T)$ ,  $|\operatorname{rshapes}(z)| \leq (m+1)^{2k}$ .

**Lemma 18.** Let  $z \in V(T)$  and let  $s \in \text{shapes}(z)$  be proper. Then  $s \in \text{rshapes}(z)$ .

Proof. Let s=(out,in). We show that there are functions  $f\in \operatorname{outfunctions}(z)$  and  $g\in \operatorname{infunctions}(z)$  such that  $out=\operatorname{union}(f)$  and  $in=\operatorname{union}(g)$ . Because s is proper we have  $out\in\operatorname{proj}(\overline{F_z},var_z)$  and  $in\in\operatorname{proj}(F_z,\overline{var_z})$ , so there must be truth assignments  $\sigma\in 2^{var_z}$  and  $\tau\in 2^{\overline{var_z}}$  such that  $out=\overline{F_z}(\sigma)$  and  $in=F_z(\tau)$ . We define f as follows. For each  $X\in \mathscr{X}_z$  we let  $f(X)=\overline{F_z}|_X(\sigma|_X)$ . The assignment  $\sigma$  is defined on  $X\subseteq var_z$ , so  $\sigma|_X\in 2^X$  and  $f(X)\in\operatorname{proj}(\overline{F_z}|_X,X)$ . That is,  $f\in \operatorname{outfunctions}(z)$ . Symmetrically, we let  $g(X)=F_z|_X(\tau|_X)$  for each  $X\in \overline{\mathscr{X}_z}$ . Since  $\tau$  is defined on  $X\subseteq \overline{var_z}$  we have  $\tau|_X\in 2^X$  and  $g(X)\in\operatorname{proj}(F_z|_X,X)$ , so  $g\in \operatorname{infunctions}(z)$ .

Pick an arbitrary  $C \in \overline{F_z}$  and let  $X = var(C) \cap var_z$ . We show that  $C \in out$  if and only if  $C \in \mathsf{union}(f)$ . Suppose  $C \in out = \overline{F_z}(\sigma)$ . The assignment  $\sigma$  has domain  $var_z$ , so  $\sigma|_X$  satisfies C because  $\sigma$  does. That is,  $C \in \overline{F_z}(\sigma|_X)$ . By choice of X we have  $C \in F_z|_X$ , so  $C \in \overline{F_z}(\sigma|_X) \cap F_z|_X$ . Since  $F_z|_X \subseteq F_z$  we get  $\overline{F_z}(\sigma|_X) \cap F_z|_X = \overline{F_z}|_X(\sigma|_X)$ . So  $C \in \overline{F_z}|_X(\sigma|_X) = f(X)$  and thus  $C \in \mathsf{union}(f)$ . For the converse direction, suppose  $C \in \mathsf{union}(f)$ . That is,  $C \in f(Y) = \overline{F_z}|_Y(\sigma|_Y)$  for some  $Y \in \mathscr{X}_z$ . Then in particular  $C \in \overline{F_z}(\sigma) = out$ . We conclude that  $\mathsf{union}(f) = out$ . The proof of  $\mathsf{union}(g) = in$  is symmetric.  $\square$ 

This shows that if we can determine the values  $n_z(s)$  for every  $z \in V(T)$  and  $s \in \mathsf{rshapes}(z)$ , we can determine the values  $n_z(s')$  for every proper shape  $s' \in \mathsf{shapes}(z)$ . More specifically, as long as we can determine lower bounds for  $n_z(s)$  for every  $s \in \mathsf{rshapes}(z)$  and the exact values of  $n_z(s)$  for proper s, we can compute the correct values for all proper shapes for every tree node.

**Definition 19.** For  $z \in V(T)$ , a lower bounding function (for z) associates with each  $s \in \mathsf{rshapes}(z)$  a value  $l_z(s)$  such that  $l_z(s) \leq n_z(s)$  and  $l_z(s) = n_z(s)$  if s is proper.

Let  $x, y, z \in V(T)$  such that x and y are the children of z. For each  $s \in \mathsf{shapes}(z)$  we write  $\mathsf{restricedgen}_z(s) = \mathsf{generators}_z(s) \cap (\mathsf{rshapes}(x) \times \mathsf{rshapes}(y))$ .

**Lemma 20.** Let  $x, y, z \in V(T)$  such that x and y are the children of z. Let  $l_x$  and  $l_y$  be lower bounding functions for x and y. Let  $l_z$  be the function defined as follows. For each  $s \in \mathsf{rshapes}(z)$ , we let

$$l_z(s) = \sum_{(s_x, s_y) \in \mathsf{restricedgen}_z(s)} l_x(s_x) \, l_y(s_y). \tag{3}$$

Then  $l_z$  is a lower bounding function for z.

Proof. The inequality  $l_z(s) \leq n_z(s)$  follows from  $\operatorname{rshapes}(x) \subseteq \operatorname{shapes}(x)$  and  $\operatorname{rshapes}(y) \subseteq \operatorname{shapes}(y)$ , in combination with equality (1) and the fact that  $l_x$  and  $l_y$  are lower bounding functions for x and y. By Lemma 18 the set  $\operatorname{restricedgen}_z(s)$  contains all pairs  $(s_x, s_y) \in \operatorname{generators}_z(s)$  such that  $s_x$  and  $s_y$  are proper. It follows from Corollary 14 and  $l_x(s_x) = n_x(s_x)$ ,  $l_y(s_y) = n_y(s_y)$  for proper  $s_x, s_y$  that  $l_z(s) \geq n_z(s)$  and thus  $l_z(s) = n_z(s)$  for proper s. We conclude that  $l_z$  is a lower bounding function for z.

**Lemma 21.** There is a polynomial p such that for any  $z \in V(T)$ , the set  $\mathsf{rshapes}(z)$  can be computed in time  $m^{2k}p(l)$ , where l is the length of F.

Proof. To compute  $\operatorname{rshapes}(z)$ , we compute all pairs  $(\operatorname{union}(f), \operatorname{union}(g))$  for  $(f,g) \in \operatorname{outfunctions}(z) \times \operatorname{infunctions}(z)$ . To compute the set  $\mathscr{X}_z$ , we run through all clauses  $C \in \overline{F_z}$  and determine  $\operatorname{var}(C) \cap \operatorname{var}_z$ . This can be done in time polynomial in l, and the same holds for the set  $\overline{\mathscr{X}_z}$ . A function  $f \in \operatorname{outfunctions}(z)$  maps each  $X \in \mathscr{X}_z$  to a set  $f(X) \in \operatorname{proj}(\overline{F_z}|_X, X)$ . By Proposition 6 the set  $\operatorname{proj}(\overline{F_z}|_X, X)$  can by computed in time polynomial in l for each  $X \in \mathscr{X}_z$ . Going through all possible pairs  $(f,g) \in \operatorname{outfunctions}(z) \times \operatorname{infunctions}(z)$  amounts to going through all possible combinations of choices of  $f(X) \in \operatorname{proj}(\overline{F_z}|_X, X)$  for each  $X \in \mathscr{X}_z$  and  $g(X') \in \operatorname{proj}(F_z|_{X'}, X')$  for each  $X' \in \overline{\mathscr{X}_z}$ , of which there are at most  $(m+1)^{2k}$ . For each such pair (f,g) we compute the sets  $\operatorname{union}(f)$  and  $\operatorname{union}(g)$ , which can be done in time polynomial in l.

**Lemma 22.** Let  $x, y, z \in V(T)$  such that x and y are the children of z. Let  $s_x \in \mathsf{shapes}(x), s_y \in \mathsf{shapes}(y), \ and \ s_z \in \mathsf{shapes}(z).$  It can be decided in time  $O(l^2)$  whether  $s_x$  and  $s_y$  generate  $s_z$ , where l is the length of F.

*Proof.* We only have to check conditions (1) to (3), which can easily be done in time quadratic in l since the sets of clauses involved have length at most l.  $\square$ 

**Lemma 23.** For any leaf node  $z \in V(T)$  a lower bounding function for z can be computed in time O(l), where l is the length of F.

Proof. Every leaf  $z \in V(T)$  is either associated with a clause  $C \in F$  or a variable  $v \in var(F)$ . In the first case,  $var_z = \emptyset$  and so  $\mathscr{X}_z = \emptyset$  if  $\overline{F_z} = \emptyset$  or  $\mathscr{X}_z = \{\emptyset\}$ . It follows that the set outfunctions(z) only contains the empty function or the function f with domain  $\{\emptyset\}$  such that  $f(\emptyset) = \emptyset$ . For the set  $\overline{\mathscr{X}_z}$  we get  $\overline{\mathscr{X}_z} = \{var(C)\}$  for the unique clause  $C \in F_z$ . Since  $F_z|_{var(C)} = \{C\}$  we have  $\operatorname{proj}(F_z|_{var(C)}, var(C)) = \{\{C\}, \emptyset\}$  and thus  $\operatorname{infunctions}(z) = \{g, g'\}$ , where g is the function with domain  $\{var(C)\}$  such that  $g(var(C)) = \{C\}$  and g' is the function with domain  $\{var(C)\}$  such that  $g'(var(C)) = \emptyset$ . It follows that  $\operatorname{rshapes}(z)$  only contains the shapes  $(\emptyset, \emptyset)$  and  $(\emptyset, \{C\})$ . The set  $var_z$  is empty, so  $2^{var_z}$  contains only the empty assignment which does not satisfy any clause. Hence  $n_z((\emptyset, \emptyset)) = 0$  and  $n_z((\emptyset, \{C\})) = 1$ .

In the second case,  $var_z = \{v\}$  for some variable  $v \in var(F)$ . Since  $F_z = \emptyset$  we have  $\overline{\mathscr{Z}_z} = \emptyset$  and so infunctions(z) only contains the empty function. The set  $\mathscr{Z}_z$  contains  $\{v\}$ , and the empty set if there is a clause  $C \in F$  with  $v \notin var(C)$ . We get  $\text{proj}(\overline{F_z}|_{\{v\}}, \{v\}) = \{F_v^+, F_v^-\}$ , where  $F_v^+$  is the set of clauses of

F with a positive occurrence of v, and  $F_v^-$  is the set of clauses F with a negative occurrence of v. Moreover,  $\operatorname{proj}(\overline{F_z}|_{\emptyset},\emptyset)=\{\emptyset\}$ . It follows that  $\operatorname{rshapes}(z)=\{(F_v^+,\emptyset),(F_v^-,\emptyset)\}$ . The set  $2^{var_z}$  only contains the assignments  $\tau_0$  with  $\tau_0(v)=0$  and  $\tau_1$  with  $\tau_1(v)=1$ , and  $\overline{F_z}(\tau_0)=F_v^-$  and  $\overline{F_z}(\tau_1)=F_v^+$ . This implies  $n_z((F_v^+,\emptyset))=1$  and  $n_z((F_v^-,\emptyset))=1$ .

In either case the set rshapes(z) and the values  $n_z(s)$  for each  $s \in rshapes(z)$  can be computed in time O(l). These values trivially provide a lower bounding function for z.

**Lemma 24.** There is a polynomial p such that for any inner node  $z \in V(T)$ , a lower bounding function for z can be computed in time  $m^{6k}p(l)$ , provided that lower bounding functions have already been computed for both children of z, where l denotes the length of F.

Proof. By Lemma 21, there is a polynomial q (independent of z) such that the set  $\operatorname{rshapes}(z)$  can be computed in time  $O(m^{2k}q(l))$ . Let x and y denote the children of z, and let  $l_x$  and  $l_y$  be lower bounding functions for x and y. We compute a lower bounding function  $l_z$  for z as follows. Initially, we set  $l_z(s_z) = 0$  for all  $s_z \in \operatorname{rshapes}(z)$ . We then run through all triples of shapes  $s_x \in \operatorname{rshapes}(x)$ ,  $s_y \in \operatorname{rshapes}(y)$ , and  $s_z \in \operatorname{rshapes}(z)$  and check whether  $s_x$  and  $s_y$  generate  $s_z$ . If that is the case, we add  $l_x(s_x) l_y(s_y)$  to  $l_z(s_z)$ .

Correctness follows from Lemma 20 and the fact that  $l_x, l_y$  are lower bounding functions for x and y. The bound on the runtime is obtained as follows. By Corollary 17 there are at most  $(m+1)^{6k}$  triples  $(s_x, s_y, s_z)$  of shapes that have to be considered. For each one, one can decide whether  $s_x$  and  $s_y$  generate  $s_z$  in time  $O(l^2)$  by Lemma 22. Depending on the outcome of that decision we may have to multiply two integers  $l_x(s_x)$  and  $l_y(s_y)$ , adding the result to  $l_z(s_z)$ . These values are bounded from above by  $2^{|var(F)|} \leq 2^l$ , so their binary representations have size O(l) and these arithmetic operations can be carried out in time polynomial in l.

**Lemma 25.** There is a polynomial p such that a lower bounding function for z can be computed for every  $z \in V(T)$  in time  $m^{6k}p(l)$ , where l is the length of F.

Proof. By Lemma 23, a lower bounding function for a leaf of T can be computed in time O(l). The number of leaves of T is in O(l), so we can compute lower bounding functions for all of them in time  $O(l^2)$ . By Lemma 24, we can then compute lower bounding functions for each inner node  $z \in V(T)$  in a bottom up manner. For each inner node z, a lower bounding function can computed in time  $m^{6k}q(l)$  by Lemma 24, where q is a polynomial independent of z. The number of inner nodes of T is in O(l), so this requires  $O(m^{6k}l\,q(l))$  time in total.

**Proposition 26.** There is a polynomial p and an algorithm  $\mathbb{A}$  such that  $\mathbb{A}$ , given a CNF formula F and a decomposition tree  $(T, \delta)$  of I(F), computes the number of satisfying total truth assignments of F in time  $m^{6k}p(l)$ . Here, m denotes the number of clauses of F, l denotes the length of F, and  $k = index(T, \delta)$ .

*Proof.* By Lemma 25 a lower bounding function  $l_r$  for the root r of T can be computed in time  $m^{6k} \ q(l)$ , where q is a polynomial independent of F. By Lemma 8, the value  $n_r((\emptyset, \emptyset))$  corresponds to the number of satisfying total truth assignments of F, and the shape  $(\emptyset, \emptyset)$  is proper. Since  $l_r$  is a lower bounding function for r it follows that  $l_r((\emptyset, \emptyset)) = n_r((\emptyset, \emptyset))$ .

Proof (of Theorem 1). Let  $\mathcal C$  be a graph class of bounded symmetric clique-width and F a CNF formula of length l with m clauses such that  $I(F) \in \mathcal C$ . Let k be an upper bound for the symmetric clique-width of any graph in  $\mathcal C$ . We compute a decomposition tree  $(T,\delta)$  of I(F) such that  $rankw(T,\delta) = rankw(I(F))$  as follows. Initially, we set k' := 1. We then repeatedly run the algorithm of Theorem 5 and increment k' by one until we find a decomposition of rank-width k'. This will be the case after at most k steps since  $rankw(I(F)) \leq scw(I(F))$  by Corollary 4. Since  $\mathcal C$  is fixed, we can consider k (and every  $k' \leq k$ ) a constant, so  $(T,\delta)$  can be obtained in time  $O(|V(I(F))|^3)$  by Theorem 5. Because 2l is an upper bound on the number of vertices of I(F), this is in  $l^{O(1)}$  (assuming that  $l \geq 2$ ). By Lemma 3,  $index(T,\delta) \leq 2^{rankw(I(F))}$  and thus  $index(T,\delta) \leq 2^{scw(I(F))} \leq 2^k$ . By Proposition 26, the number of satisfying total truth assignments of F can be computed in time  $m^{6 index(T,\delta)}p(l)$  for some polynomial p independent of F, that is, in time  $m^{O(2^k)}p(l)$ . Since k is a constant, this is in  $l^{O(1)}$ , as is the total runtime.

## 4 Conclusion

We have shown that #SAT is polynomial-time tractable for classes of formulas with incidence graphs of bounded symmetric clique-width (or bounded clique-width, or bounded rank-width). It would be interesting to know whether this problem is tractable under even weaker structural restrictions. For instance, it is currently open whether #SAT is polynomial-time tractable for classes of formulas of bounded  $\beta$ -hypertree width [10] (if a corresponding decomposition is given).

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