# Conjugacy in Baumslag's group, generic case complexity, and division in power circuits

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**Abstract.** The conjugacy is the following question in algorithmic group theory: given two words x, y over generators of a fixed group G, decide whether x and yare conjugated, i.e., whether there exists some z such that  $zxz^{-1} = y$  in G. The conjugacy problem is more difficult than the word problem, in general. We investigate the conjugacy problem for two prominent groups: the Baumslag-Solitar group  $BS_{1,2}$  and the Baumslag(-Gersten) group  $G_{1,2}$ . The conjugacy problem in  $\mathbf{BS}_{1,2}$  is  $\mathsf{TC}^0$ -complete. To the best of our knowledge  $\mathbf{BS}_{1,2}$  is the first natural infinite non-commutative group where such a precise and low complexity is shown. The Baumslag group  $G_{1,2}$  is an HNN extension of  $BS_{1,2}$ . We show that the conjugacy problem is decidable (which has been known before); but our results go far beyond decidability. In particular, we are able to show that conjugacy in  $G_{1,2}$  can be solved in polynomial time in a strongly generic setting. This means that essentially for all inputs conjugacy in  $G_{1,2}$  can be decided efficiently. In contrast, we show that under a plausible assumption the average case complexity of the same problem is non-elementary. Moreover, we provide a lower bound for the conjugacy problem in  $G_{1,2}$  by reducing the division problem in power circuits to the conjugacy problem in  $G_{1,2}$ . The complexity of the division problem in power circuits is an open and interesting problem in integer arithmetic. To date it is believed that this problem has non-elementary time complexity. Another contribution of the paper concerns a general statement about HNN extension of the form  $G = \langle H, b \mid bab^{-1} = \varphi(a), a \in A \rangle$  with a finitely generated base group H. We show that the complement of H is strongly generic if and only if  $A \neq H \neq B$ . This is the situation for  $G_{1,2}$ ; and yields an important piece of information why it is possible to solve conjugacy for  $G_{1,2}$  in strongly generic polynomial time. Note also that the complement of H is strongly generic if and only if the Schreier graph of G with respect to the subgroup H is non-amenable.

#### Introduction

More than 100 years ago Max Dehn introduced the word problem and the conjugacy problem as fundamental decision problems in group theory. Let G be a finitely generated group. Word problem: Given two words x, y written in generators, decide whether x = y in G. Conjugacy problem: Given two words x, y written in generators, decide whether  $x \sim_G y$  in G, i.e., decide whether there exists z such that  $zxz^{-1} = y$  in G. In recent years, conjugacy played an important role in non-commutative cryptography, see e.g. [7,11,22]. These applications use that is is easy to create elements which are conjugated, but to check whether two given elements are conjugated might be difficult even if

the word problem is easy. In fact, there are groups where the word problem is easy but the conjugacy problem is undecidable [18]. Frequently, in cryptographic applications the ambient group is fixed. The focus in this paper is on the conjugacy problem in  $G_{1,2}$ . In 1969 Gilbert Baumslag defined the group  $G_{1,2}$  as an example of a one-relator group which enjoys certain remarkable properties. It was introduced as an infinite non-cyclic group all of whose finite quotients are cyclic [2]. In particular, it is not residually finite; but being one-relator it has a decidable word problem [17]. The group  $G_{1,2}$  is generated by generators a and b subject to a single relation  $bab^{-1}a = a^2bab^{-1}$ . Another way to understand  $G_{1,2}$  is to view it as an HNN extension of the even more prominent Baumslag-Solitar group  $BS_{1,2}$ . The group  $BS_{1,2}$  is defined by a single relation  $tat^{-1} = a^2$  where a and t are generators<sup>3</sup>. The complexity of the word problem and conjugacy problem in  $BS_{1,2}$  are very low; indeed, we show that they are  $TC^0$ -complete. However, such a low complexity does not transfer to the complexity of the corresponding problems in HHN-extensions like  $G_{1,2}$ . Gersten showed that the Dehn function of  $G_{1,2}$  is non-elementary [9]. Moreover, Magnus' break-down procedure [16] on  $G_{1,2}$ is non-elementary, too. This means that the time complexity for the standard algorithm to solve the word problem in  $G_{1,2}$  cannot be bounded by any fixed tower of exponentials. Therefore, for many years,  $G_{1,2}$  was the simplest candidate for a group with an extremely difficult word problem. However, Myasnikov, Ushakov, and Won showed in [20] that the word problem of the Baumslag group is solvable in polynomial time! In order to achieve a polynomial time bound they introduced a versatile data structure for integer arithmetic which they called *power circuit*. The data structure supports  $+, -, \leq$ , and  $(x,y) \mapsto 2^x y$ , a restricted version of multiplication which includes exponentiation  $x\mapsto 2^x$ . Thus, by iteration it is possible to represent huge values (involving the tower function) by very small circuits. Still, all operations above can be performed in polynomial time. On the other hand there are notoriously difficult arithmetical problems in power circuits, too. A very important one is division. The input are power circuits Cand C' representing integers m and m'; the question is whether m divides m'. The problem is clearly decidable by converting m and m' into binary; but this procedure is non-elementary. So far, no idea for any better algorithm is known. It is plausible to assume that the problem "division in power circuits" has no elementary time complexity at all.

In the present paper we show a tight relation between the problems "division in power circuits" and conjugacy in  $G_{1,2}$ . Our results concerning the Baumslag-Solitar group  $BS_{1,2}$ , the Baumslag group  $G_{1,2}$ , its generic case complexity, and division in power circuits are as follows.

- The conjugacy problem of  $BS_{1,2}$  is  $TC^0$ -complete.
- There is a strongly generic polynomial time algorithm for the conjugacy problem in  $\mathbf{G}_{1,2}$ . This means, the difficult instances for the algorithm are exponentially sparse, and therefore, on random inputs, conjugacy can be solved efficiently.
- If "division in power circuits" is non-elementary in the worst case, then the conjugacy problem in  $G_{1,2}$  is non-elementary on the average.

<sup>&</sup>lt;sup>3</sup> Adding a generator b and a relation  $bab^{-1} = t$  results in  $G_{1,2}$ . Indeed, due to  $bab^{-1} = t$ , we can remove t and we obtain exactly the presentation of  $G_{1,2}$  above.

- A random walk in the Cayley graph of  $G_{1,2}$  ends with exponentially decreasing probability in  $BS_{1,2}$ . In other terms, the Schreier graph of  $G_{1,2}$  with respect to  $BS_{1,2}$  is non-amenable.

Decidability of the conjugacy problem in  $G_{1,2}$  is not new, it was shown in [3]<sup>4</sup> and decidability outside a so-called "black hole" follows already from [4]. Our work improves Beese's work leading to a polynomial time algorithm outside a proper subset of the "black hole" (and decidability everywhere). Thus, our result underlines that in special cases like  $G_{1,2}$  much better results than stated in [4] are possible. Let us also note that there are undecidable problems (hence no finite average case complexity is defined), like the halting problem for certain encodings of Turing machines, which have generically linear time partial solutions. However, many of these examples depend on encodings and special purpose constructions. In our case we consider a natural problem where the average case complexity is defined, but the only known algorithm to solve it runs in non-elementary time on the average. Nevertheless, there is a polynomial p (roughly of degree 4) such that the probability that the same algorithm requires more than p(n) steps on random inputs converges exponentially fast to zero. The main technical difficulty in establishing a strongly generic polynomial time complexity is to show that a random walk of length n in the Cayley graph of  $G_{1,2}$  ends with probability less than  $(1-\varepsilon)^n$ in the subgroup  $\mathbf{BS}_{1,2}$  for some  $\varepsilon > 0$ . Random walks in infinite graphs are widely studied in various areas, see e.g. [24] or the textbook [25]. In Section 5 we prove a general statement about HNN extension of the form  $G = \langle H, b \mid bab^{-1} = \varphi(a), a \in A \rangle$ with a finitely generated base group H and  $\Delta$  a finite symmetric set of generators for G. We show that the complement of H (inside  $\Delta^*$ ) is strongly generic if and only if  $A \neq H \neq B$ . With other words, the Schreier graph  $\Gamma(G, H, \Delta)$  is non-amenable if and only if  $A \neq H \neq B$ . (For a definition of amenability and its equivalent characterizations see e.g. [5,14].) This applies to  $G_{1,2}$  because it is an HNN extension where  $A \neq H \neq B$ . However, in the special case of  $G_{1,2}$  we can also apply a technique quite different from the general approach. In Section 4.1 we define a "pairing" between random walks in the Cayley graph and Dyck words. We exhibit an  $\varepsilon > 0$  such that for each Dyck word w of length 2n the probability that a pairing with w evaluates to 1 is bounded by  $(1/4 - \varepsilon)^n$ . The result follows since there are at most  $4^n$  Dyck words.

#### Notation and preliminaries

**Words.** An *alphabet* is a (finite) set  $\Sigma$ ; an element  $a \in \Sigma$  is called a *letter*. The set  $\Sigma^n$  forms the set of *words* of length n. The length of  $w \in \Sigma^n$  is denoted by |w|. The set of all words is denoted by  $\Sigma^*$ . It is the free monoid over  $\Sigma$ . Let  $a \in \Sigma$  be a letter and  $w \in \Sigma^*$ . The number of occurrences of a in w is denoted by  $|w|_a$ . Clearly,  $|w| = \sum_{a \in \Sigma} |w|_a$ . If we can write w = uxv, then we call x a factor of w; and we say that w = uxv is a factorization.

**Functions.** We use standard  $\mathcal{O}$ -notation for functions from  $\mathbb{N}$  to non-negative reals  $\mathbb{R}^{\geq 0}$ . (This includes of course  $\Omega$ - and  $\Theta$ -notation.) The *tower function*  $\tau: \mathbb{N} \to \mathbb{N}$  is defined by  $\tau(0) = 0$  and  $\tau(i+1) = 2^{\tau(i)}$  for  $i \geq 0$ . It is primitive recursive. We

<sup>&</sup>lt;sup>4</sup> It is unknown whether the conjugacy problem in one-relator groups is decidable, in general.

say that a function  $f: \mathbb{N} \to \mathbb{R}^{\geq 0}$  is *elementary*, if the growth of f can be bounded by a fixed number of exponentials. It is called *non-elementary* if it is not elementary, but  $f(n) \in \tau(\mathcal{O}(n))$ . Thus, in our paper non-elementary means a lower and an upper bound.

**Circuit complexity.** We deal with various complexity measures. On the lowest level we are interested in problems which can be decided by (uniform) TC<sup>0</sup>-circuits. These are circuits of polynomial size with constant depth where we allow Boolean gates and majority gates, which evaluate to 1 if and only if the majority of inputs is 1. For a precise definition and uniformity conditions we refer to the textbook [23]. TC<sup>0</sup> circuits can be simulated by NC<sup>1</sup> circuits, i.e., circuits of logarithmic depth where only Boolean gates of constant fan-in are allowed. Thus, TC<sup>0</sup> is a very low parallel complexity class. Still it is amazingly powerful with respect to arithmetic. In particular, we shall use Hesse's result that division of binary integers can be computed by a uniform family of TC<sup>0</sup>-circuits [12,13].

**Time complexity.** A uniform family of  $\mathsf{TC}^0$ -circuits computes a polynomial time computable function. We use a standard notion for worst-case and for average case complexity and random access machines (RAMs) as machine model. An algorithm  $\mathcal A$  computes a function between domains D and D'. In our applications D comes always with a natural partition  $D = \bigcup \left\{D^{(n)} \mid n \in \mathbb{N}\right\}$  where each  $D^{(n)}$  is finite. The time complexity  $t_{\mathcal A}$  is defined by  $t_{\mathcal A}(n) = \max \left\{t_{\mathcal A}(w) \mid w \in D^{(n)}\right\}$ . Assuming a uniform distribution among elements in  $D^{(n)}$ , the average case complexity is defined by  $\operatorname{av}_{\mathcal A}(n) = \frac{1}{|D^{(n)}|} \sum_{w \in D^{(n)}} t_{\mathcal A}(w)$ .

Generic case complexity. For many practical applications the "generic-case behavior" of an algorithm is more important than its average-case or worst-case behavior. We refer to [14,15] where the foundations of this theory were developed and to [19] for applications in cryptography. The notion of *generic complexity* refers to partial algorithms which are defined on a (strongly) generic set  $I \subseteq D$ . Thus, they may refuse to give an answer outside I, but if they give an answer, the answer must always be correct. In our context it is enough to deal with totally defined algorithms and strongly generic sets. Thus, the answer is always computed and always correct, but the runtime is measured by a worst-case behavior over a strongly generic set  $I \subseteq D$ . Here a set I is called strongly generic, if there exists an  $\varepsilon > 0$  such that  $\left|D^{(n)} \setminus I\right|/\left|D^{(n)}\right| \le 2^{-\varepsilon n}$  for almost all  $n \in \mathbb{N}$ . This means the probability to find a random string outside I converges exponentially fast to zero. Thus, if an algorithm A runs in polynomial time on a strongly generic set, then, for practical purposes, A behaves as a polynomial time worst-case algorithm. This is true although the average time complexity of A can be arbitrarily high.

**Group theory.** We use standard notation and facts from group theory as found in the classical text book [16]. Groups G are generated by some subset  $S \subseteq G$ . We let  $\overline{S} = S^{-1}$  and we view  $S \cup \overline{S}$  as an alphabet with involution; its elements are called *letters*. We have  $\overline{\overline{a}} = a$  for letters and also for words by letting  $\overline{a_1 \cdots a_n} = \overline{a_n} \cdots \overline{a_1}$  where  $a_i \in S \cup \overline{S}$  are letters. Thus, if  $g \in G$  is given by a word w, then  $\overline{w} = g^{-1}$  in the group G. For a word w we denote by |w| its length. We say that w is reduced if there is no factor  $a\overline{a}$  for any letter. It is called *cyclically reduced* if ww is reduced. For words (or group elements) we write  $x \sim_G y$  to denote conjugacy, i.e.,  $x \sim_G y$  if and only if there exists some  $z \in G$  such that  $zx\overline{z} = y$  in G. For the decision problem "conjugacy in G"

we assume that the input consists of cyclically reduced words x and y if not explicitly stated otherwise. We apply the standard (so called Magnus break-down) procedure for solving the word problem in HNN extensions. Our calculations are fully explicit and accessible with basic knowledge in combinatorial group theory

**Glossary.** TC<sup>0</sup> circuit class.  $x \sim_G y$  conjugacy in groups.  $(\Gamma, \delta)$  power circuits.  $\varepsilon(P)$ ,  $\varepsilon(M)$  evaluation of nodes and markings.  $\tau(n)$  tower function. Baumslag-Solitar group:  $\mathbf{BS}_{1,2} = \langle a, t \mid tat^{-1} = a^2 \rangle$ . Baumslag group:  $\mathbf{G}_{1,2} = \langle a, b \mid bab^{-1}a = a^2ba^{-1}b^{-1} \rangle$ . Subgroup relations  $A = \langle a \rangle$ ,  $T = \langle t \rangle \leq \mathbf{BS}_{1,2} = \mathbb{Z}[1/2] \rtimes \mathbb{Z} = H \leq \mathbf{G}_{1,2}$ . Standard symmetric set of generators for  $\mathbf{G}_{1,2}$  is  $\Sigma = \left\{ a, \overline{a}, b, \overline{b} \right\}^*$  and  $\overline{z} = z^{-1}$  in groups.

## 1 Power circuits

In binary a number is represented as a sum  $m=\sum_{i=0}^k b_i 2^i$  with  $b_i\in\{0,1\}$ . Allowing  $b_i\in\{-1,0,1\}$  we obtain a "compact representation" of integers, which may require less non-zero  $b_i$ s than the normal representation. The notion of power circuit is due to [21]. It generalizes compact representations and goes far beyond since it allows a compact representation of tower functions. Formally: a power circuit of size n is given by a pair  $(\Gamma,\delta)$ . Here,  $\Gamma$  is a set of n vertices and  $\delta$  is a mapping  $\delta:\Gamma\times\Gamma\to\{-1,0,+1\}$ . The support of  $\delta$  is the subset  $\Delta\subseteq\Gamma\times\Gamma$  with  $(P,Q)\in\Delta\iff\delta(P,Q)\neq0$ . Thus,  $(\Gamma,\Delta)$  is a directed graph. Throughout we require that  $(\Gamma,\Delta)$  is acyclic. In particular,  $\delta(P,P)=0$  for all vertices P. A marking is a mapping  $M:\Gamma\to\{-1,0,+1\}$ . We can also think of a marking as a subset of  $\Gamma$  where each element in M has a sign (+ or -). If M(P)=0 for all  $P\in\Gamma$  then we simply write  $M=\emptyset$ . Each node  $P\in\Gamma$  is associated in a natural way with a successor marking  $\Lambda_P:\Gamma\to\{-1,0,+1\}$ ,  $Q\mapsto\delta(P,Q)$ , consisting of the target nodes of outgoing arcs from P. We define the evaluation  $\varepsilon(P)$  of a node  $(\varepsilon(M))$  of a marking resp.) bottom-up in the directed acyclic graph by induction:

$$\begin{split} \varepsilon(\emptyset) &= 0, \\ \varepsilon(P) &= 2^{\varepsilon(\varLambda_P)} & \text{for a node } P, \\ \varepsilon(M) &= \sum_P M(P) \varepsilon(P) & \text{for a marking } M. \end{split}$$

Note that leaves evaluate to 1, the evaluation of a marking is a real number, and the evaluation of a node P is a positive real number. Thus,  $\varepsilon(P)$  and  $\varepsilon(M)$  are well-defined. We have  $\varepsilon(\Lambda_P) = \log_2(\varepsilon(P))$ , thus the successor marking plays the role of a logarithm. We are interested only in power circuits where all markings evaluate to integers; equivalently all nodes evaluate to some positive natural number in  $2^{\mathbb{N}}$ .

The power circuit-representation of an integer sequence  $m_1, \ldots, m_k$  is given by a tuple  $(\Gamma, \delta; M_1, \ldots, M_k)$  where  $(\Gamma, \delta)$  is a power circuit and  $M_1, \ldots, M_k$  are markings such that  $\varepsilon(M_i) = m_i$ . (Hence, a single power circuit can store several different numbers; a fact which has been crucial in the proof of Proposition 9, see [8].)

Example 1. We can represent every integer in the range [-n,n] as the evaluation of some marking in a power circuit with node set  $\{P_{0,n},\ldots,P_{\ell}\}$  such that  $\varepsilon(P_i)=2^i$  for

 $0 \le i \le \ell$  and  $\ell = \lfloor \log_2 n \rfloor$ . Thus, we can convert the binary notation of an integer n into a power circuit with  $\mathcal{O}(\log |n|)$  vertices and  $\mathcal{O}((\log |n|) \log \log |n|)$  arcs.

Example 2. A power circuit of size n can realize  $\tau(n)$  since a chain of n nodes represents  $\tau(n)$  as the evaluation of the last node.

**Proposition 3** ([20,8]). The following operations can be performed in quadratic time. Input a power circuit  $(\Gamma, \delta)$  of size n and two markings  $M_1$  and  $M_2$ . Decide whether  $(\Gamma, \delta)$  is indeed a power circuit, i.e., decide whether all markings evaluate to integers. If "yes":

- Decide whether  $\varepsilon(M_1) \leq \varepsilon(M_2)$ .
- Compute a new power circuit with markings M, X and U such that
  - 1.  $\varepsilon(M) = \varepsilon(M_1) \pm \varepsilon(M_2)$ .
  - 2.  $\varepsilon(M) = 2^{\varepsilon(M_1)} \cdot \varepsilon(M_2)$ .
  - 3.  $\varepsilon(M_1) = 2^{\varepsilon(X)} \cdot \varepsilon(U)$  and either  $U = \emptyset$  or  $\varepsilon(U)$  is odd.

Let us mention that the complexity of the division problem in power circuits is open. Here, the division problem is as follows. Given a power circuit of size n and two markings  $M_1$  and  $M_2$ , decide whether  $\varepsilon(M_1) \mid \varepsilon(M_2)$ , i.e.,  $\varepsilon(M_1)$  divides  $\varepsilon(M_2)$ . We suspect that the division problem in power circuits is extremely difficult. The only known general algorithm transforms  $\varepsilon(M_1)$  and  $\varepsilon(M_2)$  first in binary and solves division after that. So, the first part involves a non-elementary explosion.

# 2 Conjugacy in the Baumslag-Solitar group $BS_{1,2}$

The solution of the conjugacy problem in the Baumslag group  $\mathbf{G}_{1,2}$  relies on the simpler solution for the Baumslag-Solitar group  $\mathbf{BS}_{1,2}$ . The aim of this section is to show that the conjugacy problem in  $\mathbf{BS}_{1,2}$  is  $\mathsf{TC}^0$ -complete. The group  $\mathbf{BS}_{1,2}$  is given by the presentation  $\langle a,t \mid tat^{-1}=a^2 \rangle$ . We have  $ta=a^2t$  and  $at^{-1}=t^{-1}a^2$ . This allows to represent all group elements by words of the form  $t^{-p}a^rt^q$  with  $p,q\in\mathbb{N}$  and  $r\in\mathbb{Z}$ . However, for  $q\geq 0$ , transforming  $t^qa^r$  into this form leads to  $a^st^q$  with  $s=2^qr$ , so the word  $a^st^q$  is exponentially longer than the word  $t^qa^r$ . We denote by  $\mathbb{Z}[1/2]=\{p/2^q\in\mathbb{Q}\mid p,q\in\mathbb{Z}\}$  the ring of dyadic fractions. Multiplication by 2 is an automorphism of the underlying additive group and therefore we can define the semi-direct product  $\mathbb{Z}[1/2]\rtimes\mathbb{Z}$  as follows. Elements are pairs  $(r,m)\in\mathbb{Z}[1/2]\times\mathbb{Z}$ . The multiplication in  $\mathbb{Z}[1/2]\rtimes\mathbb{Z}$  is defined by

$$(r,m) \cdot (s,q) = (r + 2^m s, m + q).$$

Inverses can be computed by the formula  $(r,m)^{-1}=(-r\cdot 2^{-m},-m)$ . It is straightforward to show that  $a\mapsto (1,0)$  and  $t\mapsto (0,1)$  defines an isomorphism between  $\mathbf{BS}_{1,2}$  and  $\mathbb{Z}[1/2]\rtimes\mathbb{Z}$ . In the following we abbreviate  $\mathbf{BS}_{1,2}$  (=  $\mathbb{Z}[1/2]\rtimes\mathbb{Z}$ ) by H. There are several options to represent a group element  $g\in H$ . In a unary representation we write g as a word over the alphabet with involution  $\{a, \overline{a}, t, \overline{t}\}$ . Another way is to write g=(r,m) with  $r\in\mathbb{Z}[1/2]$  and  $m\in\mathbb{Z}$ . In the following we use both notations interchangeably. The binary representation of (r,m) consists of r written in

binary (as floating point number) and m in unary. Let us write (r,m) with  $r=2^ks$  and  $k,s,m\in\mathbb{Z}$ . We then have  $(2^ks,m)=(0,k)\cdot(s,m-k)$  and the corresponding triple  $[k,s,m-k]\in\mathbb{Z}^3$  is called the *triple-representation* of (r,m); it is not unique. The *power circuit representation* of g=[k,s,m-k] is given by a power circuit and markings K, S, L such that  $\varepsilon(K)=k$ ,  $\varepsilon(S)=s$ , and  $\varepsilon(L)=m-k$ . Note that if  $g\in\{a,\overline{a},t,\overline{t}\}^n$  satisfies  $g=(r,m)\in H$ , then  $|r|\leq 2^n$  and  $|m|\leq n$ . Thus, a transformation from unary to binary notation is on the safe side.

**Proposition 4.** Let  $(r_1, m_1), \ldots, (r_n, m_n) \in \mathbb{Z}[1/2] \rtimes \mathbb{Z}$  given in binary representation for all i. Then there is a uniform construction of a  $\mathsf{TC}^0$ -circuit which calculates  $(r, m) = (r_1, m_1) \cdots (r_n, m_n)$  in  $\mathbb{Z}[1/2] \rtimes \mathbb{Z}$ .

*Proof.* The statements concerning computations in  $\mathsf{TC}^0$  are standard and can be found e.g. in the textbook [23]. Let  $N = \max\{m_i, \lfloor |\log_2 r_i| \rfloor + 1, n \mid 1 \le i \le n\}$ . Since the  $m_i$  are written in unary, we may assume for simplicity that all  $|m_i| \le 1$  (hence, requiring 2 bits) and all  $r_i$  are written in binary using exactly 2N bits (N bits for the mantissa and N for the exponent). Thus, we may assume that the input is a bit-string of length exactly  $2(N^2+N)$ . We have  $m=\sum_{i=1}^n m_i$ . By induction, using the equality  $(r,m)(s,q)=(r+s\cdot 2^m,m+q)$ , we see  $r=\sum_{i=1}^n r_i\cdot 2^{k_i}$  where  $k_i=\sum_{i=1}^{k-1} m_i$ . Since the numbers  $k_i$  are bounded by N, they can be calculated by the iterated addition of the unary numbers  $m_j$  for j < i, which is in  $\mathsf{TC}^0$ . In particular, m can be calculated by a  $\mathsf{TC}^0$ -circuit. The bit shift  $r_i\mapsto r_i\cdot 2^{k_i}$  can be computed by a  $\mathsf{TC}^0$ -circuit. It remains to calculate the iterated addition of binary numbers which is possible in  $\mathsf{TC}^0$ .

The next proof uses a deep result of Hesse: integer division is in uniform  $TC^0$ .

**Proposition 5.** Let  $f = (r, m), g = (s, q) \in \mathbb{Z}[1/2] \rtimes \mathbb{Z}$  be given in binary representation. Then there is a uniform construction of a  $\mathsf{TC}^0$ -circuit which decides  $f \sim_H g$ .

*Proof.* Let  $(r,m) \sim_H (s,q)$ , i.e., there are  $k \in \mathbb{Z}$ ,  $x \in \mathbb{Z}[1/2]$  with (x,k)(r,m) = (s,q)(x,k). In particular,  $(r,m) \sim_H (s,q)$  if and only if m=q and there are  $k \in \mathbb{Z}$ ,  $x \in \mathbb{Z}[1/2]$  such that

$$s = r \cdot 2^k - x \cdot (2^m - 1). \tag{1}$$

We have  $(r,m)\sim_H(s,m)$  if and only if  $(-r,-m)\sim_H(-s,-m)$  since  $(-p,-m)\sim_H(-p2^{-m},-m)=(p,m)^{-1}$  for all  $p\in\mathbb{Z}[1/2]$ . Therefore, without restriction  $m\in\mathbb{N}$ . Since a conjugation with  $t^k$  maps (r,m) to  $(2^kr,m)$ , we may assume that  $r,s\in\mathbb{Z}$  and  $m\in\mathbb{N}$ . For m=0 this means  $(r,0)\sim_H(s,0)$  if and only if there is some  $k\in\mathbb{Z}$  such that  $s=r\cdot 2^k$ . This can be decided in  $\mathrm{TC}^0$ . For m=1 we can choose x=r-s and the answer is "yes". For  $m\geq 2$  we can multiply (1) by  $2^\ell$  such that  $x\cdot 2^\ell\in\mathbb{Z}$ . We obtain  $2^\ell\cdot (r\cdot 2^k-s)=2^\ell x\cdot (2^m-1)$ , i.e.,  $2^\ell\cdot (r\cdot 2^k-s)\equiv 0 \mod (2^m-1)$ . The number 2 is invertible modulo  $2^m-1$  and its order is m. Hence, actually for  $m\geq 1$ :

$$(r,m) \sim_H (s,m) \iff \exists k \in \mathbb{N} : 0 \le k < m \land r \cdot 2^k - s \equiv 0 \mod (2^m - 1).$$
 (2)

It can be checked whether such a k exists using Hesse's result for division [12,13].  $\square$ 

**Theorem 6.** The word problem as well as the conjugacy problem in  $BS_{1,2}$  is  $TC^0$ -complete.

*Proof.* By Proposition 4 and Proposition 5, the conjugacy problem can be solved in  $TC^0$ . The word problem is a special instance of the conjugacy problem and the word problem in  $\mathbb{Z}$  is  $TC^0$ -hard in unary notation. This follows because the  $TC^0$ -hard problem *MAJORITY* (see [23]) reduces uniformly to the unary word problem in  $\mathbb{Z}$ .

Remark 7. Let us highlight that integer division can be reduced to the conjugacy problem in  $\mathbf{BS}_{1,2}$ . For  $m \geq 1$  we obtain as a special case of (2) and a well-known fact from elementary number theory

$$(0,m) \sim_H (2^s - 1, m) \iff 2^m - 1 \mid 2^s - 1 \iff m \mid s.$$
 (3)

If we allow a power circuit representation for integers, then this reduction from division to conjugacy can be computed in polynomial time. Hence, no elementary algorithm is known to solve the conjugacy problem in  $\mathbf{BS}_{1,2}$  in power circuit representation, whereas the word problem remains solvable in cubic time by [8].

# 3 Conjugacy in the Baumslag group $G_{1,2}$

The Baumslag group  $G_{1,2}$  is an HNN extension of the Baumslag-Solitar group  $BS_{1,2}$ . We make this explicit. We let  $\mathbf{BS}_{1,2}$  be our base group, generated by a and t. Again,  $\mathbf{BS}_{1,2}$  is abbreviated as H. The group H contains infinite cyclic subgroups  $A = \langle a \rangle$ and  $T = \langle t \rangle$  with  $A \cap T = \{1\}$ . Let b be a fresh letter which is added as a new generator together with the relation  $bab^{-1} = t$ . This defines the Baumslag group  $G_{1,2}$ . It is generated by a, t, b with defining relations  $tat^{-1} = a^2$  and  $bab^{-1} = t$ . However, the generator t is now redundant and we obtain  $G_{1,2}$  as a group generated by a, bwith a single defining relation  $bab^{-1}a = a^2bab^{-1}$ . We represent elements of  $G_{1,2}$ by  $\beta$ -factorizations. A  $\beta$ -factorization is written as a word  $z = \gamma_0 \beta_1 \gamma_1 \dots \beta_k \gamma_k$  with  $\beta_i \in \{b, \overline{b}\}$  and  $\gamma_i \in \{a, \overline{a}, t, \overline{t}\}^*$ . The number k is called the  $\beta$ -length and is denoted as  $|z|_{\beta}$  (i.e.,  $|z|_{\beta} = |z|_{b} + |z|_{\overline{b}}$ ). A transposition of a  $\beta$ -factorization  $z = \gamma_0 \beta_1 \gamma_1 \dots \beta_k \gamma_k$  is given as  $z' = \beta_i \gamma_i \dots \beta_k \gamma_k \gamma_0 \beta_1 \gamma_1 \dots \beta_{i-1} \gamma_{i-1}$  for some  $1 \le i \le k$ . Clearly,  $z \sim_{\mathbf{G}_{1,2}}$ z' in this case. Throughout we identify a power  $c^{-\ell}$  with  $\overline{c}^{\ell}$  for letters c and  $\ell \in \mathbb{N}$ . **Britton reductions.** A *Britton reduction* considers some factor  $\beta \gamma \overline{\beta}$  with  $\gamma \in \{a, \overline{a}, t, \overline{t}\}^*$ . There are two cases. First, if  $\beta = b$  and  $\gamma = a^{\ell}$  in H for some  $\ell \in \mathbb{Z}$  then the factor  $b\gamma \overline{b}$ is replaced by  $t^{\ell}$ . Second, if  $\beta = \overline{b}$  and  $\gamma = t^{\ell}$  in H for some  $\ell \in \mathbb{Z}$  then the factor  $\overline{b}\gamma b$ is replaced by  $a^{\ell}$ . At most  $|z|_{\beta}$  Britton reduction are possible on a word z. Be aware! There can be a non-elementary blow-up in the exponents, see Example 8. If no Britton reduction is possible, then the word x is called Britton-reduced. It is called cyclically Britton-reduced if xx is Britton-reduced. Britton reductions are effective because we can check whether  $\gamma = a^{\ell}$  (resp.  $\gamma = t^{\ell}$ ) in H. Thus, on input  $x \in \{a, \overline{a}, t, \overline{t}, b, \overline{b}\}^*$  we can effectively calculate a Britton-reduced word  $\hat{x}$  with  $x = \hat{x}$  in  $G_{1,2}$ . The following assertions are standard facts for HNN extensions, see [16]:

- 1. If x is Britton-reduced then  $x \in H$  if and only if  $|x|_{\beta} = 0$ .
- 2. If x is Britton-reduced and  $|x|_{\beta} = 0$  then x = 1 in  $G_{1,2}$  if and only if x = 1 in H.
- 3. Let  $\beta_1 \gamma_1 \dots \beta_k \gamma_k$  and  $\beta_1' \gamma_1' \dots \beta_k' \gamma_k'$  be  $\beta$ -factorizations of Britton-reduced words x and y such that  $k \geq 2$  and x = y in  $\mathbf{G}_{1,2}$ . Then we have k = k' and  $(\beta_1, \dots, \beta_k) = (\beta_1', \dots, \beta_{k'}')$ . Moreover,  $\gamma_1' \in \gamma_1 T$  if  $\beta_2 = b$  and  $\gamma_1' \in \gamma_1 A$  if  $\beta_2 = \overline{b}$ .

Example 8. Define words  $w_0 = t$  and  $w_{n+1} = b w_n a \overline{w_n} \overline{b}$  for  $n \ge 0$ . Then we have  $|w_n| = 2^{n+2} - 3$  but  $w_n = t^{\tau(n)}$  in  $G_{1,2}$ .

The power circuit-representation of a  $\beta$ -factorization  $\gamma_0\beta_1\gamma_1\ldots\beta_k\gamma_k$  is the sequence  $(\beta_1,\ldots,\beta_k)$  and a power circuit  $(\Gamma,\delta)$  together with a sequence of markings  $K_0,S_0,L_0,\ldots,K_k,S_k,L_k$  such that  $[\varepsilon(K_i),\varepsilon(S_i),\varepsilon(L_i)]=[k_i,s_i,\ell_i]$  is the triple representation of  $\gamma_i\in H$  for  $1\leq i\leq k$ . It is known that the word problem of  $\mathbf{G}_{1,2}$  is decidable in cubic time. Actually a more precise statement holds.

**Proposition 9** ([20,8]). There is a cubic time algorithm which computes on input of a power circuit representation of  $x = \gamma_0 \beta_1 \gamma_1 \dots \beta_k \gamma_k$  a power circuit representation of a Britton-reduced word (resp. cyclically Britton-reduced word)  $\hat{x}$  such that  $x = \hat{x}$  in  $\mathbf{G}_{1,2}$  (resp.  $x \sim_{\mathbf{G}_{1,2}} \hat{x}$ ). Moreover, the size for the power circuit representation of  $\hat{x}$  is linear in the size of the power circuit representation of x.

*Remark 10.* A polynomial time algorithm for the result in Proposition 9 has been given first in [20], it has been estimated by  $\mathcal{O}(n^7)$ . This was lowered in [8] to cubic time.

**Theorem 11.** The following computation can be performed in time  $\mathcal{O}(n^4)$ . Input: words  $x,y \in \{a,\overline{a},b,\overline{b}\}^*$ . Decide whether  $|\widehat{x}|_{\beta} > 0$  for a cyclically Britton-reduced form  $\widehat{x}$  of x. If "yes", decide  $x \sim_{\mathbf{G}_{1,2}} y$  and, in the positive case, compute a power circuit representation of some z such that  $zx\overline{z} = y$  in  $\mathbf{G}_{1,2}$ .

*Proof.* Due to Proposition 9, we may assume that input words x and y are given as cyclically Britton-reduced words. In particular,  $\widehat{x}=x$  and  $|\widehat{x}|_{\beta}=n>0$ . Let us write  $x=\gamma_0b^{\varepsilon_1}\gamma_1\dots b^{\varepsilon_n}\gamma_n$  as its  $\beta$ -factorization where  $\varepsilon_i=\pm 1$ . If all  $\varepsilon_i=+1$  then we replace x and y by  $\overline{x}$  and  $\overline{y}$ . Hence, without restriction there exists some  $\varepsilon_i=-1$ . After a possible transposition we may assume that  $x=b^{\varepsilon_1}\gamma_1\dots b^{\varepsilon_n}\gamma_n$  with  $\varepsilon_1=-1$ . Since y is cyclically Britton-reduced, too, Collins' Lemma ([16, Thm. IV.2.5]) tells us several things: If  $x\sim_{\mathbf{G}_{1,2}}y$  then  $|y|_{\beta}=n$  and after some transposition the  $\beta$ -factorization of y can be written as  $b^{\varepsilon_1}\gamma_1'\dots b^{\varepsilon_n}\gamma_n'$ . Moreover, still by Collins' Lemma, we now have  $x\sim_{\mathbf{G}_{1,2}}y\iff\exists k\in\mathbb{Z}:y=a^kxa^{-k}$  in  $\mathbf{G}_{1,2}$ . The key is that k is unique and that we find an efficient way to calculate it.  $\delta$ 

Case n=1. We have  $x=\overline{b}(r,m)$  and  $y=\overline{b}(s,q)$  for some  $(r,m),(s,q)\in\mathbb{Z}[1/2]\rtimes\mathbb{Z}$ . Now,  $a^kx=ya^k$  in  $\mathbf{G}_{1,2}$  if and only if (0,k)(r,m)=(s,q)(k,0). This forces k=q-m. Hence

$$x \sim_{\mathbf{G}_{1,2}} y \iff 2^{q-m}r = s + 2^q(q-m) \text{ for } n = 1.$$
 (4)

Case  $n \geq 2$  and  $\varepsilon_2 = +1$ . Then  $x = \overline{b}(r,m)b\gamma_2\cdots b^{\varepsilon_n}\gamma_n$  and  $y = \overline{b}(s,q)b\gamma_2'\cdots b^{\varepsilon_n}\gamma_n'$ . We have  $r \neq 0 \neq s$  since x and y are Britton-reduced. For every  $k \in \mathbb{Z}$  and every Britton-reduced  $\beta$ -factorization  $\overline{b}\widetilde{\gamma}_1b\dots b^{\varepsilon_n}\widetilde{\gamma}_n$  for  $a^kx\overline{a}^k$  we have  $\widetilde{\gamma}_1 \in t^k(r,m)T$ , and hence  $\widetilde{\gamma}_1 = (2^kr,p)$  for some  $p \in \mathbb{Z}$ . We conclude that there is a unique  $k \in \mathbb{Z}$  such that  $a^kx\overline{a}^k = \overline{b}(2^kr,p)b\cdots b^{\varepsilon_n}\widetilde{\gamma}_n \in \mathbf{G}_{1,2}, p \in \mathbb{Z}$ , and  $2^kr$  is an odd integer. This means we may assume from the very beginning that r and s are odd integers. Under this assumption, if  $a^kxa^{-k} = y$  in  $\mathbf{G}_{1,2}$  then necessarily k = 0 and hence x = y in  $\mathbf{G}_{1,2}$ . We obtain the following algorithm to decide  $x \sim_{\mathbf{G}_{1,2}} y$ .

<sup>&</sup>lt;sup>5</sup> Beese calculates in [3] this value *k* and computes certain normal forms which are checked for equivalence. This leads to an exponential time algorithm.

- For  $\gamma_1=(r,m)$  and  $\gamma_1'=(s,q)$  calculate unique  $k,\ell\in\mathbb{Z}$  such that  $2^kr$  and  $2^\ell s$  are odd integers.
- Decide whether  $a^k x \overline{a}^k = a^\ell y \overline{a}^\ell \in \mathbf{G}_{1,2}$ . If "yes" then  $x \sim_{\mathbf{G}_{1,2}} y$  otherwise  $x \not\sim_{\mathbf{G}_{1,2}} y$ .

Case  $n \geq 2$  and  $\varepsilon_2 = -1$ . Then  $x = \overline{b}(r,m)\overline{b}\,\gamma_2\cdots b^{\varepsilon_n}\gamma_n$  and  $y = \overline{b}(s,q)\overline{b}\,\gamma_2'\cdots b^{\varepsilon_n}\gamma_n'$ . For every  $k \in \mathbb{Z}$  we can write  $a^kx\overline{a}^k$  in some Britton-reduced form which looks like  $\overline{b}\,\widetilde{\gamma}_1\overline{b}\cdots b^{\varepsilon_n}\widetilde{\gamma}_n$ . Now,  $\widetilde{\gamma}_1 \in t^k(r,m)A$ . Thus, there is a unique  $k \in \mathbb{Z}$  (necessarily k = -m) such that  $\widetilde{\gamma}_1 = (p,0)$  for some  $p \in \mathbb{Z}[1/2]$ . Using the same arguments as above, we obtain the following algorithm. For  $\gamma_1 = (r,m)$  and  $\gamma_1' = (s,q)$  decide whether  $a^{-m}xa^m = a^{-q}ya^q \in \mathbf{G}_{1,2}$ . If "yes" then  $x \sim_{\mathbf{G}_{1,2}} y$  otherwise  $x \not\sim_{\mathbf{G}_{1,2}} y$ .

By Proposition 9, the tests  $a^k x \overline{a}^k = y \in \mathbf{G}_{1,2}$  can be performed in cubic time. All other computations can be done in quadratic time by Proposition 4. Since all transpositions of the  $\beta$ -factorization for y have to be considered this yields an  $\mathcal{O}(n^4)$ -algorithm.

For the remainder of the section the situation is as follows: We have  $x=(r,m)\in \mathbb{Z}[1/2]\rtimes \mathbb{Z}$  and  $y=(s,q)\in \mathbb{Z}[1/2]\rtimes \mathbb{Z}$ , both can be assumed to be in power circuit representation. We may assume  $x\neq 1\neq y$  in  $G_{1,2}$ . After conjugation with some  $t^k$  where k is large enough we may assume that  $r,m,s,q\in \mathbb{Z}$ . If m=0 then we replace x by  $bx\bar{b}$ . Hence,  $m\neq 0$  and, by symmetry,  $q\neq 0$ , too. By (2) and "division in power circuits", we are able to to test whether  $(r,m)\sim_H(0,m)$  and  $(s,q)\sim_H(0,q)$ . Assume that one of the answers is "no". Say,  $(r,m)\not\sim_H(0,m)$ . Then there is no  $h\in A\cup T\subseteq H$  such that  $(r,m)\sim_H h$ . Since then  $\beta\gamma(r,m)\overline{\gamma}\overline{\beta}$  is Britton-reduced for all  $\beta\in \{b,\overline{b}\}$ ,  $\gamma\in \{a,\overline{a},t,\overline{t}\}^*$  we obtain:

**Proposition 12.** Let  $r, m \in \mathbb{Z}$ ,  $m \neq 0$ . If  $(r, m) \not\sim_H (0, m)$  then

$$(r,m) \sim_{\mathbf{G}_{1,2}} (s,q) \iff (r,m) \sim_H (s,q).$$

By Proposition 12, we may assume  $(r,m) \sim_H (0,m)$ ,  $(s,q) \sim_H (0,q)$ , and  $(r,m) \not\sim_H (s,q)$ . This involves perhaps non-elementary procedures. However, it remains to decide  $(0,m) \sim_{\mathbf{G}_{1,2}} (0,q)$ , only. The last test is polynomial time again, even for power circuits.

**Proposition 13.** Let  $m, q \in \mathbb{Z}$ . Then we have

$$(0,m) \sim_{\mathbf{G}_{1,2}} (0,q) \iff (m,0) \sim_H (q,0) \iff \exists k \in \mathbb{Z} : m = 2^k q.$$

Proof. The assertion  $(m,0) \sim_H (q,0) \iff \exists k \in \mathbb{Z} : m = 2^k q$  is clear since  $(m,0) = a^m$  and  $(q,0) = a^q$  in  $H = \mathbf{BS}_{1,2}$ . Let  $(0,m) \sim_{\mathbf{G}_{1,2}} (0,q)$ . We have to show  $(m,0) \sim_H (q,0)$  since the other direction is trivial. We have  $(q,0) \sim_{\mathbf{G}_{1,2}} (0,q)$ . Let  $\gamma_0 b^{\varepsilon_1} \gamma_1 \cdots b^{\varepsilon_n} \gamma_n$  be a β-factorization of some z with  $n \in \mathbb{N}$  minimal such that  $\overline{z}(q,0)z = (0,m)$ . Since  $\overline{\gamma_0}(q,0)\gamma_0 = (p,0)$  for some  $p \neq 0$ , we have  $n \geq 1$  and  $\varepsilon_1 = -1$  because there has to occur a Britton reduction. Thus,  $b\overline{\gamma_0}(q,0)\gamma_0\overline{b} = t^p$  in  $\mathbf{G}_{1,2}$ . Now,  $\overline{\gamma_1}(0,p)\gamma_1 \in A \cup T$  if and only if  $\overline{\gamma_1}(0,p)\gamma_1 = (0,p)$ . Thus, we may assume  $\gamma_1 = 1$  in H. Since n is minimal we cannot have  $\varepsilon_2 = +1$ . Thus, we must have n = 1 and we may choose  $z = \gamma \overline{b}$  for some  $\gamma \in H$ . This means  $\overline{z}(q,0)z = b\overline{\gamma}(q,0)\gamma \overline{b} = (0,m)$  which implies  $(m,0) \sim_H (q,0)$ .

**Corollary 14.** The following problem is decidable in at most non-elementary time. Input: Power circuit representations x, y for elements of  $G_{1,2}$ . Question:  $x \sim_{G_{1,2}} y$ ?

**Corollary 15.** If there is no elementary algorithm to solve the division problem in power circuits then the conjugacy problem in the Baumslag group  $G_{1,2}$  is non-elementary in the average case even for a unary representation of group elements.

Proof. Assume that the conjugacy problem in the Baumslag group  $\mathbf{G}_{1,2}$  is elementary on the average. We give an elementary algorithm to solve division in power circuits. Let  $(\Gamma, \delta)$  be a power circuit of size n with markings M and S such that  $\varepsilon(M) = m$  and  $\varepsilon(S) = s$ . For each node in  $P \in \Gamma$  it is easy to construct a word  $w(P) \in \left\{a, \overline{a}, b, \overline{b}\right\}^*$  such that  $t^{\varepsilon(P)} = w(P)$  in  $\mathbf{G}_{1,2}$  and  $|w(P)| \leq n^n$ . Just follow the scheme from Example 8. Hence, in time  $2^{\mathcal{O}(n\log n)}$  we can construct words x and y such that x = (0, m) and  $y = (2^s - 1, m)$  in  $\mathbf{G}_{1,2}$ . Now by Remark 7 we have  $m \mid s$  if and only if  $x \sim_{\mathbf{G}_{1,2}} y$ . The number of words of length  $2^{\mathcal{O}(n\log n)}$  is at most  $2^{2^{\mathcal{O}(n\log n)}}$ .

## 4 Generic case analysis

Let us define a preorder between functions from  $\mathbb{N}$  to  $\mathbb{R}^{\geq 0}$  as follows. We let  $f \leq g$  if there exist  $k \in \mathbb{N}$  and  $\varepsilon > 0$  such that for almost all n we have

$$f(n) \le n^k g(n) + 2^{-\varepsilon n}$$
.

Moreover, we let  $f \approx g$  if both,  $f \leq g$  and  $g \leq f$ . We are mainly interested in functions  $f \approx 0$ . These functions form an ideal in the ring of functions which are bounded by polynomial growth. Moreover, if  $f \approx 0$  then  $g \approx 0$  for  $g(n) \in f(\theta(n))$ . The notion  $f \approx g$  is therefore rather flexible and simplifies some formulae. We consider cyclically reduced words over  $\mathcal{L} = \{a, \overline{a}, b, \overline{b}\}$  of length n with uniform distribution. This yields a function  $p(n) = \Pr\left[\exists y : x \sim_{\mathbf{G}_{1,2}} y \land y \in H\right]$ . We prove  $p(n) \approx 0$ . More precisely, we are interested in the following result.

**Theorem 16.** There is a strongly generic algorithm that decides in time  $\mathcal{O}(n^4)$  on cyclically reduced input words  $x, y \in \{a, \overline{a}, b, \overline{b}\}^*$  with  $|xy| \in \theta(n)$  whether  $x \sim_{\mathbf{G}_{1,2}} y$ .

In the preceding section we have described the algorithm for the conjugacy problem. Hence, it remains to show that it runs strongly generically in  $\mathcal{O}(n^4)$ . We give two proofs of Theorem 16. The first one is given in Section 4.1. It uses a pairing by Dyck words. It is a little bit tedious, but self-contained and elementary. The second proof is given in Section 5. It is based on a more general characterization which applies to all finitely generated HNN extensions, see Theorem 20. To the best of our knowledge this characterization has not been stated elsewhere. The proof is not very hard, but in order to derive Theorem 16 we need additional results from the literature.

#### 4.1 Pairing with Dyck words: First proof of Theorem 16

*Proof.* By Theorem 11, there is an algorithm deciding  $x \sim_{\mathbf{G}_{1,2}} y$  which runs in time  $\mathcal{O}(n^4)$  for inputs which cannot be conjugated to elements in H. Hence, we only have

to bound the number of cyclically reduced words of length  $m \in \theta(n)$  which can be conjugated to some element in H. For simplicity of notation we assume m=n. A reduced word in  $\Sigma^n$  can be identified with a random walk without backtracking in the Cayley graph of  $G_{1,2}$  with generators a and b. We encode reduced words over  $\Sigma$ of length n in a natural way as words in  $\Omega = \Sigma \cdot \{1, 2, 3\}^{n-1}$ . On  $\Omega$  we choose a uniform probability (e.g., if the i-th letter is b then the i+1-st letter is a,  $\bar{a}$ , or b with equal probability 1/3). Because we are interested in conjugacy, we compute the probability under the condition that  $x \in \Omega$  is cyclically reduced. (Actually this does not change the results but makes the analysis smoother.) The probability that  $x \in \Omega$ is cyclically reduced is at least 2/3 for all n. Let  $C \subseteq \Omega$  be the subset of cyclically reduced words. We show  $\Pr\left[\exists y: x \sim_{\mathbf{G}_{1,2}} y \land y \in H \mid x \in C\right] \approx 0$ . The question whether there exists some y with  $x \sim_{\mathbf{G}_{1,2}} y$  is answered by calculating Britton reductions for a transposition of x. The set C is closed under transpositions and it is no restriction to assume that  $|x|_{\beta} \geq 1$ . Therefore, we can choose the transposition that x' = vu where x = uv such that the first letter of x' is  $\beta \in \{b, \overline{b}\}$ . There are at most n such transpositions. Hence,

$$\Pr\left[\exists y: x \sim_{\mathbf{G}_{1,2}} y \land y \in H \mid x \in C\right] \approx \Pr\left[x \in H \mid x \in C\right]$$
$$= \Pr\left[x \in H \land x \in C\right] \cdot \Pr\left[x \in C\right]^{-1} \le \Pr\left[x \in H\right] \cdot \Pr\left[x \in C\right]^{-1} \le \frac{3}{2} \Pr\left[x \in H\right].$$

It is therefore enough to prove  $\Pr[x \in H] \approx 0$ . We switch the probability space and we embed  $\Omega$  into the space  $\Sigma^*$  with a measure  $\mu_{0,n}$  on  $\Sigma^*$  which concentrates on  $\Omega$ , i.e.,  $\mu_{0,n}(\Omega)=1$ . Within  $\Omega$  we still have a uniform distribution for  $\mu_{0,n}$ . In order to emphasize this change of view, we write  $\Pr[\cdots]=\Pr_{0,n}[\cdots]$ . We are now interested in words  $x \in \{b, \overline{b}\} \cdot \Sigma^*$  which contain exactly 2m letters  $\beta \in \{b, \overline{b}\}$  for  $m \geq 1$ . (The number  $|x|_{\beta}$  must be even if  $x \in H$ .) Each such word can be written as a  $\beta$ -factorization of the form  $x=\beta_1\alpha_1\dots\beta_{2m}\alpha_{2m}$  where  $\alpha_i=a^{e_i}$  with  $e_i\in\mathbb{Z}$ . This defines a new measure  $\mu_m$  on  $\Sigma^*$  which is defined as follows. We start a random walk without backtracking with either b or  $\overline{b}$  with equal probability. For the next letter there are always 3 possibilities, each is chosen with probability 1/3. We continue as long as the random walk contains at most 2m letters from  $\{b, \overline{b}\}$ . This gives a corresponding probability on  $\Sigma^*$  which is concentrated on those words with  $|x|_{\beta}=2m$ . We denote the corresponding probability by  $\Pr_m[\cdots]$ . In order to switch from  $\Pr_{0,n}[\cdots]$  to  $\Pr_m[\cdots]$  we consider the block structure B(x) of a word  $x \in \{b, \overline{b}\} \cdot \Sigma^*$ . We define B(x) as the tuple  $(e_1, e'_1, \dots, e_k, e'_k)$  for  $x = \beta_1^{e_1} \alpha_1^{e'_1} \cdots \beta_k^{e_k} \alpha_k^{e'_k}$  where  $e_i, e'_i > 0$ , with the exception that possibly  $e'_k = 0$ ,  $\beta_i \in \{b, \overline{b}\}$ , and  $\alpha_i \in \{a, \overline{a}\}$ .

Let  $\widetilde{E}_{k,m} = \left\{ (e_1,e'_1,\dots,e_k,e'_k) \;\middle|\; \sum_{i=1}^k e_i = 2m \land \sum_{i=1}^k e'_i = n-2m \right\}$ . For each  $\widetilde{e} \in \widetilde{E}_{k,m}$  we obtain  $\Pr_{0,n} \left[ B(x) = \widetilde{e} \right] \in \theta(2^{2k}3^{-n})$  and  $\Pr_m \left[ B(x) = \widetilde{e} \right] \in \theta(2^{2k}3^{-n})$ . In particular, we have  $\sum_{m=0}^{\lfloor n/4 \rfloor} \sum_k \sum_{\widetilde{e} \in \widetilde{E}_{k,m}} \Pr_{0,n} \left[ B(x) = \widetilde{e} \right] \leq n2^n3^{-n} \approx 0$  because  $k \leq 2m$  for  $\widetilde{e} \in \widetilde{E}_{k,m}$ . Moreover,  $\Pr_0 \left[ x \in H \mid B(x) = \widetilde{e} \right] = \Pr_m \left[ x \in H \mid B(x) = \widetilde{e} \right]$ . Indeed, both values are equal to  $2^{-2k}$  for  $e'_k > 0$  and equal to  $2^{1-2k}$  for  $e'_k = 0$ . This

yields:

$$\begin{split} \Pr_{0,n}\left[x\in H\right] &\approx \sum_{m=\lceil n/4\rceil}^{n} \sum_{k} \sum_{\widetilde{e}\in \widetilde{E}_{k,m}} \Pr_{0,n}\left[x\in H \wedge B(x) = \widetilde{e}\right] \\ &\approx \sum_{m=\lceil n/4\rceil}^{n} \sum_{k} \sum_{\widetilde{e}\in \widetilde{E}_{k,m}} \Pr_{m}\left[x\in H \wedge B(x) = \widetilde{e}\right] \\ &= \sum_{m=\lceil n/4\rceil}^{n} \Pr_{m}\left[x\in H \wedge |x| = n\right] \leq \sum_{m=\lceil n/4\rceil}^{n} \Pr_{m}\left[x\in H\right] \\ &\approx \Pr_{\lceil n/4\rceil}\left[x\in H\right] \approx \ 0 \text{ by Lemma 17}. \end{split}$$

Hence, the proof of Theorem 16 is reduced to show Lemma 17.

From now on we work with the measure  $\mu_n$  and the corresponding probability  $\Pr_n[\cdots]$  for  $n \geq 1$ . Thus, we may assume that our probability space contains only those words x which have  $\beta$ -factorizations of the form  $x = \beta_1 \alpha_1 \dots \beta_{2n} \alpha_{2n}$  with  $\alpha_i \in a^{\mathbb{Z}}$ . The following result is the main lemma for the analysis of the generic case.

**Lemma 17.** We have 
$$\Pr_n [x \in H] \le (8/9)^n$$
.

The proof of Lemma 17 is based on a "pairing" with Dyck words: Define a new alphabet  $B = \{|,|\}$  where | is an opening left-bracket and | is the corresponding closing right-bracket. The set of Dyck words  $D_n$  is the set of words in  $B^{2n}$  with correct bracketing. The number of Dyck words is well-understood, we have  $|D_n| = \frac{1}{n+1} {2n \choose n} \le$  $4^n$ . Thus,  $|D_n| = C_n$ , where  $C_n$  is the *n*-th Catalan number. The connection between Dyck words and Britton reductions is as follows. Britton reductions are defined for words  $\{a, \overline{a}, t, \overline{t}, b, \overline{b}\}^*$ . Consider a  $\beta$ -factorization of the form  $x = \beta_1 \alpha_1 \dots \beta_{2n} \alpha_{2n}$ with  $\alpha_i \in a^{\mathbb{Z}}$ . If  $x \in H$ , then there exists a sequence of Britton reductions which transforms x into  $\hat{x} \in \{a, \bar{a}, t, \bar{t}\}^*$ . We call such a sequence a successful Britton reduction. Every successful Britton reduction defines in a natural way a Dyck word by assigning an opening bracket to position i and a closing bracket to position j if  $\beta_i u \beta_j$ is replaced by a Britton reduction. Moreover, Britton reductions are confluent on H. In particular, this means that for  $x \in H$  we can start a successful Britton reduction by replacing all factors  $\beta_i a^e \beta_{i+1}$  with  $\beta_i = b = \beta_{i+1}$  and  $e \in \mathbb{Z}$  by  $t^e$  where  $1 \le i < 2n$ . Thus, if such a successful Britton reduction is described by d, then we may assume that  $d_i d_{i+1} = \lfloor \rceil$  whenever  $\beta_i a^e \beta_{i+1} = b a^e \overline{b}$ . Vice versa, if  $d_i d_{i+1} = \lfloor \rceil$ , then we must have  $\beta_i = b = \overline{\beta_{i+1}}$ , otherwise d is no description of any Britton reduction for x at all. Note that for each i with  $d_i = |$  there is exactly one j which matches  $d_i$ . The characterization of j is that  $d_{i+1}\cdots d_{j-1}$  is a Dyck word and  $d_j=$  . If d describes a Britton reduction for x and (i,j) is a matching pair for d then  $\beta_i\overline{\beta_j}=\beta\,\overline{\beta}$  for some  $\beta\in\{b,\overline{b}\}$ . We therefore say that x and d match if the following two conditions are satisfied:

- 1. For all  $1 \leq i < 2n$  we have  $d_i d_{i+1} = \lfloor \rceil \iff \beta_i \beta_{i+1} = b \, \overline{b}$ .

  2. For all  $1 \leq i < j \leq 2n$  where  $d_i d_j = \lfloor \rceil$  is a matching pair we have  $\beta_i \beta_j = \beta \, \overline{\beta}$ .

We define  $\langle x, d \rangle_{\beta} = 1$  if x and d match and  $\langle x, d \rangle_{\beta} = 0$  otherwise. We refine this pairing by defining  $\langle x, d \rangle = 1$  if  $\langle x, d \rangle_{\beta} = 1$  and d describes a successful Britton reduction proving  $x \in H$ . Otherwise we let  $\langle x, d \rangle = 0$ . Clearly,

$$\Pr_{n} [x \in H] \le \sum_{d \in D_{n}} \Pr_{n} [\langle x, d \rangle = 1].$$
 (5)

Since  $|D_n| \leq 4^n$ , the proof of Lemma 17 reduces to show that for every  $d \in D_n$  we have

$$\Pr_n\left[\langle x\,,\,d\rangle=1\right] \le (2/9)^n. \tag{6}$$

**Lemma 18.** Let  $d \in D_n$  be a Dyck word and  $k = |\{i \mid d_i d_{i+1} = \lfloor \}\}|$ . Then we have  $\Pr_n[\langle x, d \rangle_\beta = 1] \leq (2/3)^{n-k}(2/9)^k$ .

*Proof.* Let x be given as its  $\beta$ -factorization  $x=\beta_1\alpha_1,\ldots,\beta_{2n}\alpha_{2n}$ . In order to compute  $\langle x\,,\,d\rangle_{\beta}$ , we scan  $d=d_1\cdots d_{2n}$  from left to right with  $d_i\in\{\lfloor\rceil\}$ . We stop at each j where  $d_j=\lceil$ . Let i be the corresponding index such that  $d_id_j$  is a matching pair in the Dyck word d. We have i< j. For fixed j, the probability that  $\beta_j=\overline{\beta_i}$  depends on  $\beta_{j-1}$ , only. We have  $\Pr_n\left[\beta_j=\overline{\beta_i}\mid\beta_{j-1}=\beta_i\right]=1/3$  and  $\Pr_n\left[\beta_j=\overline{\beta_i}\mid\overline{\beta_{j-1}}=\beta_i\right]=2/3$ . Thus,  $\Pr_n\left[\beta_j=\overline{\beta_i}\right]\leq 2/3$ . Moreover, for j=i+1 we obtain  $\Pr_n\left[\beta_j=\overline{\beta_i}\right]=1/3$ . Now,  $\Pr_n\left[\langle x\,,\,d\rangle_{\beta}=1\right]$  implies in addition that for j=i+1 we must have  $\beta_i=b$ . In that case we calculate

$$\Pr_n\left[\beta_i = b \land \beta_{i+1} = \overline{b}\right] = \Pr_n\left[\beta_{i+1} = \overline{b} \mid \beta_i = b\right] \Pr_n\left[\beta_i = b\right] \le (1/3) \cdot (2/3).$$

The result follows.

**Lemma 19.** Let  $d \in D_n$  be a Dyck word and  $k = |\{i \mid d_i d_{i+1} = |\}\}|$ . Then we have

$$\Pr_k \left[ \langle x, d \rangle = 1 \mid \langle x, d \rangle_\beta = 1 \right] \le (5/16)^{n-k}.$$

*Proof.* For real valued random variables X we let  $\|X\| = \sqrt{\sum_{k \in \mathbb{Z}} \Pr{[X = k]^2}}$ . Let us consider first an integer valued random variable X which is given by some word of the form  $u\beta a^X\beta'v$ . The distribution  $\Pr{[X = k]}$  depends on  $\beta, \beta'$ , only. If  $\beta = \beta'$  then  $\Pr{[X = k]} = \frac{3^{-|k|}}{2}$  for  $k \in \mathbb{Z}$ . If  $\beta \neq \beta'$  then  $\Pr{[X = 0]} = 0$  and  $\Pr{[X = k]} = 3^{-|k|}$  for  $k \neq 0$ . Thus, if  $\beta = \beta'$  then  $\|X\|^2 = 5/16$ ; and if  $\overline{\beta} = \beta'$  then  $\|X\|^2 = 1/4$ . Hence:

$$||X||^2 \le 5/16.$$
 (7)

Next, consider a word of the form  $u\beta a^X\beta'w\beta''a^Y\overline{\beta}v$  with  $\beta,\beta',\beta''\in\{b,\overline{b}\}$  under the assumption that  $\beta'w\beta''=(r,m)$  in  $\mathbf{G}_{1,2}$  where  $(r,m)\in\mathbb{Z}[1/2]\rtimes\mathbb{Z}=H$ . The random variables X and Y are independent and define another random variable Z (with values in  $\mathbb{Z}[1/2]$ ) by the equation  $(X,0)\cdot(r,m)\cdot(Y,0)=(Z,m)$  in  $\mathbf{BS}_{1,2}$ , i.e.,  $Z=X+r+2^mY$ . Hence, for  $k\in\mathbb{Z}$  we obtain

$$\Pr[Z = k] = \sum_{i \in \mathbb{Z}} \Pr[X = i] \Pr[Y = 2^{-m}(k - r - i)].$$
 (8)

Note that  $\Pr[Y=2^{-m}(k-r-i)]=0$  unless  $2^{-m}(k-r-i)\in\mathbb{Z}$ . The numbers  $m,k,r\in\mathbb{Z}$  are fixed and  $2^{-m}(k-r-i)=2^{-m}(k-r-j)$  implies i=j.

Thus, we can define a new random variable Y' with the distribution  $\Pr[Y'=i] = \Pr[Y=2^{-m}(k-r-i)]$ . By (8) and Cauchy-Schwarz inequality

$$\Pr\left[Z=k\right] = \sum_{i \in \mathbb{Z}} \Pr\left[X=i\right] \Pr\left[Y'=i\right] \le \|X\| \|Y'\|.$$

Since  $||Y'|| \le ||Y||$ , we obtain  $\Pr[Z = k] \le ||X|| ||Y||$ . Finally, by (7)

$$\Pr\left[Z = k\right] \le 5/16. \tag{9}$$

Now, let  $d=d_1\cdots d_{2n}$  with  $d_i\in B$  be a Dyck word and consider indices i< j-1 such that (i,j) is a matching pair. (This means  $d_id_j=\lfloor\rceil$  and  $d_{i+1}\cdots d_{j-1}$  is a nonempty Dyck word.) Let  $n'=\frac{j-i+1}{2}$  and  $d'=d_{i+1}\cdots d_{j-1}$ . Next, we claim that

$$\Pr_{n'} \left[ \langle x, d_i d' d_j \rangle = 1 \mid \langle y, d' \rangle = 1 \land x = \overline{b} y b \right] \le 5/16. \tag{10}$$

Note that (10) refers to the measure  $\mu_{n'}$  and thus, x runs over those reduced words in  $\Sigma^*$  with  $|x|_{\beta}=2n'$ . In order to see this inequality, consider a word  $\overline{b}yb$  such that  $\langle y\,,\,d'\rangle=1$ . The word y must contain two positions where letters from  $\{b,\overline{b}\}$  appear because j>i+1. Thus, we can write  $y=\overline{b}a^X\beta w\beta'a^Yb$  such that  $\beta w\beta'=(r,m)$  in  $\mathbf{G}_{1,2}$ ; and we can read X and Y as integer valued random variables as before. For the derived random variable Z defined by  $Z=X+r+2^mY$  we obtain  $\Pr\left[Z=0\right]\leq 5/16$  by (9). But  $\Pr\left[Z=0\right]$  is equal to  $\Pr_{n'}\left[\langle\beta y\tilde{\beta}\,,\,d_id'd_j\rangle=1\;\Big|\;\langle y\,,\,d'\rangle=1\wedge\beta\tilde{\beta}=\overline{b}b\right]$ . Hence, the claim.

The other situation considers words of the form  $x=by\bar{b}$ . Again, we want to show

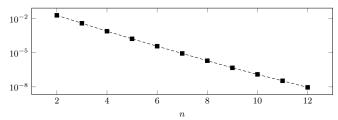
$$\Pr_{n'} \left[ \langle x, d_i d' d_j \rangle = 1 \mid \langle y, d' \rangle = 1 \land x = by \overline{b} \right] \le 5/16. \tag{11}$$

This is a more complicated situation and we need a case distinction about the structure of  $d'=d_{i+1}\cdots d_{j-1}$ . We let k denote the index which matches i+1 and  $\ell$  matches the index j-1. For  $\langle by\bar{b}\ ,\ d_id'd_j\rangle_\beta=1$ , we can write  $by\bar{b}=ba^e\beta u\bar{\beta}y''\bar{b}$ . (Throughout we let  $\beta\in\{b,\bar{b}\}$  and  $u,v,w,y\in\Sigma^*$ ). But actually more is true. Assume  $\beta=\bar{b}$  then index i must match index i+1, but here we have i+1< j, a contradiction. Hence, we conclude  $\beta=b$ . By symmetry, it follows that we can write  $by\bar{b}=ba^ebw\bar{b}a^f\bar{b}$ .

Case k>i+2. In this case we consider words  $by\bar{b}$  which can be written as  $by\bar{b}=ba^eba^X\beta u\beta'a^Y\bar{b}v\bar{b}$  such that  $\langle ba^X\beta u\beta'a^Y\bar{b}$ ,  $d_{i+1}\cdots d_k\rangle=1$ . This implies  $\beta u\beta'=(r,0)\in\mathbb{Z}[1/2]\rtimes\mathbb{Z}=H$  and  $v=(s,q)\in H$ . Here, X and Y are random variables as above. In this setting,  $\langle by\bar{b}$ ,  $d_id_j\rangle=1$  forces Z=0 where Z=X+r+Y-q. Inequality (9) yields  $\Pr[Z=0]\leq 5/16$ . This shows (11) in the case k>i+2.

Case  $\ell < j-2$ . Symmetric to the precedent case.

Case k=i+2 and  $\ell=j-2$ . We claim that this implies  $k<\ell$ . Indeed, assume  $\ell \leq k$  then we must have  $i+1=\ell$  and therefore i+1=j-2. Thus,  $d'=d_{i+1}d_{i+2}$ . But then  $\langle by\bar{b}\,,\,d_i\cdots d_{i+3}\rangle=1$  implies  $by\bar{b}=ba^eba^m\bar{b}a^f\bar{b}$  with  $m\neq 0$ , i.e.,  $y=a^et^ma^f\in H$  with  $m\neq 0$ . A contradiction because for  $m\neq 0$  we have  $by\bar{b}\notin H$  and d is not successful. Thus,  $i< k<\ell < j$ . Now,  $\langle by\bar{b}\,,\,d_id'd_j\rangle=1$  implies  $by\bar{b}=ba^eba^X\bar{b}\,u\,ba^Y\bar{b}a^f\bar{b}$ . Again, X and Y are random variables as above. Let  $u=(r,m)\in \mathbb{Z}[1/2]\rtimes \mathbb{Z}=H$ . We have  $ba^X\bar{b}=t^X$  and  $ba^Y\bar{b}=t^Y$  in  $\mathbf{G}_{1,2}$ . Thus,



**Fig. 1.** Portion of reduced words  $x \in H$  with  $|x|_{\beta} = 2n$ , sampling  $11 \cdot 10^9$  words.

 $\langle by\overline{b}\,,\,d_id'd_j\rangle=1$  implies Z+m=0 where Z=X+Y. With the same arguments as in (9) we derive  $\Pr\left[Z=-m\right]\leq 5/16.$  This shows (11) in the final case k=i+2 and  $\ell=j-2$ , too.

Now, Lemma 19 follows from (10) and (11) since n-k matching pairs (i,j) exist in d with i+1 < j.

Lemma 18 and Lemma 19 enable us to calculate  $\Pr_n[\langle x, d \rangle = 1]$  as follows:

$$\Pr_{n} [\langle x , d \rangle = 1] = \Pr_{k} [\langle x , d \rangle = 1 \mid \langle x , d \rangle_{\beta} = 1] \cdot \Pr_{n} [\langle x , d \rangle_{\beta} = 1]$$

$$\leq (5/16)^{n-k} \cdot (2/3)^{n-k} (2/9)^{k} \leq (2/9)^{n}.$$

This shows (6) and therefore Lemma 17 which in turn implies Theorem 16.

#### 4.2 Computer Experiments

We have conducted computer experiments with a sample of  $11\cdot 10^9$  (i.e., 11 billion) random words  $x\in \varSigma^*$  with  $4\le |x|_\beta=2n\le 24$ , see Figure 1. Moreover, for n=14 our random process did not find a single  $x\in H$ . The experiments confirm  $\Pr_n\left[x\in H\right]\approx 0$ . The initial values seem to suggest  $\Pr_n\left[x\in H\right]\in \mathcal{O}(0.25^n)$ . This is much better than the upper bound of Lemma 17, but our proof used very rough estimations in (5) and (6), only. Hence, a difference is no surprise.

# 5 Back-to-base probability in HNN extensions: Second proof of Theorem 16

This section has been added to the arXiv version in November 2014, only. The motivation has been to give an alternative proof of Theorem 16 which uses some known results from literature. For convenience of the reader there is some overlap with material in Section 4.1. This allows an independent reading. In the following we investigate the general situation of an HNN extension G which is given as  $G = \langle H, b \mid bab^{-1} = \varphi(a), a \in A \rangle$  with a finitely generated base group H. By the Back-to-base probability we mean the probability that a random walk in the associated Cayley graph of G ends in the base group H. In order to make the statement precise we fix the following notation. We let H be the base group which is generated by some finite subset  $\Sigma \subseteq H$  such that  $\Sigma = \Sigma^{-1}$ . We use a symmetric set of generators in order to apply Proposition 22. (In fact, Proposition 22 is false for non-symmetric generating sets, in general.) We let A

and B be isomorphic subgroups of H and  $\varphi:A\to B$  be a fixed isomorphism between them. Then, as usual,  $G=\left\langle H,b\mid bab^{-1}=\varphi(a),a\in A\right\rangle$  denotes the corresponding HNN extension of H with stable letter b. By  $\Delta$  we denote the set  $\Delta=\Sigma\cup\left\{b,\overline{b}\right\}$  where  $\overline{b}=b^{-1}$ . Thus, the "evaluation of words over  $\Delta$ " defines a monoid presentation  $\eta:\Delta^*\to G$ , which is induced by the inclusion  $\Delta\subseteq G$ . Recall that for  $x\in\Delta^*$  and  $a\in\Delta$  we denote by  $|x|_a$  the number of occurrences of the letter a in the word x, and we let  $|x|_\beta=|x|_b+|x|_{\overline{b}}$ . For  $x\in\Delta^*$  let  $\widehat{x}\in\Delta^*$  denote a Britton-reduced word such that  $\eta(x)=\eta(\widehat{x})$  in G. Using this notation let us define  $\|x\|_\beta$  by  $\|x\|_\beta=|\widehat{x}|_\beta$ .

For each  $n \in \mathbb{N}$  we view  $\Delta^n$  as a probability space with a uniform distribution. Thus, we consider random walks in the Cayley graph of G w.r.t. the generating set  $\Delta$  where each outgoing edge is chosen with equal probability. In contrast to Section 4.1 random walks may backtrack, i.e., they are not necessarily reduced words. We aim to show the following result.

**Theorem 20.** Let  $G = \langle H, b \mid bab^{-1} = \varphi(a), a \in A \rangle$  be an HNN extension of H and  $\eta : \Delta^* \to G$  as above. Then we have  $A \neq H \neq B$  if and only if  $\{x \in \Delta^* \mid \eta(x) \notin H\}$  is strongly generic in  $\Delta^*$ .

Remark 21. In terms of amenability of Schreier graphs (see e.g., [5,14]) we can restate Theorem 20 as follows: Let  $G = \langle H, b \mid bab^{-1} = \varphi(a), a \in A \rangle$  be an HNN extension of H and  $\eta: \Delta^* \to G$  as above. The Schreier graph  $\Gamma(G, H, \Delta)$  is non-amenable if and only if  $A \neq H \neq B$ .

Before we prove Theorem 20 let us show how to derive Theorem 16 from Theorem 20. We use the following two propositions (see also [14]).

**Proposition 22** ([5, **Prop. 38, Thm. 51**]). Let G be a finitely generated group and  $H \leq G$  be a subgroup. Let  $\eta: \Delta^* \to G$ ,  $\eta': \Delta'^* \to G$  two monoid presentations of G. Then,  $\Delta^* \setminus \eta^{-1}(H)$  is strongly generic in  $\Delta^*$  if and only if  $\Delta'^* \setminus \eta'^{-1}(H)$  is strongly generic in  $\Delta'^*$ .

**Proposition 23** ([1,6,10]). Let G be a finitely generated group,  $H \leq G$  be a subgroup, and  $\eta: \Delta^* \to G$  be a monoid presentation of G. Let  $\Xi$  be the set of reduced words of  $\Delta^*$ . Then,  $\Delta^* \setminus \eta^{-1}(H)$  is a strongly generic in  $\Delta^*$  if and only if  $\Xi \setminus \eta^{-1}(H)$  is strongly generic in  $\Xi$ .

In order to see Theorem 16 we proceed as follows: Let  $\Xi$  denote the set of reduced words in  $\{a, \overline{a}, b, \overline{b}\}^*$  and  $\eta: \{a, \overline{a}, b, \overline{b}\}^* \to \mathbf{G}_{1,2}$  the canonical presentation. Then Theorem 20, Proposition 22, and Proposition 23 show that  $\Xi \setminus \eta^{-1}(H)$  is strongly generic in  $\Xi$ . Now, with the same arguments as in Section 4.1 it follows that elements which cannot be conjugated into H form a strongly generic set in  $\Xi$ .

Now, we turn to the proof of Theorem 20. It covers the rest of this section. First, we consider A=H=B. Then G is a semidirect product  $G=H\rtimes \mathbb{Z}$ . Let  $\pi_2:G\to \mathbb{Z}$  the projection onto the second component. Then we have  $\eta(x)\in H$  if and only if  $\pi_2(\eta(x))=0$ . Since  $\Delta$  can be viewed as a constant, it is not hard to see that we have  $\Pr\left[\eta(x)\in H\right]\in\Theta(1/\sqrt{n})$ . (Actually, if  $|\Delta|$  is not viewed as a constant we obtain a more precise estimation. Since the expected value for  $|x|_\beta$  is  $n/2\,|\Delta|$  one can show  $\Pr\left[\eta(x)\in H\right]\in\Theta(\sqrt{|\Delta|/n})$ . But we do not need this for our purpose.)

The second case is  $A=H\neq B$ . For example, G is the Baumslag-Solitar group  $\mathbf{BS}_{1,2}$ . We content ourselves with a lower bound on  $\Pr\left[\eta(x)\in H\right]$ . We begin with a the conditional probability:

$$\Pr\left[\eta(x) \in H \mid |x|_{\beta} = 2m\right] \ge \frac{\binom{2m+1}{m}}{(m+1)2^m} \in \Theta(m^{-1.5}). \tag{12}$$

To see this observe that, due to A=H, a Britton reduction on a word  $x\in \Delta^*$  leads always to H if both,  $|x|_b=|x|_{\overline{b}}$  and for every prefix y of x we have  $|y|_b\geq |y|_{\overline{b}}$ . Thus,  $\eta(x)\in H$  as soon as the projection of x onto  $\left\{b,\overline{b}\right\}^*$  is a Dyck word. As we noticed earlier, the number of Dyck words of length 2m is the m-th Catalan number  $\frac{1}{m+1}\binom{2m}{m}\in\Theta(m^{-1.5})$ . We obtain a trivial estimation  $\Pr\left[\eta(x)\in H\right]\in\Omega(n^{-2.5})$  which is good enough because it means that for A=H the set  $\{x\in\Delta^*\mid\eta(x)\not\in H\}$  is not strongly generic in  $\Delta^*$ . However, using some standard Chernoff bounds and the fact that the expected value for  $|x|_\beta$  is  $n/2|\Delta|$ , we can state for A=H a more precise upper and lower bound as follows:

$$\Pr\left[\eta(x) \in H\right] \in \mathcal{O}(\sqrt{|\Delta|/n}) \cap \Omega((|\Delta|/n)^{1.5}). \tag{13}$$

Finally, let us consider the most interesting case  $A \neq H \neq B$ . This is the situation e.g. in the Baumslag group  $\mathbf{G}_{1,2}$ . In order to finish the proof of Theorem 20 we have to show  $\Pr\left[\eta(x) \in H\right] \approx 0$ . This covers the rest of this section. As we have done in Section 4 we switch the probability space. We embed  $\Delta^n$  into the space  $\Delta^*$  with a measure  $\mu_{0,n}$  on  $\Delta^*$  which concentrates its mass on  $\Delta^n$  (i.e.,  $\mu_{0,n}(\Delta^n) = 1$ ) with corresponding probability  $\Pr_{0,n}\left[\cdots\right]$ . We now have to show that  $\Pr_{0,n}\left[\eta(x) \in H\right] \approx 0$  if  $A \neq H \neq B$ . Let  $\mu_m$  be the measure on  $\Delta^*$  which is defined by reading letters from  $\Delta$  each with equal probability as long as the random walk contains at most m letters  $\beta \in \left\{b, \overline{b}\right\}$ . This gives a corresponding probability on  $\Delta^*$  which is concentrated on those words with  $|x|_{\beta} = m$ . We denote the corresponding probability by  $\Pr_m\left[\cdots\right]$ . Still there is a close connection between these probabilities. In particular:

$$\Pr_{0,n}\left[|x|_{\beta} = m\right] = \binom{n}{m} \cdot (2/|\Delta|)^m \cdot (1 - 2/|\Delta|)^{n-m} = \Pr_m\left[|x| = n\right]$$
 (14)

$$\Pr_{0,n}\left[\eta(x) \in H \mid |x|_{\beta} = m\right] = \Pr_{m}\left[\eta(x) \in H \mid |x| = n\right] \tag{15}$$

Since  $\Pr_{0,n}\left[|x|_{\beta}=m\right]pprox 0$  for  $m\leq n/\left|\Delta\right|$  we can perform a similar computation as in Section 4.1:

$$\begin{split} \Pr_{0,n}\left[\eta(x)\in H\right] &= \sum_{m=0}^{n} \Pr_{0,n}\left[\eta(x)\in H \wedge |x|_{\beta} = m\right] \\ &\approx \sum_{m=\lceil n/|\Delta|\rceil}^{n} \Pr_{0,n}\left[\eta(x)\in H \wedge |x|_{\beta} = m\right] \\ &= \sum_{m=\lceil n/|\Delta|\rceil}^{n} \Pr_{0,n}\left[\eta(x)\ \Big|\ |x|_{\beta} = m\right] \cdot \Pr_{0,n}\left[|x|_{\beta} = m\right] \end{split}$$

$$= \sum_{m=\lceil n/|\Delta|\rceil}^{n} \Pr_{m} \left[ \eta(x) \in H \mid |x| = n \right] \cdot \Pr_{m} \left[ |x| = n \right]$$

$$= \sum_{m=\lceil n/|\Delta|\rceil}^{n} \Pr_{m} \left[ \eta(x) \in H \land |x| = n \right]$$

$$\leq \sum_{m=\lceil n/|\Delta|\rceil}^{n} \Pr_{m} \left[ \eta(x) \in H \right].$$

Therefore, it is enough to show that  $\Pr_m [\eta(x) \in H] \approx 0$  as a function in m.

There is also a natural probability distribution on  $\Sigma^*$  which is formally defined by  $\mu_0$  (N.B.  $\mu_0$  is different from  $\mu_{0,n}$ !) Indeed, we have  $\mu_0(\Sigma^*)=1$  and the distribution on  $\Sigma^*$  is given by a random walk which stops with probability  $2/|\Delta|$  and, if it does not stop, then it chooses the next letter with equal probability. In order to emphasize that the mass of  $\mu_0$  is on  $\Sigma^*$  we also write  $\Pr_{\Sigma}[y] = \Pr_0[y]$  for  $y \in \Sigma^*$ .

**Lemma 24.** For all  $\gamma \in \Sigma^*$  and  $\beta \in \{b, \overline{b}\}$  we have

$$\Pr_{\Sigma} \left[ \eta(\beta \gamma y \overline{\beta}) \notin H \right] \ge \frac{2}{|\Delta|^2}.$$

*Proof.* By symmetry we may assume  $\beta = b$ . We have to show that  $\Pr_{\Sigma} [\eta(\gamma y) \notin A] \ge 2/|\Delta|^2$ . We consider the cases  $\eta(\gamma) \notin A$  and  $\eta(\gamma) \in A$  separately. For  $\eta(\gamma) \notin A$  we obtain

$$\Pr_{\Sigma} \left[ \eta(\gamma y) \notin A \right] \ge \Pr_{\Sigma} \left[ y = 1 \right] = 2/\left| \Delta \right| \ge 2/\left| \Delta \right|^2.$$

For  $\eta(\gamma) \in A$  and  $a \in \Sigma$  we obtain  $\eta(\gamma a) \in A$  if and only if  $\eta(a) \notin A$ . Since  $A \neq H$  and  $\Sigma$  generates H, there must be some letter  $a \in \Sigma$  with  $\eta(a) \notin A$ . Therefore, in the second case

$$\Pr_{\Sigma} [\eta(\gamma y) \notin A] \ge \Pr_{\Sigma} [y = a] = 2/|\Delta|^2.$$

As before a  $\beta$ -factorization of  $x\in \Delta^*$  with  $|x|_{\beta}=m$  is written as a word  $x=\gamma_0\beta_1\gamma_1\dots\beta_m\gamma_m$  such that  $\beta_i\in\{b,\bar{b}\}$  and  $\gamma_i\in\Sigma^*$  for  $1\leq i\leq m$ . Using the notion of  $\beta$ -factorization we define for all  $0\leq\ell\leq m$  a random variable  $X_\ell:\Delta^*\to\mathbb{N}$  as follows. We let  $X_\ell(x)=\|\gamma_0\beta_1\gamma_1\dots\beta_\ell\gamma_\ell\|_{\beta}$ . Another way to explain  $X_\ell(x)$  is as follows. Choose any prefix z of x such that  $|z|_{\beta}=\ell$ , compute the Britton reduction  $\widehat{z}$  of z and let  $X_\ell(x)=\|\widehat{z}\|_{\beta}$ , i.e.,  $X_\ell(x)=\|z\|_{\beta}$ . The differences  $Y_i=X_i-X_{i-1}$  define random variables  $Y_i$  for  $1\leq i\leq m$  with values in  $\{-1,1\}$ . Clearly,  $X_\ell=\sum_{i=1}^\ell Y_i$  for all  $0\leq\ell\leq m$ . Note that  $X_0=0$  and  $X_1=Y_1=1$  are constant functions.

Consider a  $\beta$ -factorization  $x=\gamma_0\beta_1\gamma_1\ldots\beta_m\gamma_m$  for x with  $|x|_\beta=m$ . For  $1\leq i\leq m$  let  $z_{i-1}$  be Britton-reduced such that  $\eta(z_{i-1})=\eta(\gamma_0\beta_1\ldots\gamma_{i-2}\beta_{i-1})$ . Then the  $\beta$ -factorization of  $z_{i-1}$  becomes  $\gamma_0'\beta_1'\gamma_1'\ldots\beta_j'\gamma_j'$  for some  $j\leq i-1$ . Note that the last factor  $\gamma_j'$  can be, a priori, any word in  $\Sigma^*$ . Now, it depends only on the factors  $\beta_j'\gamma_j'$  and  $\gamma_{i-1}\beta_i$  whether or not the  $\beta$ -length of the Britton-reduced word increases or decreases

when reading the next factor  $\gamma_{i-1}\beta_i$ . The probability for that is described by the random variable  $Y_i$ . For all  $\varepsilon \in \{-1, 1\}^{i-1}$  Lemma 24 shows

$$\Pr\left[Y_i = 1 \mid Y_j = \varepsilon_j \text{ for } j < i\right] \ge 1/2 + 1/2 \cdot 2/\left|\Delta\right|^2 = 1/2 + 1/\left|\Delta\right|^2. \tag{16}$$

Let  $\{Z_i \mid i=1,\ldots,m\}$  be a set of m independent random variables taking values in  $\{-1,1\}$  such that  $\Pr[Z_i=1]=1/2+1/|\Delta|^2$  for  $1\leq i\leq m$ . By (16) it follows that for every  $\varepsilon=(\varepsilon_j)\in\{-1,1\}^{k-1}$  and  $1\leq k\leq m$  we have

$$\Pr_{m} \left[ Y_{k} = -1 \mid Y_{j} = \varepsilon_{j} \,\forall j < k \right] \leq \Pr \left[ Z_{k} = -1 \right]. \tag{17}$$

This observation is crucial in the proof of the next lemma.

#### Lemma 25. We have

$$\Pr_{m}[X_{m} = 0] \le \left(1 - \frac{4}{|\Delta|^{4}}\right)^{m/2}.$$

*Proof.* The assertion is trivial for m=0 or m odd. Hence, let  $m\geq 2$  be even. First, let us show that for all  $p \in \mathbb{Z}$ ,  $1 \le k \le \ell \le m$ , and  $\varepsilon = (\varepsilon_i) \in \{-1,1\}^{k-1}$  we have

$$\Pr_{m} \left[ \sum_{i=k}^{\ell} Y_{i} \leq p \,\middle|\, Y_{j} = \varepsilon_{j} \,\forall j < k \right] \leq \Pr_{m} \left[ \sum_{i=k}^{\ell} Z_{i} \leq p \right]. \tag{18}$$

We prove (18) by induction on  $k - \ell$ . The case  $\ell = k$  is trivial, hence let  $\ell < k$ .

$$\Pr_{m} \left[ \sum_{i=k}^{\ell} Y_{i} \leq p \mid Y_{j} = \varepsilon_{j} \, \forall j < k \right]$$

$$= \sum_{\varepsilon_{k} = \pm 1} \Pr_{m} \left[ Y_{k} = \varepsilon_{k} \mid Y_{j} = \varepsilon_{j} \, \forall j < k \right] \cdot \Pr_{m} \left[ \sum_{i=k+1}^{\ell} Y_{i} \leq p - \varepsilon_{k} \mid Y_{j} = \varepsilon_{j} \, \forall j \leq k \right]$$

$$\leq \sum_{\varepsilon_{k} = \pm 1} \Pr_{m} \left[ Y_{k} = \varepsilon_{k} \mid Y_{j} = \varepsilon_{j} \, \forall j < k \right] \cdot \Pr_{m} \left[ \sum_{i=k+1}^{\ell} Z_{i} \leq p - \varepsilon_{k} \right]$$

$$\leq \sum_{\varepsilon_{k} = \pm 1} \Pr_{m} \left[ Z_{k} = \varepsilon_{k} \right] \cdot \Pr_{m} \left[ \sum_{i=k+1}^{\ell} Z_{i} \leq p - \varepsilon_{k} \right] = \Pr_{m} \left[ \sum_{i=k}^{\ell} Z_{i} \leq p \right].$$

We have to explain the inequality leading to the last line above. By (17) there is some  $\delta_{\varepsilon,k} \geq 0$  such that  $\Pr_m [Y_k = -1 \mid Y_j = \varepsilon_j \ \forall j < k] + \delta_{\varepsilon,k} = \Pr_m [Z_k = -1]$ . Thus, by definition,  $\Pr_m [Y_k = 1 \mid Y_j = \varepsilon_j \ \forall j < k] - \delta_{\varepsilon,k} = \Pr_m [Z_k = 1]$ . Hence, the inequality follows from  $\Pr_m\left[\sum_{i=k+1}^\ell Z_i \leq p-1\right] \leq \Pr_m\left[\sum_{i=k+1}^\ell Z_i \leq p+1\right]$ . As a special case for k=1 and  $\ell=m$  we obtain

$$\Pr_{m}\left[X_{m} \le p\right] = \Pr_{m}\left[\sum_{i=1}^{m} Y_{i} \le p\right] \le \Pr_{m}\left[\sum_{i=1}^{m} Z_{i} \le p\right]. \tag{19}$$

In order to prove the lemma it is enough to consider p = 0. We get

$$\Pr_{m} [X_{m} = 0] \leq \Pr \left[ \sum_{i=1}^{m} Z_{i} \leq 0 \right] = \sum_{\substack{\varepsilon = (\varepsilon_{j}) \in \{-1,1\}^{m} \\ |\{j| \varepsilon_{j} = 1\}| \leq m/2}} \prod_{i=1}^{m} \Pr [Z_{i} = \varepsilon_{i}]$$

$$\leq 2^{m} \cdot \left( \frac{1}{2} - \frac{1}{|\Delta|^{2}} \right)^{m/2} \cdot \left( \frac{1}{2} + \frac{1}{|\Delta|^{2}} \right)^{m/2} = \left( 1 - \frac{4}{|\Delta|^{4}} \right)^{m/2}.$$

Hence, we have concluded the proof of Theorem 20 because Lemma 25 implies in particular  $\Pr_m [X_m = 0] \approx 0$ .

#### Conclusion

We have investigated the complexity of the conjugacy problem in two important groups in combinatorial group theory. The conjugacy problem in  $\mathbf{BS}_{1,2}$  is  $\mathsf{TC}^0$ -complete. If division in power circuits is non-elementary in the worst case, then the conjugacy problem in  $\mathbf{G}_{1,2}$  is non-elementary on the average, but solvable in  $\mathcal{O}(n^4)$  on a strongly generic subset. This is a striking contrast underlying the importance of generic case complexity on natural examples. In order to derive the result about generic case complexity, we proved a more general result about HNN extensions. We showed that  $G = \langle H, b \mid bab^{-1} = \varphi(a), a \in A \rangle$  has a non-amenable Schreier graph with respect to the base group H if and only if  $A \neq H \neq B$ .

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