# Block Interpolation: <br> A Framework for Tight Exponential-Time Counting Complexity 

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#### Abstract

We devise a framework for proving tight lower bounds under the counting exponential-time hypothesis \#ETH introduced by Dell et al. (ACM Transactions on Algorithms, 2014). Our framework allows us to convert classical \#P-hardness results for counting problems into tight lower bounds under \#ETH, thus ruling out algorithms with running time $2^{o(n)}$ on graphs with $n$ vertices and $O(n)$ edges. As exemplary applications of this framework, we obtain tight lower bounds under \#ETH for the evaluation of the zero-one permanent, the matching polynomial, and the Tutte polynomial on all non-easy points except for one line. This remaining line was settled very recently by Brand et al. (IPEC 2016).


## 1 Introduction

Counting complexity is a classical subfield of complexity theory, launched by Valiant's seminal paper 30] that introduced the class \#P and proved \#P-hardness of the zero-one permanent, a problem equivalent to counting perfect matchings in a bipartite graph. This initial breakthrough spawned an ongoing research program that systematically studies the complexity of computational counting problems, and many results in this area can be organized as dichotomy results. Such results show that, among problems that can be expressed in certain rich frameworks, each problem is either polynomial-time solvable or \#P-hard. Moreover, these results often give criteria for deciding which side of the dichotomy a given problem occupies. For instance, a full dichotomy was shown for the problems of counting solutions to constraint-satisfaction problems [5, 6, and similar results are known for large subclasses of so-called Holant problems [9, 7, and for the evaluation of graph polynomials such as the Tutte polynomial [22] and the cover polynomial [2, 1].

Over the course of the counting complexity program, it became clear that most interesting counting problems are \#P-hard, and that the class of polynomial-time solvable problems is rather limited, nevertheless containing some surprising examples, such as counting perfect matchings in planar graphs, counting spanning trees, and problems amenable to holographic algorithms [32]. To attack the large body of hard problems, several relaxations were studied, such as approximate counting [23, 16, 15], counting modulo fixed numbers [17, 31], and counting on restricted graph classes, such as planar and/or 3-regular graphs [29, 8].

In this paper, we follow an avenue of relaxations recently introduced by Dell et al. [14] and consider the possibility of sub-exponential exact algorithms for counting problems. More precisely, we rule out such algorithms for various counting problems under popular complexity-theoretic assumptions. For instance, we can clearly count perfect matchings on $m$-edge graphs in time $2^{O(m)}$ by brute-force, but is there a chance of obtaining a running time of $2^{o(m)}$ ? An unconditional negative answer would imply the separation of FP and \#P, so our results need to rely upon additional hardness assumptions: We build upon the exponential-time

[^0]hypothesis \#ETH, introduced in [14], which we may consider for now as the hypothesis that the satisfying assignments to 3 -CNF formulas $\varphi$ on $n$ variables cannot be counted in time $2^{o(n)}$. This hypothesis is trivially implied by the better-known and widely-believed decision version ETH, introduced in [21, 20], which assumes the same lower bound for deciding the satisfiability of $\varphi$.

Dell et al. [14] were able to prove almost-tight lower bounds under \#ETH for a variety of counting problems: For instance, they could rule out algorithms with running time $2^{o(n / \log n)}$ for the zero-one permanent on graphs with $n$ vertices and $O(n)$ edges. Similar lower bounds were shown for counting vertex covers, and for most points of the Tutte polynomial.

### 1.1 Hardness via polynomial interpolation

The lower bounds in [14] are obtained via polynomial interpolation, one of the most prominent techniques for non-parsimonious reductions between counting problems [26, 22, 29, 8, 19, 18, 14, To illustrate this technique, and for the purposes of further exposition, let us reduce counting perfect matchings to counting matchings (that are not necessarily perfect), using a standard argument similar to [29]. In the following, let $G$ be a graph with $n$ vertices. We wish to obtain the number of perfect matchings in $G$ by querying an oracle for counting matchings in arbitrary graphs.

Step 1 - Set up interpolation: For $k \in \mathbb{N}$, let $m_{k}$ denote the number of matchings with exactly $k$ unmatched vertices in $G$. In particular, $m_{0}$ is equal to the number of perfect matchings in $G$. For an indeterminate $x$, define a polynomial $\mu$ via

$$
\begin{equation*}
\mu(x)=\sum_{k=0}^{n} m_{k} \cdot x^{k} \tag{1}
\end{equation*}
$$

and observe that its degree is $n$. Hence, we could use Lagrange interpolation to recover all its coefficients if we were given the evaluations of $\mu$ at $n+1$ distinct input points. In particular, this would give us the constant coefficient $m_{0}$, which counts the number of perfect matchings in $G$.

Step 2 - Evaluate the polynomial with gadgets: We can evaluate $\mu(t)$ at points $t \in \mathbb{N} \backslash\{0\}$ by a reduction to counting matchings: For $t \in \mathbb{N}$ with $t \geq 1$, define a graph $G_{t}$ from $G$ by adding, for each vertex $v \in V(G)$, a gadget that consists of an independent set of $t-1$ fresh vertices together with edges from all of these vertices to $v$. Then it can be checked that $\mu(t)$ is equal to the number of matchings in $G_{t}$ : Each matching in $G$ with exactly $k$ unmatched vertices can be extended to $t^{k}$ matchings in $G_{t}$ by including up to one gadget edge at each unmatched vertex.

In summary, by evaluating the polynomial $\mu(t)$ for all $t \in\{1, \ldots, n+1\}$ via gadgets and an oracle for counting matchings in $G_{t}$, we can use Lagrange interpolation to obtain $m_{0}$. This gives a polynomial-time Turing reduction from counting perfect matchings to counting matchings, transferring the \#P-hardness of the former problem to the latter.

Furthermore, the above argument can also be used to derive a lower bound for counting matchings, which is however far from being tight: If the running time for counting perfect matchings on $n$-vertex graphs has a lower bound of $2^{\Omega(n)}$, then only a $2^{\Omega(\sqrt{n})}$ lower bound for counting matchings follows from the above argument, since the reduction incurs a quadratic blowup. This is because $G_{n+1}$ has a gadget of size $O(n)$ at each vertex, and thus $O\left(n^{2}\right)$ vertices in total.

Following the same outline as above, but using more sophisticated gadgets with $O\left(\log ^{c} n\right)$ vertices, similar reductions for various problems were obtained in [18, [19, [14], implying $2^{\Omega\left(n / \log ^{c} n\right)}$ lower bounds for these problems, which are however still not tight. In particular, these reductions share the somewhat unsatisfying commonality that they "leak" hardness: Tight lower bounds for the source problems of computing specific hard coefficients in a polynomial became less tight over the course of the reduction.

### 1.2 The limits of interpolation

Let us say that a reduction is gadget-interpolation-based if it proceeds along the two steps sketched above: First encode a hard problem into the coefficients of a polynomial $p$, then find gadgets that can be "locally" placed at vertices or edges so as to evaluate $p(\xi)$ at sufficiently many points $\xi$. Finally use Lagrange interpolation to recover $p$ from these evaluations. As remarked before, this is a well-trodden route for \#P-hardness proofs. However, when taking this route to prove lower bounds under \#ETH, we run into the following obstacles:

1. Gadget-interpolation-based reductions typically yield polynomials $p$ of degree $n=|V(G)|$, hence require $n+1$ evaluations of $p$ at distinct points, and thus in turn require $n+1$ distinct gadgets to be placed at vertices of $G$. But since there are only finitely many simple graphs on $O(1)$ vertices, the size of such gadgets must necessarily grow as some unbounded function $\alpha(n)$. Thus, any gadget-interpolation-based reduction can only yield $2^{\Omega(n / \alpha(n))}$ time lower bounds for some unbounded function $\alpha \in \omega(1)$, but such bounds are typically not tight.
2. Additionally, such reductions issue only polynomially many queries to the target problem. This is required for the setting of \#P-hardness, but it is nonessential in exponential-time complexity: To obtain a lower bound of $2^{\Omega(n)}$, we might as well use a reduction that requires $2^{o(n)}$ time and issues $2^{o(n)}$ queries to the target problem, provided that the graphs used in the oracle queries have only $O(n)$ vertices. However, it is a priori not clear how to exploit this additional freedom.

These two issues are immanent to every known lower bound under \#ETH and have ruled out tight lower bounds of the form $2^{\Omega(n)}$ so far.

### 1.3 The block-interpolation framework

In this paper, we circumvent the barriers mentioned above by introducing a framework that allows to apply the full power of sub-exponential reductions to counting problems. To this end, we use a simple trick based on multivariate polynomial interpolation, which we dub block-interpolation: In this setting, we do not use a univariate polynomial $p$ of degree $n$ in the reduction, but rather a multivariate polynomial $\mathbf{p}$ that we can easily obtain from $p$. This polynomial $\mathbf{p}$ also has total degree $n$, but it has only maximum degree $1 / \epsilon$ in each individual indeterminate, for any $\epsilon>0$ we choose. By making sure that $\epsilon$ is small enough, we can interpolate the polynomial $\mathbf{p}$ from $2^{o(n)}$ evaluations.

While this increases the number of evaluations significantly, we obtain the following crucial benefit: Each evaluation $\mathbf{p}(\xi)$ required for the interpolation can be performed at a point $\xi$ whose individual entries are contained in a fixed set of size $1 / \epsilon+1$. This will enable us to compute $\mathbf{p}(\xi)$ by attaching only $1 / \epsilon+1$ distinct gadgets to $G$. The catch here is that different vertices may obtain different gadgets, which was not feasible in the univariate setting.

This way, we overcome the two above-mentioned limitations of gadget-interpolation-based reductions while simultaneously staying as close as possible to the outline of such reductions. Consequently, we can phrase our technique as a general framework that can be used to convert a large body of existing \#P-hardness proofs into tight lower bounds under \#ETH. Curiously enough, the growth of the gadgets used in the original proofs is irrelevant to our framework, as only a constant number of gadgets will be used throughout the reduction. This allows us to use luxuriously large gadgets and it shortcuts the need for the involved and resourceful gadget constructions used, e.g., for simulating weights in the Tutte polynomial [14, 19] or in the independent set polynomial [18]. More importantly, our bounds are tight.

### 1.4 Applications of the framework

To showcase our framework, we show that \#ETH rules out algorithms with running time $2^{o(n)}$ for several classical problems on unweighted simple graphs $G$ with $n$ vertices and $O(n)$ edges, which we call sparse graphs. All of the considered problems admit trivial $2^{O(n)}$ time algorithms on such graphs. It should be noted that it is crucial to obtain hardness results for sparse graphs: Many reductions between counting problems
proceed by placing gadgets at edges, and this would map non-sparse graphs with $\omega(n)$ edges to target graphs on $\omega(n)$ vertices, thus failing to provide $2^{\Omega(n)}$ time lower bounds for the target problem.

More precisely, we show the following slightly stronger statements: Assuming \#ETH, each of the considered problems admits constants $\epsilon, C>0$ such that the problem cannot be solved in time $O\left(2^{\epsilon n}\right)$ on $n$-vertex graphs, even when these graphs have at most $C n$ edges. This clearly implies the claimed statements, and we will elaborate on this in Section 2.3 ,

Theorem 1.1. Assuming \#ETH, counting perfect matchings admits no $2^{o(n)}$ time algorithm, even for graphs that are bipartite, sparse, and unweighted.

In [14], only a lower bound of $2^{\Omega(n / \log n)}$ under \#ETH was shown for this problem. Tight lower bounds of $2^{\Omega(n)}$ were obtained only (a) under the randomized version rETH of ETH, which implies ETH and thus in turn \#ETH, but no converse direction is known, or (b) under \#ETH, but by introducing negative edge weights. Such edge weights are generally worrying, because it is a priori unclear how to remove them in reductions to other problems.

By reduction from Theorem 1.1. we then obtain a hardness result for the matching polynomial, as defined in (1), and a similar graph polynomial, the independent set polynomial. We will provide the precise definitions of the matching and independent set polynomials and their associated evaluation problems in Section 2.1

Theorem 1.2. Assuming \#ETH, the problem of evaluating the matching polynomial $\mu(G ; \xi)$ admits no $2^{o(n)}$ time algorithm at all fixed $\xi \in \mathbb{Q}$, even on graphs that are sparse and unweighted. The same holds for the independent set polynomial $I(G ; \xi)$ at all fixed $\xi \in \mathbb{Q} \backslash\{0\}$.

Both statements hold in particular at $\xi=1$, where these polynomials simply count matchings, and independent sets, respectively. No lower bounds for $\mu(G ; \xi)$ are stated in the literature. In 18, a lower bound of $2^{\Omega\left(n / \log ^{3} n\right)}$ for $I(G ; \xi)$ was shown at general $\xi \in \mathbb{Q} \backslash\{0\}$, and a lower bound of $2^{\Omega(n)}$ was shown at $\xi=1$, but neither of these bounds apply to sparse graphs.

Finally, we show lower bounds for the Tutte polynomial. Again, the formal definition of this graph polynomial will be provided in Section 2.1.

Theorem 1.3. Assuming \#ETH, the Tutte polynomial $T(x, y)$ for fixed points $(x, y) \in \mathbb{Q}^{2}$ cannot be evaluated in time $2^{o(n)}$ on sparse simple graphs if

- $y \neq 1$, and
- $(x, y) \notin\{(1,1),(-1,-1),(0,-1),(-1,0)\}$, and
- $(x-1)(y-1) \neq 1$.

In [14], only lower bounds of the type $2^{\Omega\left(n / \log ^{c} n\right)}$ could be shown for the Tutte polynomial on sparse simple graphs. Please consider [14, Figure 1] for a plot of the points covered by Theorem 1.3. Our lower bound applies at all points for which these authors show any super-polynomial lower bound. While our theorem leaves open the non-easy points on the line $y=1$, that is, all points $(x, 1)$ with $x \neq 1$, tight lower bounds for these points have been found by Brand et al. [3] since the conference version of the present paper was published. We thus have:

Theorem $1.4([3])$. If $(x, y) \in \mathbb{Q}^{2}$ is such that $(x-1)(y-1)=1$ or $(x, y) \in\{(1,1),(-1,-1),(0,-1),(-1,0)\}$, then $T(x, y)$ can be computed in polynomial time. Otherwise, there is no $2^{o(n)}$ time algorithm for computing $T(x, y)$ on sparse simple graphs unless \#ETH fails.

## Organization of this paper

In Section 2, we survey the necessary preliminaries from the theory of graph polynomials, polynomial interpolation, and exponential-time complexity. Then, in Section 3, we introduce the interpolation framework used for later results. In Section 4, we prove Theorems 1.1, 1.3 as applications of this framework.

## 2 Preliminaries

For $k \in \mathbb{N}$, we abbreviate $[k]=\{1, \ldots, k\}$. For sets $X$, write $\binom{X}{2}$ for the set of all unordered pairs with elements from $X$. The graphs in this paper are finite, undirected, and simple. If $G$ is a graph, we implicitly assume that $V(G)=[n]$ for some $n \in \mathbb{N}$, and consequently $E(G) \subseteq\binom{[n]}{2}$. Graphs may feature edge- or vertex-weights within intermediate steps of arguments, but all such weights will ultimately be removed to obtain hardness results on unweighted graphs.

For simplicity, we phrase our results using only rational numbers, but they could be easily adapted to the real or complex numbers, provided that these numbers are represented properly. We also write $x \leftarrow y$ for substituting the expression $y$ into an indeterminate $x$.

### 2.1 Graph polynomials

Our arguments and statements of results use graph polynomials, which are functions that map graphs $G$ to polynomials $p(G) \in \mathbb{Q}[\mathbf{x}]$, where $\mathbf{x}$ is some set of indeterminates. They are usually defined such that isomorphic graphs $G$ and $G^{\prime}$ are required to satisfy $p(G)=p\left(G^{\prime}\right)$, but we ignore this restriction for our purposes. As a notational convention, we abbreviate $p(G ; \xi)=(p(G))(\xi)$ for the evaluation of the polynomial $p(G)$ at a point $\xi$.

The arguably most famous graph polynomial is the Tutte polynomial [4], which we define in the following, along with the matching polynomial and the independent set polynomial [25].

Definition 2.1 (Matching and independent set polynomials). Let $G$ be a graph and let $\mathcal{M}[G]$ denote the set of (not necessarily perfect) matchings in $G$, that is, edge-subsets that are vertex-disjoint. For $M \in \mathcal{M}[G]$, let usat $(G, M)$ denote the set of unmatched vertices of $G$ in $M$. Then we define the matching polynomial $\mu$ (also called matching defect polynomial) as

$$
\mu(G ; x)=\sum_{M \in \mathcal{M}[G]} x^{|\mathrm{usat}(G, M)|}
$$

Similarly, let $\mathcal{I}[G]$ denote the independent sets of $G$. Then the independent-set polynomial $I$ is

$$
I(G ; x)=\sum_{S \in \mathcal{I}[G]} x^{|S|}
$$

Note that both the matching polynomial and the independent-set polynomial are weighted sums over its eponymous structures. A similar definition applies in the case of the Tutte polynomial, but the weights are more intricate.

Definition 2.2 (Tutte polynomial). For a subset $A \subseteq E(G)$, let $k(G, A)$ denote the number of connected components in the edge-induced subgraph $G[A]$. Then define the classical parameterization of the Tutte polynomial as

$$
T(G ; x, y)=\sum_{A \subseteq E(G)}(x-1)^{k(G, A)-k(G, E)}(y-1)^{k(G, A)+|A|-|V|}
$$

We also use the random-cluster formulation of the Tutte polynomial:

$$
Z(G ; q, w)=\sum_{A \subseteq E(G)} q^{k(G, A)} w^{|A|}
$$

The polynomials $Z$ and $T$ are essentially different parameterizations of each other: As noted in [14], with $q=(x-1)(y-1)$ and $w=y-1$, we have

$$
\begin{equation*}
T(G ; x, y)=(x-1)^{-k(G, E)}(y-1)^{-|V(G)|} \cdot Z(G ; q, w) \tag{2}
\end{equation*}
$$

We will mostly use the random-cluster formulation of the Tutte polynomial.

Definition 2.3 (Perfect matching polynomial and permanent). Recall that we assume that for each graph $G$, there is some $n \in \mathbb{N}$ such that $V(G)=[n]$ and $E(G) \subseteq\binom{V(G)}{2}$. For $e \in\binom{\mathbb{N}}{2}$, let $x_{e}$ be an indeterminate. We write $\mathcal{P} \mathcal{M}[G]$ for the set of perfect matchings of $G$. Then the perfect matching polynomial is defined as

$$
\operatorname{PerfMatch}(G)=\sum_{M \in \mathcal{P} \mathcal{M}[G]} \prod_{e \in M} x_{e} .
$$

See also [32, Section 3], and note that only finitely many indeterminates are present in $\operatorname{PerfMatch}(G)$. If $G$ is bipartite, we also denote $\operatorname{PerfMatch}(G)$ by the permanent $\operatorname{perm}(G)$. In doing so, we slightly abuse notation, since the permanent is defined on matrices $A$, but we implicitly consider the bi-adjacency matrix $A$ of the bipartite graph $G$ when speaking of the permanent of $G$.

For any graph polynomial $p$, we define two computational problems $\operatorname{Coeff}(p)$ and $\operatorname{Eval}(p)$, and a family of problems Eval ${ }_{S}(p)$ for subsets $S \subseteq \mathbb{Q}$.
$\operatorname{Coeff}(p)$ : On input $G$, output the list of all coefficients of $p(G)$. In this paper, we will consider this problem only for univariate graph polynomials.
$\operatorname{Eval}(p)$ : On input $G$ and an arbitrary point $\xi$, evaluate $p(G ; \xi)$. Here, $\xi$ is to be considered as a rationalvalued assignment to the indeterminates of $p(G)$. We will often consider $\xi$ as vertex- or edge-weights that are substituted into the indeterminates of $p(G)$.
$\operatorname{Eval}_{S}(p)$ : On input $G$ and a point $\xi$ whose coordinate-wise entries are all from $S$, evaluate $p(G ; \xi)$. The problem differs from $\operatorname{Eval}(p)$ in that only specific points $\xi$ are allowed as input. Note that, if $p$ is univariate and $S=\{a\}$ is a singleton set, then $\operatorname{Eval}_{S}(p)$ simply asks to compute $p(G ; a)$ for fixed $a$ on input $G$. We write $\operatorname{Eval}_{a}(p)$ in this case.

Example 2.4. For the matching polynomial $\mu$, the problem $\operatorname{Eval}(\mu)$ asks to evaluate $\mu(G ; \xi)$ when given as input a graph $G$ and a number $\xi$. For fixed $a$, the problem $\operatorname{Eval}_{a}(\mu)$ asks to evaluate $\mu(G ; a)$ on input $G$. For instance, the problem $\operatorname{Eval}_{0}(\mu)$ asks to count perfect matchings in a graph.

Rather than evaluating a multivariate graph polynomial $\mathbf{p}$ like PerfMatch on an unweighted graph $G$ and a point $\xi$, we often annotate edges/vertices of $G$ with the entries of $\xi$, assuming $V(G)$ and $E(G)$ to be ordered. We then speak of evaluating $\mathbf{p}\left(G^{\prime}\right)$ on the weighted graph $G^{\prime}$ derived from $G$ and $\xi$ this way.

### 2.2 Multivariate polynomial interpolation

Given a univariate polynomial $p \in \mathbb{Q}[x]$ of degree $n$, we can use Lagrange interpolation to compute the coefficients of $p$ when provided with the set of evaluations $\{(\xi, p(\xi)) \mid \xi \in \Xi\}$ for any set $\Xi \subseteq \mathbb{Q}$ of size $n+1$. It is known that polynomial interpolation can be generalized to multivariate polynomials $p \in \mathbb{Q}[\mathbf{x}]$, for instance, if $\Xi$ is a sufficiently large grid.

Lemma 2.5. Let $p \in \mathbb{Q}\left[x_{1}, \ldots, x_{n}\right]$ be a multivariate polynomial such that, for all $i \in[n]$, the degree of $x_{i}$ in $p$ is bounded by $d_{i} \in \mathbb{N}$. Furthermore, assume we are given sets $\Xi_{i} \subseteq \mathbb{Q}$ for $i \in[n]$ such that $\left|\Xi_{i}\right|=d_{i}+1$ for all $i \in[n]$. Consider the cartesian product of these sets, that is,

$$
\Xi:=\Xi_{1} \times \ldots \times \Xi_{n}
$$

Then we can compute the coefficients of $p$ with $O\left(|\Xi|^{3}\right)$ arithmetic operations when given as input the set

$$
\{(\xi, p(\xi)) \mid \xi \in \Xi\}
$$

Proof. For $s, t, s^{\prime}, t^{\prime} \in \mathbb{N}$ and matrices $A \in \mathbb{Q}^{s \times t}$ and $B \in \mathbb{Q}^{s^{\prime} \times t^{\prime}}$, we write $A \otimes B$ for the Kronecker product of $A$ and $B$, which is the matrix $A \otimes B \in \mathbb{Q}^{s \cdot s^{\prime} \times t \cdot t^{\prime}}$ whose rows are indexed by $[s] \times\left[s^{\prime}\right]$, whose columns are indexed by $[t] \times\left[t^{\prime}\right]$, and which satisfies

$$
(A \otimes B)_{\left(i, i^{\prime}\right),\left(j, j^{\prime}\right)}=A_{i, j} \cdot B_{i^{\prime}, j^{\prime}} \quad \text { for }\left(i, i^{\prime}\right) \in[s] \times\left[s^{\prime}\right] \text { and }\left(j, j^{\prime}\right) \in[t] \times\left[t^{\prime}\right]
$$

For $\ell \in[n]$, enumerate $\Xi_{\ell}=\left\{a_{1}^{(\ell)}, \ldots, a_{d_{\ell}+1}^{(\ell)}\right\}$ and let $A^{(\ell)}$ denote the Vandermonde matrix of dimensions $\left(d_{\ell}+1\right) \times\left(d_{\ell}+1\right)$ with $A_{i, j}^{(\ell)}=\left(a_{i}^{(\ell)}\right)^{j}$ for all $i, j \in\left[d_{\ell}+1\right]$. It is well-known that each Vandermonde matrix $A^{(\ell)}$ for $\ell \in[n]$ has full rank, provided that $a_{i}^{(\ell)} \neq a_{i^{\prime}}^{(\ell)}$ for all $i \neq i^{\prime}$. This condition is guaranteed in our setting. Now define

$$
A:=A^{(1)} \otimes \ldots \otimes A^{(n)}
$$

Since each matrix $A^{(\ell)}$ for $\ell \in[n]$ has full rank, so does the matrix $A$, by an elementary property of the rank of Kronecker products [24, Corollary 13.11].

Let $\mathbf{c}$ denote the vector that lists the coefficients of $p$ in lexicographic order ${ }^{1}$, and let $\mathbf{v}$ denote the vector that lists the evaluations $p(\xi)$ for $\xi \in \Xi$ in lexicographic order. Then it can be verified that $A \mathbf{c}=\mathbf{v}$. Since $A$ has full rank, this system of linear equations can be solved with $O\left(|\Xi|^{3}\right)$ arithmetic operations for co, and we obtain the coefficients of $p$.

Remark 2.6. By exploiting faster methods for solving linear systems of equations, the running time above could be lowered from $O\left(|\Xi|^{3}\right)$ to $O\left(|\Xi|^{\omega}\right)$ operations, where $\omega<2.4$ is the exponent of matrix multiplication. This is however non-essential for our reductions.

### 2.3 Exponential-time complexity

We build upon the counting exponential-time hypothesis \#ETH introduced in [14], which is a variant of the corresponding hypothesis ETH for decision problems [20, 21].

Definition 2.7. The counting exponential-time hypothesis \#ETH is the following claim: There is a constant $\epsilon>0$ such that no deterministic algorithm with running time $O\left(2^{\epsilon n}\right)$ can count the satisfying solutions of 3-CNF formulas $\varphi$ with $n$ variables.

Note that \#ETH rules out $2^{o(n)}$ time algorithms for counting satisfying assignments of 3-CNF formulas with $n$ variables. In fact, \#ETH is often stated as claiming precisely this lower bound. However, this latter statement is a priori not equivalent to \#ETH, as there could be, say, an uncomputable sequence of $O\left(2^{\epsilon n}\right)$ time algorithms with $\epsilon \rightarrow 0$ for counting satisfying assignments. For this reason, some authors choose to characterize the original definitions of ETH and \#ETH as nonuniform [10].

A particularly useful tool for proving lower bounds under \#ETH is the sparsification lemma, which was first shown for the decision version ETH [21, Corollary 1] and later adapted to counting problems [14] Lemma A.1]. This result allows us to assume that the formulas $\varphi$ in Definition 2.7 are sparse, i.e., even an $2^{o(m)}$ time algorithm would refute $\# \mathrm{ETH}$, where $m$ is the number of clauses of $\varphi$. Note that this indeed strengthens \#ETH, as we may assume $n \leq m$, whereas we can a priori only guarantee $m=O\left(n^{3}\right)$ for 3-CNF formulas.

Theorem 2.8. Assuming \#ETH, there is a constant $\epsilon>0$ such that no deterministic algorithm with running time $O\left(2^{\epsilon m}\right)$ can count the satisfying solutions of $3-C N F$ formulas $\varphi$ with $m$ clauses.

This theorem is shown by an application of so-called sub-exponential reduction families [21, Section 1.1.4]. In the following definition, we adapt these reductions for our particular applications involving graph polynomials. That is, we restrict our definition to problems $A$ and $B$ that receive graphs as inputs, and we ensure that the instances generated by the reduction are sparse.

Definition 2.9. A sub-exponential reduction family from problem $A$ to $B$ is an algorithm $\mathbb{T}$ with oracle access for B . Its inputs are pairs $(G, \epsilon)$ where $G$ is an input graph for A , and $\epsilon$ with $0<\epsilon \leq 1$ is a running time parameter, such that

1. $\mathbb{T}$ computes $\mathrm{A}(G)$, and it does so in time $f(\epsilon) \cdot 2^{\epsilon \cdot|V(G)|}$, and
2. $\mathbb{T}$ only invokes the oracle for B on graphs $G^{\prime}$ with at most $g(\epsilon) \cdot(|V(G)|+|E(G)|)$ vertices and edges.
[^1]In these statements, $f$ and $g$ are computable functions that depend only on $\epsilon$. We write $\mathrm{A} \leq_{\text {serf }}^{T} \mathrm{~B}$ if such a reduction exists.

That is, the running time of $\mathbb{T}$ (and hence, the number of oracle queries) can be chosen as $O\left(2^{\epsilon n}\right)$ for arbitrarily small $\epsilon$, but this comes at the cost of incurring a "blowup factor" of $g(\epsilon)$ in the reduction images. It can be verified that sub-exponential reductions preserve lower bounds, see [21, Section 1.1.4]:
Lemma 2.10. Let A and B be problems that satisfy $\mathrm{A} \leq_{\text {serf }}^{T} \mathrm{~B}$, and assume that for all $\epsilon, C>0$, there is an $O\left(2^{\epsilon n}\right)$ time algorithm for B on graphs with $n$ vertices and at most $C n$ edges. Then, for all $\delta, D>0$, there is an $O\left(2^{\delta n}\right)$ time algorithm for A on graphs with $n$ vertices and at most Dn edges.

Corollary 2.11. If $\mathrm{A} \leq_{\text {serf }}^{T} \mathrm{~B}$ and there are $\epsilon, C>0$ such that A cannot be solved in time $O\left(2^{\epsilon n}\right)$ on graphs with $n$ vertices and at most $C n$ edges, then there are $\delta, D>0$ such that B cannot be solved in time $O\left(2^{\delta n}\right)$ on graphs with $n$ vertices and at most $D n$ edges.

If the prerequisites of the above corollary hold, then it is in particular true that there is a constant $D$ such that B cannot be solved in time $2^{o(n)}$ on graphs with $n$ vertices and at most $D n$ edges.

## 3 The Block Interpolation Framework

In this section, we show how to obtain tight lower bounds for evaluating graph polynomials under \#ETH by means of multivariate polynomial interpolation. More specifically, for a general class of univariate graph polynomials $p$, we show that, for certain fixed $\xi \in \mathbb{Q}$, we can reduce the coefficient problem of $p$ to the evaluation problem of $p$ on $\xi$.

$$
\begin{equation*}
\operatorname{Coeff}(p) \leq_{\text {serf }}^{T} \operatorname{Eval}_{\xi}(p) \tag{3}
\end{equation*}
$$

This is useful due to the following reasons: Firstly, many counting problems can be expressed as evaluation problems $\operatorname{Eval}_{\xi}(p)$ for adequate graph polynomials $p$ and points $\xi$. For instance, the Tutte polynomial collects an abundance of such examples. Secondly, as discussed in the introduction, many classical \#P-hardness proofs for $\operatorname{Eval}_{\xi}(p)$ first establish \#P-hardness for $\operatorname{Coeff}(p)$ and then reduce this to the evaluation problem by some form of interpolation. In many cases, the classical \#P-hardness proof for Coeff $(p)$ already yields a tight lower bound under \#ETH. Our technique then allows to transfer this lower bound to $\operatorname{Eval}_{\xi}(p)$.

The remainder of this section is structured as follows: In Section 3.1, we first describe the "format" required from a univariate graph polynomial $p$ for our framework to apply. Then we show in Section 3.2 how to perform the reduction

$$
\begin{equation*}
\operatorname{Coeff}(p) \leq_{s e r f}^{T} \operatorname{Eval}_{S}(\mathbf{p}) \tag{4}
\end{equation*}
$$

where $\mathbf{p}$ is a certain "multivariate version" of $p$, as mentioned in the introduction, and $S \subseteq \mathbb{Q}$ is a set whose size depends only upon the running time parameter $\epsilon$ in the sub-exponential reduction family, but not on the size of the input graph. In Section 3.3 we then show how to reduce

$$
\begin{equation*}
\operatorname{Eval}_{S}(\mathbf{p}) \leq_{s e r f}^{T} \operatorname{Eval}_{\xi}(p) \tag{5}
\end{equation*}
$$

by means of gadget families, provided that these families exist. Pipelining the reductions (4) and (5) then gives the full reduction (3).

### 3.1 Admissible Graph Polynomials

Our framework applies to univariate graph polynomials $p$ that admit a canonical multivariate generalization. More specifically, we call $p$ subset-admissible if $p$ is induced by a sieving function $\chi$ which filters the structures counted by $p$, and a weight selector $\omega$ which assigns a particular kind of weight to each of these structures. While this definition may seem abstract at first, we will soon observe that various popular graph polynomials can be expressed naturally in this form.

Definition 3.1. Let $\mathcal{G}$ denote the set of all unweighted simple graphs and recall that, for each graph $G \in \mathcal{G}$, we assume that there exists some $n \in \mathbb{N}$ such that $V(G)=[n]$ and $E(G) \subseteq\binom{[n]}{2}$. Let $\mathcal{V}=\mathbb{N}$ denote the set of all possible vertices of unweighted simple graphs, and let $\mathcal{E}=\binom{\mathbb{N}}{2}$ denote the set of all possible edges of such graphs. We also write $\mathcal{F}=\mathcal{V} \cup \mathcal{E}$.

For a sieve function $\chi: \mathcal{G} \times 2^{\mathcal{F}} \rightarrow \mathbb{Q}$ and a weight selector $\omega: \mathcal{G} \times 2^{\mathcal{F}} \rightarrow 2^{\mathcal{F}}$, we define the graph polynomial induced by the pair $(\chi, \omega)$ as

$$
\begin{equation*}
p_{\chi, \omega}(G ; x)=\sum_{A \subseteq V(G) \cup E(G)} \chi(G, A) \cdot x^{|\omega(G, A)|} \tag{6}
\end{equation*}
$$

We say that $p$ is subset-admissible if $p=p_{\chi, \omega}$ for some $(\chi, \omega)$ as above.
Remark. Note that $\chi$ and $\omega$ may be partial functions in the above definition, since, e.g., the value of $\chi(G, A)$ is irrelevant if $A \nsubseteq V(G) \cup E(G)$.

In the following, we observe that the matching polynomial $\mu$ and the independent set polynomial $I$ from Definition 2.1 are subset-admissible. It would be nice to show the same for the Tutte polynomials $T$ and $Z$, but this fails for syntactic reasons, since we defined admissible polynomials to be univariate. Instead, we will work with restrictions of $Z$ to $Z_{q}:=Z(q, \cdot)$ for fixed $q \in \mathbb{Q}$.

Example 3.2. Given a sentence $\phi$, define $[\phi]=1$ if $\phi$ is true, and $[\phi]=0$ otherwise. With this notion, the matching polynomial $\mu$ is induced by

$$
\begin{array}{ll}
\chi: & (G, A) \mapsto[A \in \mathcal{M}[G]], \\
\omega: & (G, A) \mapsto \operatorname{usat}(G, A)
\end{array}
$$

and $I$ is induced similarly by $\chi:(G, A) \mapsto[A \in \mathcal{I}[G]]$ and $\omega:(G, A) \mapsto A$.
For $q \in \mathbb{Q} \backslash\{0\}$, the polynomial $Z_{q}=Z(q, \cdot)$ is induced by $\chi:(G, A) \mapsto q^{k(G, A)}$ and $\omega:(G, A) \mapsto A$. We stress again that $Z_{q} \in \mathbb{Q}[x]$ is a univariate restriction of $Z$ for fixed $q \in \mathbb{Q}$.

Every subset-admissible graph polynomial of the form $p_{\chi, \omega}$ admits a canonical multivariate generalization $\mathbf{p}_{\chi, \omega}$ on indeterminates $\mathbf{x}=\left\{x_{a} \mid a \in \mathcal{F}\right\}$, which is given by

$$
\begin{equation*}
\mathbf{p}_{\chi, \omega}(G ; \mathbf{x})=\sum_{A \subseteq V(G) \cup E(G)} \chi(G, A) \prod_{a \in \omega(G, A)} x_{a} \tag{7}
\end{equation*}
$$

Please note that only finitely many indeterminates from $\mathbf{x}$ are present in $\mathbf{p}_{\chi, \omega}(G)$ for any (finite) graph $G$. Compare (7) to (6): It is clear that, when substituting $x_{a} \leftarrow x$ for all $a \in \mathcal{F}$, the multivariate polynomial $\mathbf{p}_{\chi, \omega}$ coincides with the univariate polynomial $p_{\chi, \omega}$. Note also that $\mathbf{p}$ is multilinear by definition. Similar multivariate generalizations were known, e.g., for the Tutte polynomial [28].

Example 3.3. Consider the polynomial $p=p_{\chi, \omega}$ induced by

$$
\begin{array}{ll}
\chi: & (G, A) \mapsto[A \in \mathcal{P} \mathcal{M}[G]] \\
\omega: & (G, A) \mapsto A
\end{array}
$$

This polynomial $p$ admits the simple expression $p(G)=m_{G} \cdot x^{|V(G)| / 2}$, where $m_{G}$ denotes the number of perfect matchings in $G$. Note that $p(G)$ contains at most one monomial. Its multivariate generalization however gives us the richer structure $\mathbf{p}(G)=\operatorname{PerfMatch}(G)$.

Furthermore, for fixed $q \in \mathbb{Q}$, the multivariate generalization of $Z(q, \cdot)$ yields the so-called multivariate Tutte polynomial considered in [28]:

$$
\mathbf{Z}(G ; q, \cdot)=\sum_{A \subseteq E(G)} q^{k(G, A)} \prod_{e \in A} x_{e}
$$

If $p$ is a univariate subset-admissible polynomial and $\mathbf{p}$ is its multivariate generalization, then the following simple relation holds between the coefficients of $p$ and $\mathbf{p}$ :
Lemma 3.4. For any monomial $\theta$, let $c_{\theta}$ denote the coefficient of $\theta$ in $\mathbf{p}$. For $k \in \mathbb{N}$, let $C_{k}$ denote the set of monomials with total power $k$ in $\mathbf{p}$. Then for any $k \in \mathbb{N}$, the coefficient of $x^{k}$ in $p$ is equal to $\sum_{\theta \in C_{k}} c_{\theta}$.
Proof. When substituting $x_{a} \leftarrow x$ for all $a \in \mathcal{F}$, we obtain $p$ from $\mathbf{p}$, and the monomials transformed to $x^{k}$ are precisely the members of $C_{k}$. This proves the claim by collecting the coefficients of these monomials.

### 3.2 First Reduction Step: Multivariate Interpolation

Let $p=p_{\chi, \omega}$ be a subset-admissible polynomial with multivariate generalization $\mathbf{p}$. For ease of presentation, we assume for now that $\omega: \mathcal{G} \times 2^{\mathcal{F}} \rightarrow 2^{\mathcal{E}}$, that is, $\omega$ maps only into edge-subsets rather than subsets of edges and vertices. The general case is shown identically, with more notational overhead.

We perform the reduction $\operatorname{Coeff}(p) \leq_{s e r f}^{T} \operatorname{Eval}(\mathbf{p})$ by means of interpolation. Recall that, in the univariate case, to obtain $p(G)$ for an $m$-edge graph $G$, we require the evaluations of $p(G ; \xi)$ at $m+1$ distinct points $\xi$. For the multivariate generalization $\mathbf{p}$, we could in principle interpolate via Lemma 2.5 Since $\mathbf{p}$ is multilinear, this requires the evaluations of $\mathbf{p}(G ; \xi)$ on a grid $\Xi$ with two distinct values per coordinate, say $\Xi=[2]^{m}$. By Lemma 3.4, the coefficients of $p$ can be obtained from those of $\mathbf{p}$, so we could interpolate $\mathbf{p}$ to recover $p$.

While this detour seems extremely wasteful due to its $2^{m}$ (rather than $m+1$ ) incurred evaluations, it yields the following reward: For each variable $x_{e}$ in $\mathbf{p}$, Lemma 2.5 only requires us to substitute two distinct values (or weights) into $x_{e}$, whereas interpolation on $p$ requires $m+1$ distinct substitutions to its only variable $x$. The small number of distinct weights will prove very useful, since each such weight will be simulated by a certain gadget. If there are only two weights to simulate, then we require only two fixed gadgets, whose sizes are trivially bounded by $O(1)$.

However, to interpolate $\mathbf{p}$, we still need the prohibitively large number of $2^{m}$ evaluations. To overcome this, we trade off the number of evaluations with the numbers of distinct values per edge, and thus, with the size of the gadgets ultimately required. To this end, we group the edges into blocks and treat all edges within each block identically, similar to [12]. At this point, the full power of sub-exponential reduction families from Definition 2.9 is used crucially.
Lemma 3.5. Let $p$ be subset-admissible, with multivariate generalization $\mathbf{p}$, and let $W=\left(w_{0}, w_{1}, \ldots\right)$ be an infinite recursively enumerable sequence of pairwise distinct numbers in $\mathbb{Q}$.

Then $\operatorname{Coeff}(p) \leq_{\text {serf }}^{T} \operatorname{Eval}_{W}(\mathbf{p})$ holds by a reduction that satisfies the following for all inputs $(G, \epsilon)$ : There is some number $d=d(\epsilon)$ depending only upon $\epsilon$, such that all oracle queries $\mathbf{p}\left(G^{\prime}\right)$ are asked only on graphs $G^{\prime}$ obtained from $G$ by introducing edge-weights from $W_{d}=\left\{w_{0}, \ldots, w_{d}\right\}$.

When invoking Lemma 3.5 the list $W$ contains the weights that can be simulated by gadgets. Note that any such list $W$ can be used if $W$ is infinite and recursively enumerable. Furthermore, note that $\mathbf{p}$ is evaluated only on edge-weighted versions of $G$ itself; properties such as bipartiteness of $G$ or its size are hence trivially preserved.

Proof of Lemma 3.5. Let $d \in \mathbb{N}$ be a parameter, to be chosen later depending on $\epsilon$, and let $G=(V, E)$ be an $m$-edge graph for which we want to determine the coefficients of $p=p(G)$. Let

$$
\mathbf{x}=\left\{x_{e} \mid e \in E\right\}
$$

denote the indeterminates of $\mathbf{p}$ and note that both $p$ and $\mathbf{p}$ have maximum degree $m$ by definition.
In the first step, partition $E$ into $t:=\lceil m / d\rceil$ sets $E_{1}, \ldots, E_{t}$ of size at most $d$ each (which we call blocks), using an arbitrary equitable assignment of edges to blocks. Define new indeterminates

$$
\mathbf{y}=\left\{y_{1}, \ldots, y_{t}\right\}
$$

and a new multivariate polynomial $\mathbf{q} \in \mathbb{Q}[\mathbf{y}]$ from $\mathbf{p}$ by substituting

$$
x_{e} \leftarrow y_{i} \quad \text { for all } i \in[t] \text { and } e \in E_{i} .
$$

|  | $p \in \mathbb{Q}[x]$ | $\mathbf{p} \in \mathbb{Q}[\mathbf{x}]$ | $\mathbf{q} \in \mathbb{Q}[\mathbf{y}]$ |
| :---: | :---: | :---: | :---: |
| number of indeterminates | 1 | $m$ | $t=\lceil m / d\rceil$ |
| max. degree per indeterminate | $m$ | 1 | $d$ |
| max. number of monomials | $m+1$ | $2^{m}$ | $(d+1)^{t}$ |

Table 1: The polynomials $p, \mathbf{p}$ and $\mathbf{q}$ appearing in the proof of Lemma 3.5

We are working with three polynomials, namely $p, \mathbf{p}$ and $\mathbf{q}$, summarized in Table 1 for convenience. While the total degree of $\mathbf{q}$ is bounded by $m$, the degree of each indeterminate $y_{i}$ in $\mathbf{q}$ is bounded by $d$, since each block contains at most $d$ edges. Hence, the number of monomials in $\mathbf{q}$ is at most $(d+1)^{t}=2^{d^{\prime} m}$ with $d^{\prime}=O(\log (d) / d)$. Note that $d^{\prime} \rightarrow 0$ as $d \rightarrow \infty$.

We will ultimately obtain the coefficients of $\mathbf{q}$ via interpolation, but first, let us observe that the coefficients of $\mathbf{q}$ allow us to determine those of the univariate version $p$. Write $c_{k}^{p}$ for the coefficient of $x^{k}$ in $p$ and $c_{\theta}^{\mathbf{q}}$ for the coefficient of the monomial $\theta$ in $\mathbf{q}$. Analogously to Lemma 3.4, we have $c_{k}^{p}=\sum_{\theta \in C_{k}} c_{\theta}^{\mathbf{q}}$ where $C_{k}$ for $k \in \mathbb{N}$ is the set of all monomials with total power $k$ in $\mathbf{q}$. This allows us to compute the coefficients of $p$ from those of $\mathbf{q}$.

It remains to describe how to obtain the coefficients of $\mathbf{q}$. For this, recall the definition of $W_{d}$ from the statement of the lemma. We evaluate $\mathbf{q}$ on the grid $\Xi=\left(W_{d}\right)^{t}$ using the oracle for Eval ${ }_{W}(\mathbf{p})$ : For each $\xi \in \Xi$, substitute $y_{i} \leftarrow \xi_{i}$ for all $i \in[t]$ to obtain an edge-weighted graph $G_{\xi}$ that contains only weights from $W_{d}$, and for which we can thus compute $\mathbf{p}\left(G_{\xi}\right)$ by an oracle call to $\mathrm{Eval}_{W}(\mathbf{p})$.

Using $|\Xi|=(d+1)^{t}=2^{d^{\prime} m}$ oracle calls and grid interpolation via Lemma 2.5, we obtain all coefficients of $\mathbf{q}$ with $O\left(2^{3 d^{\prime} m}\right)$ arithmetic operations. By definition of $\mathbf{p}$ and $\mathbf{q}$ and the set $W_{d}$, each arithmetic operation involves numbers on at most $O(m) \cdot g(d)$ bits for a computable function $g$. Since $d^{\prime} \rightarrow 0$ as $d \rightarrow \infty$, we can pick $d$ large enough such that $3 d^{\prime} \leq \epsilon$, and thus achieve running time $O\left(2^{\epsilon m}\right)$. No vertices or edges were added to $G$ during this reduction.

### 3.3 Second Reduction Step: Weight Simulation by Gadgets

Lemma 3.5 gives a sub-exponential reduction family from $\operatorname{Coeff}(p)$ to $\operatorname{Eval}(\mathbf{p})$ that maps instances $(G, \epsilon)$ to edge-weighted versions obtained from $G$ by introducing $f(\epsilon)$ distinct edge-weights for some computable function $f$. For the full reduction, this latter problem must be reduced to Eval ${ }_{\xi}(p)$ for fixed $\xi \in \mathbb{Q}$.

This may not work for all $\xi \in \mathbb{Q}$ : For instance, the evaluation problem Eval $_{0}(I)$ for the independent-set polynomial $I$ at the point 0 is trivial. We must hence impose several conditions on $\xi$ to enable this reduction.
Definition 3.6. Let $p$ be subset-admissible, let $\xi \in \mathbb{Q}$ and

- let $W=\left(w_{0}, w_{1}, \ldots\right)$ be a sequence of pairwise distinct values in $\mathbb{Q}$,
- let $\mathcal{H}=\left(H_{0}, H_{1}, \ldots\right)$ be a sequence of edge-gadgets, which are triples $(H, u, v)$ with a graph $H$ and attachment vertices $u, v \in V(H)$, and
- let $F: \mathcal{G} \times \mathbb{Q} \rightarrow \mathbb{Q} \backslash\{0\}$ be a polynomial-time computable function, which we call a factor function.

If $G$ is an edge-weighted graph with weights from $W$, let $T(G)$ be the unweighted graph obtained by replacing, for $i \in \mathbb{N}$, each edge $u v \in E(G)$ of weight $w_{i}$ with a fresh copy of the edge-gadget $H_{i}$ by identifying $u, v$ across $G$ and $H_{i}$.

We say that $(\mathcal{H}, F)$ allows to reduce $\operatorname{Eval}_{W}(\mathbf{p})$ to $\operatorname{Eval}_{\xi}(p)$ if the following holds: Whenever $G$ is a graph with edge-weights from $W$, then

$$
\begin{equation*}
\mathbf{p}(G)=\frac{p(T(G) ; \xi)}{F(G, \xi)} \tag{8}
\end{equation*}
$$

Remark 3.7. The same definition applies to vertex-weighted graphs; here we use vertex-gadgets, which are pairs $(H, v)$ with an attachment vertex $v \in V(H)$. Vertex-gadgets are inserted at a vertex $v \in V(G)$ by identifying $v$ in $H$ and $G$.

At the end of this subsection, we discuss several possible extensions of this definition. As a first example for the notions introduced in Definition 3.6 we consider (well-known) edge-gadgets for PerfMatch.

Example 3.8. Let $p$ denote the polynomial from Example 3.3, whose multivariate generalization satisfies $\mathbf{p}=$ PerfMatch. For $k \in \mathbb{N}$, let $H_{k}$ be the edge-gadget obtained by placing $k$ parallel edges between two fresh vertices $u$ and $v$ and subdividing each edge twice. Let $\mathcal{H}=\left(H_{1}, H_{2}, \ldots\right)$, let $\mathbb{N}=(1,2,3, \ldots)$ and let $F$ denote the function mapping all inputs to 1 . Then it can be seen that $(\mathcal{H}, F)$ allows to reduce Eval $\mathbb{N}^{( }(\mathbf{p})$ to $\operatorname{Eval}_{1}(p)$.

We easily obtain the following lemma from Definition 3.6
Lemma 3.9. Let $W=\left(w_{0}, w_{1}, \ldots\right)$ and let $(\mathcal{H}, F)$ allow to reduce $\operatorname{Eval}_{W}(\mathbf{p})$ to $\operatorname{Eval}_{\xi}(p)$. Let $G$ be a graph with edge-weights from $W$. Then we can compute $\mathbf{p}(G)$ from $p(T(G) ; \xi)$ in polynomial time via 8). If $G$ has $n$ vertices and $m$ edges, and only contains edge-weights $w_{i}$ with $i \leq t$ for some $t \in \mathbb{N}$, then $T(G)$ has $O(n+s m)$ vertices and edges, where $s=\max _{i \in[t]}\left|V\left(H_{i}\right)\right|+\left|E\left(H_{i}\right)\right|$ depends only on $\mathcal{H}$ and $t$.

By combining Lemmas 3.5 and 3.9 we obtain the wanted reduction from $\operatorname{Coeff}(p)$ to $\operatorname{Eval}_{\xi}(p)$ at fixed points $\xi \in \mathbb{Q}$ and finish the set-up of our framework.

Theorem 3.10. Let $p$ be subset-admissible and let $\xi \in \mathbb{Q}$ be fixed. Assuming \#ETH, there are constants $\epsilon, C>0$ such that the problem $\operatorname{Eval}_{\xi}(p)$ admits no $O\left(2^{\epsilon n}\right)$ time algorithm on unweighted graphs with $n$ vertices and at most $C n$ edges, provided that the following two conditions hold:

Coefficient hardness: Assuming \#ETH, there are constants $\delta, D>0$ such that Coeff $(p)$ admits no $O\left(2^{\delta n}\right)$ time algorithm on unweighted graphs with $n$ vertices and at most Dn edges.

Weight simulation: There is a recursively enumerable sequence of pairwise distinct weights $W=\left(w_{0}, w_{1}, \ldots\right)$, a sequence of gadgets $\mathcal{H}=\left(H_{0}, H_{1}, \ldots\right)$, and a function $F$ such that $(\mathcal{H}, F)$ allows to reduce $\operatorname{Eval}_{W}(\mathbf{p})$ to $\operatorname{Eval}_{\xi}(p)$.

Proof. We present a sub-exponential reduction family from $\operatorname{Coeff}(p)$ to $\operatorname{Eval}_{\xi}(p)$. Given $\gamma>0$ and a graph $G$ with $n$ vertices and $D n$ edges, first apply Lemma 3.5. This way, we reduce Coeff $(p)$ to $O\left(2^{\gamma n}\right)$ instances of Eval $_{W}(\mathbf{p})$ on graphs $G^{\prime}$ that only use weights $w_{0}, \ldots, w_{s}$ with $s=s(\gamma)$.

Since $(\mathcal{H}, F)$ allows to reduce $\operatorname{Eval}_{W}(\mathbf{p})$ to $\operatorname{Eval}_{\xi}(p)$, we can invoke Lemma 3.9 and reduce each instance $G^{\prime}$ for $\operatorname{Eval}_{W}(\mathbf{p})$ to an instance of $\operatorname{Eval}_{\xi}(p)$ on the graph $T\left(G^{\prime}\right)$. This graph features at most $g(s) \cdot D \cdot n$ vertices and edges, where $g$ is a computable function. Note that this second reduction runs in polynomial time and also satisfies the requirements for a sub-exponential reduction. Altogether, the theorem then follows from Corollary 2.11.

Let us remark some corollaries of the reduction shown above.
Remark 3.11. If the source instance $G$ for $\operatorname{Coeff}(p)$ has maximum degree $\Delta$, then the reduction images $T\left(G^{\prime}\right)$ obtained above on parameter $\epsilon$ feature maximum degree $c \Delta$ for a constant $c=c(\epsilon)$. By suitable choice of $\mathcal{H}$, we can also ensure other properties on $T(G)$ :

- If $G$ is bipartite and all edge-gadgets $(H, u, v) \in \mathcal{H}$ can be 2-colored such that $u$ and $v$ receive different colors, then $T\left(G^{\prime}\right)$ is bipartite as well. This can be verified, e.g., for Example 3.8
- If $G$ is planar and all edge-gadgets $(H, u, v) \in \mathcal{H}$ admit planar drawings with $u$ and $v$ on their outer faces, then $T\left(G^{\prime}\right)$ is planar as well.

To conclude this subsection, we list several possible generalizations of Definition 3.6 and Theorem 3.10 that we chose not to include in our formulation.

1. We may extend Definition 3.6 to incorporate weight simulations that proceed non-locally, that is, in a less obvious way than by inserting local gadgets at edges. This route was taken in [3] since the conference version of this article was published.
2. In Lemma 3.5, we do not need to evaluate $\mathbf{p}$ on a grid $W^{t}$ for a fixed coordinate set $W$. Instead, we could as well interpolate on a grid $W_{1} \times \ldots \times W_{t}$, provided that each $W_{i}$ is large enough and that the weights do not depend on the size of $G$.
3. We may also allow $2^{o(m)}$ time for the computation of the factor function $F\left(G_{w}, \xi\right)$. Rather than allowing only a multiplicative factor, we could also allow an arbitrary function to be computed from $p(T(G) ; \xi)$ and the input.

## 4 Applications of the Framework

In the following subsections, we apply Theorem 3.10 to obtain tight lower bounds for concrete counting problems, including the unweighted permanent in Section 4.1 the matching and independent set polynomials in Section 4.2 and the Tutte polynomial in Section 4.3.

### 4.1 The Unweighted Permanent

As mentioned in the introduction, it was shown in [14] that, unless \#ETH fails, the problem Eval $_{\{-1,1\}}($ perm $)$ on graphs with $n$ vertices and $O(n)$ edges admits no algorithm with running time $2^{o(n)}$. For convenience, we recall that this is the problem of evaluating the permanent on graphs with edge-weights +1 and -1 . In the same paper, it was also shown that an algorithm for the unweighted permanent on such graphs would falsify rETH, the randomized version of ETH. We improve upon this by showing that it is sufficient to assume \#ETH, which is a priori weaker than ETH and also constitutes a more natural assumption for lower bounds on counting problems.

Theorem (Restatement of Theorem 1.1). Assuming \#ETH, there are constants $\epsilon, C>0$ such that the problem $\mathrm{Eval}_{1}(\mathrm{perm})$ of counting unweighted perfect matchings in bipartite graphs cannot be solved in time $O\left(2^{\epsilon n}\right)$ on graphs with $n$ vertices and at most $C n$ edges.

Proof. In the following, we invoke Theorem 3.10 to show

$$
\operatorname{Eval}_{\{-1,1\}}(\text { perm }) \leq_{\text {serf }}^{T} \operatorname{Eval}_{1}(\text { perm })
$$

Let $G$ be a graph with edge-weights from $\{-1,1\}$ and let $E_{-1}(G)$ denote the set of edges with weight -1 in $G$. Define a sieve function and weight selector

$$
\begin{aligned}
\chi(G, A) & =[A \in \mathcal{P M}[G]] \\
\omega(G, A) & =A \cap E_{-1}(G)
\end{aligned}
$$

and observe that these induce a univariate graph polynomial $p=p_{\chi, \omega}$ with

$$
p(G ;-1)=\operatorname{perm}(G)
$$

Since knowing all coefficients of $p(G)$ clearly allows to evaluate $p(G ;-1)$, we obtain from [14, Thm. 1.3] that there are constants $\delta, D>0$ such that $\operatorname{Coeff}(p)$ cannot be solved in time $O\left(2^{\delta n}\right)$ on $n$-vertex graphs with $D n$ edges, unless \#ETH fails. Hence the coefficient hardness condition of Theorem 3.10 is satisfied ${ }^{2}$

To check the weight simulation condition, recall the pair $(\mathcal{H}, F)$ from Example 3.8 that allows to reduce $\operatorname{Eval}_{\mathbb{N}}(\mathbf{p})$ to $\mathrm{Eval}_{1}(p)$. Hence all prerequisites for Theorem 3.10 are fulfilled, so counting perfect matchings in unweighted graphs has the desired lower bound. By Remark 3.11 the reduction images $T(G)$ constructed by the theorem are bipartite as well. This proves the theorem.

[^2]We collect a series of corollaries for other counting problems from this theorem. Let $L(G)$ denote the line graph of a graph $G=(V, E)$ : This graph has vertex set $E$, and $e, e^{\prime} \in E$ are defined to be adjacent in $L(G)$ iff $e \cap e^{\prime} \neq \emptyset$. A graph is a line graph if it is the line graph of some graph.

Corollary 4.1. Assuming \#ETH, there is a constant $\epsilon>0$ for each of the following problems such that no $O\left(2^{\epsilon n}\right)$ time algorithm solves the problem:

1. $\mathrm{Eval}_{1}(\mathrm{perm})$, that is, counting perfect matchings in bipartite graphs, in graphs with $n$ vertices and maximum degree 3 .
2. Counting maximum independent sets (or minimum vertex covers), even in line graphs with $n$ vertices and maximum degree 4.
3. Counting minimum-weight satisfying assignments to monotone $2-C N F$ formulas on $n$ variables, even if every variable appears in at most 4 clauses.

Proof. For the first statement, we use a known reduction from the permanent on general bipartite graphs to bipartite graphs of maximum degree 3, shown in [13, Theorem 6.2]. This reduction maps graphs with $n$ vertices and $m$ edges to graphs with $O(n+m)$ vertices and edges.

For the second statement, if $G$ has $m$ edges and maximum degree $\Delta=\Delta(G)$, then $L(G)$ has $m$ vertices and maximum degree $2(\Delta-1)$. The set $\mathcal{P} \mathcal{M}[G]$ corresponds bijectively to the independent sets of size $n / 2$ in $L(G)$, which are the maximum independent sets in $L(G)$, unless $G$ has no perfect matching, which we can test efficiently. The maximum independent sets in turn stand in bijection with the minimum vertex covers of $L(G)$ via complementation. We thus obtain the statement by reduction from $E^{2} \mathrm{Eva}_{1}$ (perm) on graphs of maximum degree 3 .

For the third statement, observe that the minimum vertex covers of a graph $H=(V, E)$ correspond bijectively to the minimum-weight satisfying assignments of an associated monotone 2-CNF formula $\varphi$ : To obtain this formula, create a variable $x_{v}$ for each $v \in V$ and a clause ( $x_{u} \vee x_{v}$ ) for each $u v \in E$. This is a standard reduction, noted also in [29, Proposition 2.1].

### 4.2 The Matching and Independent Set Polynomials

We prove Theorem 1.2 a tight lower bound for the evaluation problem of the matching polynomial $\operatorname{Eval}_{\xi}(\mu)$ at fixed $\xi \in \mathbb{Q}$, by invoking Theorem 3.10 . As described in the introduction, the perfect matchings of $G$ are counted by the coefficient of $x^{0}$ in $\mu(G)$, so $\operatorname{Coeff}(\mu)$ and $\operatorname{Eval}_{0}(\mu)$ have the same lower bound as Eval ${ }_{1}($ perm $)$. This settles the requirement of coefficient hardness in Theorem 3.10. In the following, we analyze the example for an interpolation-based reduction from the introduction (where we reduced counting perfect matchings to counting matchings) to show that $\mu$ also admits weight simulation.

Lemma 4.2. Let $\mathcal{H}=\left(H_{i}\right)_{i \in \mathbb{N}}$ be the following family of vertex-gadgets, where $H_{i}$ contains one attachment vertex $v$, adjacent to an independent set of $i$ vertices.


Consider $\xi \in \mathbb{Q} \backslash\{0\}$ to be fixed. Let $W=\left(w_{t}\right)_{t \in \mathbb{N}}$ with $w_{t}=1+\frac{t}{\xi^{2}}$ for $t \in \mathbb{N}$. Given a graph $G$ with vertex-weights from $W$, let $a_{t}$ for $t \in \mathbb{N}$ denote the number of vertices of weight $w_{t}$ in $G$. We define

$$
F(G, \xi)=\prod_{t \in \mathbb{N}} \xi^{t \cdot a_{t}}
$$

Then $(\mathcal{H}, F)$ allows to reduce $\operatorname{Eval}_{W}(\mu)$ to $\operatorname{Eval}_{\xi}(\mu)$.

Proof. Recall the graph transformation $T(G)$ for vertex-weighted graphs from Definition 3.6 and Remark 3.7 At every vertex of weight $w_{t}$, for $t \in \mathbb{N}$, we attach the gadget $H_{t}$. To show that $(\mathcal{H}, F)$ indeed satisfies Definition 3.6, we need to show that

$$
\begin{equation*}
\mu(T(G), \xi)=F(G, \xi) \cdot \mu(G) \tag{9}
\end{equation*}
$$

To see this, observe that every matching $M$ in $G$ can be extended locally at vertices $v \in V(G)$ by edges of vertex-gadgets to obtain a matching in $T(G)$. Let $v \in V(G)$ be a vertex of weight $w_{t}$ for $t \in \mathbb{N}$. If $v \notin \operatorname{usat}(G, M)$, then $M$ cannot be extended at the vertex $v$, and $v$ incurs the factor $\xi^{t}$ in $\mu(T(G))$. If $v \in \operatorname{usat}(G, M)$, then we can choose not to extend $v$, or we may choose to extend by one of the $t$ edges of $H_{t}$, so we obtain a factor of

$$
\xi^{t}+t \xi^{t-2}=\xi^{t}\left(1+\frac{t}{\xi^{2}}\right)
$$

at the vertex $v$. This establishes (9), and hence the lemma.
The desired theorem follows.
Theorem (Restatement of Theorem 1.2. If \#ETH holds, then for each $\xi \in \mathbb{Q}$, there are constants $\epsilon, C>0$ such that $\mathrm{Eval}_{\xi}(\mu)$ cannot be solved in time $O\left(2^{\epsilon n}\right)$ on graphs with $n$ vertices and maximum degree $C$. With $\xi=1$, this holds especially for $\operatorname{Eval}_{1}(\mu)$, which amounts to counting matchings.
Proof. By Corollary 4.1. there is a constant $\delta>0$ such that $\operatorname{Coeff}(\mu)$ cannot be solved in time $O\left(2^{\delta n}\right)$ unless \#ETH fails, even on graphs with maximum degree 3. This settles the requirement of coefficient hardness in Theorem 3.10, even on graphs of maximum degree 3.

In Lemma 4.2, we have seen that $\mu$ admits weight simulations. By Remark 3.11 and the reduction from $\operatorname{Coeff}(\mu)$ on graphs of maximum degree 3 , the queries issued by Theorem 3.10 have maximum degree $O(1)$, which implies the degree bound in the theorem.

Remark 4.3. For later use, note that the same proof yields the same lower bound for $\mu(G ; \xi)$ even when $\xi \in \mathbb{C}$ is a complex number with $\xi=\sqrt{c}$ for $c \in \mathbb{Q}$. To this end, note that we may assume that $G$ has an even number of vertices (for instance, by adding an isolated vertex and dividing $\mu(G ; \xi)$ by $\xi$ ). Then we can compute $\mu(G ; \xi)$ over the rational numbers: Every matching in $G$ has an even number of unmatched vertices, and thus only even powers of $\xi$ appear in $\mu(G ; \xi)$.

As in Corollary 4.1 we can easily obtain corollaries for the independent set polynomial $I$ and for counting monotone 2-SAT, improving upon [18, 14].
Corollary 4.4. Assuming \#ETH, there is a constant $\epsilon>0$ for each of the following problems such that no $O\left(2^{\epsilon n}\right)$ time algorithm solves the problem:

1. $\operatorname{Eval}_{\xi}(I)$ on line graphs of maximum degree $O(1)$, for $\xi \in \mathbb{Q} \backslash\{0\}$, especially at $\xi=1$, which amounts to counting independent sets (or vertex covers).
2. Counting satisfying assignments to monotone 2-CNF formulas, even if every variable appears in at most $O(1)$ clauses.

Proof. We first prove the first statement: If $G$ has $n$ vertices and $m$ edges, and satisfies $\Delta(G)=O(1)$, then $L(G)$ has $m$ vertices and $\Delta(L(G))=O(1)$. For every matching $M \in \mathcal{M}[G]$, we have

$$
2|M|+|\operatorname{usat}(M)|=n
$$

and since matchings of $G$ stand in bijection with independent sets of $L(G)$,

$$
\mu(G ; \xi)=\sum_{M \in \mathcal{M}[G]} \xi^{|\operatorname{usat}(M)|}=\xi^{n} \cdot \sum_{M \in \mathcal{M}[G]} \xi^{-2|M|}=\xi^{n} \cdot I\left(L(G) ; \xi^{-2}\right)
$$

Hence, for fixed $\xi \neq 0$, an algorithm for $\operatorname{Eval}_{\xi}(I)$ on line graphs implies one for Eval ${ }_{\xi^{\prime}}(\mu)$ on general graphs with $\xi^{\prime}=\sqrt{\xi^{-1}}$, but this is ruled out by Theorem 1.2 and Remark 4.3 .

For the second statement, recall that independent sets and vertex covers stand in bijection. We then reduce from counting vertex covers as in Corollary 4.1

### 4.3 The Tutte Polynomial

To apply the block-interpolation framework to the Tutte polynomial, we use univariate restrictions of $Z$, as discussed in Example 3.2 Let $Z_{q}=Z(q, \cdot)$ for fixed $q \in \mathbb{Q} \backslash\{0\}$. As in [14], for $q=0$, we instead consider the polynomial

$$
Z_{0}(G ; w)=\sum_{A \subseteq E(G)} 0^{k(G, A)-k(G, E)} w^{|A|}
$$

Note that terms corresponding to $A \subseteq E(G)$ with $k(G, A) \neq k(G, E)$ vanish in $Z_{0}(G ; w)$. We will use $Z_{0}$ to prove lower bounds for the Tutte polynomial on the line defined by $x=1$ : If $(x, y) \in \mathbb{Q}^{2}$ satisfies $x=1$, we can write $q=0$ and $w=y-1$ and obtain

$$
\begin{align*}
T(G ; x, y) & =(x-1)^{-k(G, E)} \cdot(y-1)^{-|V(G)|} \cdot Z(G ; q, w) \\
& =w^{-|V(G)|} \cdot Z_{0}(G ; w) . \tag{10}
\end{align*}
$$

Using Theorem 3.10, we then prove lower bounds for $\operatorname{Eval}_{w}\left(Z_{q}\right)$ at fixed $q, w \in \mathbb{Q}$. As in the previous examples, we first require a lower bound for $\operatorname{Coeff}\left(Z_{q}\right)$, which we adapt from [14].

Lemma 4.5. 14, Propositions 4.1 and 4.3] Assuming $\# E T H$, for each $q \in \mathbb{Q} \backslash\{1\}$, there are constants $\epsilon, C>0$ such that the problem Coeff $\left(Z_{q}\right)$ cannot be solved in time $O\left(2^{\epsilon n}\right)$ on n-vertex graphs with Cn edges $\underbrace{3}_{-}$

Remark 4.6. In fact, we could also use block interpolation to simplify this result from [14] by performing an interpolation step that needed to be circumvented by the authors with some tricks. However, since Lemma 4.5 was already shown in [14], we omit the self-contained proof that would still require some arguments which are very specific to the Tutte polynomial.

Note that the case $q=1$ is left uncovered by this lemma, and we consequently cannot prove lower bounds at $q=1$, where $\operatorname{Coeff}\left(Z_{1}\right)$ in fact becomes polynomial-time solvable.

In [14], the problem $\operatorname{Coeff}\left(Z_{q}\right)$ with $q \neq 1$ is reduced to unweighted evaluation via Theta graphs and wumps, families of edge-gadgets that incur only $O\left(\log ^{c} n\right)$ blowup. This economical (but still not constant) factor however requires a quite involved analysis. Using block interpolation, we can instead use mere paths, and hence perform stretching, a classical weight simulation technique for the Tutte polynomial [22, 14, 16]. In the following, please recall $\mathbf{Z}_{q}$, as defined by Example 3.2 and 7 .

Lemma 4.7. For $k \in \mathbb{N}$, let $P_{k}$ denote the path on $k$ edges with distinguished start/end vertices $u, v \in V\left(P_{k}\right)$ and let $\mathcal{P}=\left(P_{1}, P_{2}, \ldots\right)$. Let $w, q \in \mathbb{Q}$ be fixed with $w \neq 0$ and $q \notin\{1,-w,-2 w\}$. Then there is an infinite recursively enumerable sequence of pairwise distinct weights $W$ and a factor function $F$ such that $(\mathcal{P}, F)$ allows to reduce $\mathrm{Eval}_{W}\left(\mathbf{Z}_{q}\right)$ to $\operatorname{Eval}_{w}\left(Z_{q}\right)$.

Proof. We have to distinguish whether $q=0$ or $q \neq 0$ holds, and we obtain different weights and factor functions in the different cases.

If $q=0$, we define $W=\left(w_{k}\right)_{k \in \mathbb{N}}$ with the pairwise distinct weights $w_{k}=\frac{w}{k}$ for integers $k \in \mathbb{N}$. Given a graph $G$ with edge-weights from $W$, let $a_{k}(G)$ for $k \in \mathbb{N}$ denote the number of edges in $G$ with weight $w_{k}$, and define

$$
F(G)=\prod_{k \in \mathbb{N}}\left(k w^{k-1}\right)^{a_{k}(G)} .
$$

Then $(\mathcal{P}, F)$ allows to reduce $\operatorname{Eval}_{W}\left(\mathbf{Z}_{0}\right)$ to $\operatorname{Eval}_{w}\left(Z_{0}\right)$, see for instance [14, Corollary 6.7] and [15].
If $q \neq 0$, then the family of paths realizes different weights and requires a different factor function. Define $W=\left(w_{k}\right)_{k \in \mathbb{N}}$ with

$$
w_{k}=\frac{q}{\left(1+\frac{q}{w}\right)^{k}-1}
$$

[^3]and observe that these weights are pairwise distinct provided that $1+\frac{q}{w} \notin\{-1,0,1\}$, which holds by $q \neq 0$ and the prerequisites of the proposition. Given a graph $G$ with edge-weights from $W$, let $a_{k}(G)$ for $k \in \mathbb{N}$ denote the number of edges in $G$ with weight $w_{k}$ and define
$$
F(G)=q^{-|E(G)|} \prod_{k \in \mathbb{N}}\left((q+w)^{k}-w^{k}\right)^{a_{k}(G)}
$$

It is shown [14, Lemma 6.2] and [28, Prop. 2.2 and 2.3] that $(\mathcal{P}, F)$ allows to reduce $\mathbf{Z}_{q}$ on $W$ to $Z_{q}(w)$.
By combining Lemma 4.5 for the coefficient hardness and Lemma 4.7 for weight simulations, we can then invoke Theorem 3.10 and obtain:

Lemma 4.8. Let $w \neq 0$ and $q \notin\{1,-w,-2 w\}$. Assuming \#ETH, there are constants $\epsilon, C>0$ such that the problem $\operatorname{Eval}_{w}\left(Z_{q}\right)$ admits no $O\left(2^{\epsilon n}\right)$ time algorithm on graphs with $n$ vertices and at most $C n$ edges.

From this lemma, we can derive the hardness of the problem $Z(q, w)$ at most points with $q \notin\{0,1\}$ analogously as in [14, Proposition 6.4].

Lemma 4.9. Let $(q, w) \in \mathbb{Q}^{2} \backslash\{(4,-2),(2,-1),(2,-2)\}$ with $q \notin\{0,1\}$ and $w \neq 0$. Assuming \#ETH, there are constants $\epsilon, C>0$ such that the problem $\mathrm{Eval}_{w}\left(Z_{q}\right)$ admits no $O\left(2^{\epsilon n}\right)$ time algorithm on graphs with $n$ vertices and at most $C n$ edges.

Proof. By Lemma 4.8, the claim must only be shown if $q \in\{-w,-2 w\}$ holds in addition to the prerequisites of Lemma 4.9 As in [14, Proposition 6.4], we then use the operations of thickening and stretching to reduce the problem $\operatorname{Eval}_{w^{\prime}}\left(Z_{q}\right)$ for some $w^{\prime}$ with $q \notin\left\{-w^{\prime},-2 w^{\prime}\right\}$ to $\mathrm{Eval}_{w}\left(Z_{q}\right)$. The hardness of Eval ${ }_{w}\left(Z_{q}\right)$ then follows from Lemma 4.8 .

To proceed this way, let $G_{k}$ be the graph obtained from $G$ by replacing each edge with $k$ parallel edges, followed by subdividing each edge once. Then there exists a number $w_{k}$, depending on $q, w$, and $k$, such that $Z\left(G ; q, w_{k}\right)$ can be computed in polynomial time from the value $Z\left(G_{k} ; q, w\right)$, as shown in [14, Proposition 6.4]. The same reference shows that, if the prerequisites of the lemma are satisfied, a suitable value $k=k(q, w)$ can be chosen such that $q \notin\left\{-w_{k},-2 w_{k}\right\}$. Since $q, w$ are fixed, we have $k=O(1)$, and the graph $G_{k}$ hence has $O(|V(G)|+|E(G)|)$ vertices and edges. This proves the claim.

By the substitution (2) that maps $Z(\cdot, \cdot)$ to the classical parameterization $T(\cdot, \cdot)$ of the Tutte polynomial, we can rephrase this result in terms of $T$.

Theorem (Restatement of Theorem 1.3). Assuming \#ETH, there are constants $\epsilon, C>0$ such that the Tutte polynomial $T(x, y)$ cannot be evaluated in time $O\left(2^{\epsilon n}\right)$ on graphs with $n$ vertices and at most Cn edges, provided that $y \neq 1$, and $(x, y) \notin\{(-1,-1),(0,-1),(-1,0)\}$, and $(x-1)(y-1) \neq 1$.

Proof. Using (2), computing $Z(G ; q, w)$ is equivalent to computing $T(G ; x, y)$ with $x=\frac{q}{w}+1$ and $y=w+1$, provided that $q \neq 0$. We use this to rephrase the evaluations $Z(q, w)$ for $(q, w) \in \mathbb{Q}^{2}$ that are not shown to be hard by Lemma 4.9 in terms of $T(x, y)$.

1. If $w=0$, then $y=1$.
2. If $(q, w) \in\{(4,-2),(2,-1),(2,-2)\}$, then $(x, y) \in\{(-1,-1),(-1,0),(0,-1)\}$.
3. If $q=1$, then $(x-1)(y-1)=1$.

Hence, Lemma 4.9 shows a tight lower bound for all points $(x, y) \in \mathbb{Q}^{2}$ relevant for the theorem that satisfy $q=(x-1)(y-1) \neq 0$. We then consider those points with $q=0$. Since we may assume $y \neq 1$, only points $(x, y)$ with $x=1$ and $y \neq 1$ are left open. In this case, we invoke Lemma 4.8 with $q=0$ and $w=y-1$. Using (10), we then obtain $T(G ; 1, y)=w^{-|V(G)|} \cdot Z_{0}(G ; w)$. Since $w \neq 0$ and $\operatorname{Eval}_{w}\left(Z_{0}\right)$ admits a tight lower bound under \#ETH by Lemma 4.8 the theorem follows.

If either of the last two conditions of Theorem 1.3 does not hold, then the evaluation of the Tutte polynomial is known to admit a polynomial-time algorithm. The \#P-hard points on the line given by $y=1$ are however not covered by Theorem 1.3 and they actually do not fit into the block interpolation framework as defined in this paper. Nevertheless, as discussed earlier, this line was settled recently [3] by extending the block interpolation framework to a setting where gadgets are not required to be placed locally at vertices.

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[^1]:    ${ }^{1}$ This vector includes the coefficients of all monomials with degree at most $d_{i}$ in $x_{i}$, even if some of these coefficients may be zero.

[^2]:    ${ }^{2}$ In fact, the authors of [14] state their lower bound as ruling out $2^{o(n)}$ time algorithms for Coeff $(p)$ on graphs with $n$ vertices and $O(n)$ edges. This is however only to simplify the presentation of their result. Their reduction from counting satisfying assignments for 3 -CNFs to $\mathrm{Eval}_{\{-1,1\}}\left(\right.$ perm ) is in fact a $\leq_{\text {serf }}^{T}$ reduction and hence also supports the stronger claim needed for the coefficient hardness condition of Theorem 3.10

[^3]:    ${ }^{3}$ In [14], this result is stated as $\operatorname{Coeff}\left(Z_{q}\right)$ not having $2^{o(m)}$ time algorithms for $q \in \mathbb{Q} \backslash\{1\}$. However, the paper actually shows the slightly stronger claim in the statement of the lemma.

