# New Deterministic Algorithms for Solving Parity Games* 

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#### Abstract

We study parity games in which one of the two players controls only a small number $k$ of nodes and the other player controls the $n-k$ other nodes of the game. Our main result is a fixed-parameter algorithm that solves bipartite parity games in time $k^{O(\sqrt{k})} \cdot O\left(n^{3}\right)$, and general parity games in time $(p+k)^{O(\sqrt{k})} \cdot O(p n m)$, where $p$ is the number of distinct priorities and $m$ is the number of edges. For all games with $k=o(n)$ this improves the previously fastest algorithm by Jurdziński, Paterson, and Zwick (SICOMP 2008).


We also obtain novel kernelization results and an improved deterministic algorithm for graphs with small average degree.

## 1 Introduction

A parity game [5] is a two-player game of perfect information played on a directed graph $G$ by two players, even and odd, who move a token from node to node along the edges of $G$ so that an infinite path is formed. The nodes of $G$ are partitioned into two sets $V_{0}$ and $V_{1}$; the even player moves if the token is at a node in $V_{0}$ and the odd player moves if the token is at a node in $V_{1}$. The nodes of $G$ are labeled by a priority function $p: V \rightarrow \mathbb{N}_{0}$, and the players compete for the parity of the highest priority occurring infinitely often on the infinite path $v_{0}, v_{1}, v_{2} \ldots$ describing a play: the even player wins if $\lim \sup _{i \rightarrow \infty} p\left(v_{i}\right)$ is even, and the odd player wins if it is odd.

The winner determination problem for parity games is the algorithmic problem to determine for a given parity game $G=\left(V_{0} \uplus V_{1}, E, p\right)$ and an initial node $v_{0} \in V_{0} \cup V_{1}$, whether the even player has a winning strategy in the game if the token is initially placed on node $v_{0}$. We say that an algorithm for this problem solves parity games. Parity games have various applications in computer science and the theory of formal languages and automata in particular. They are closely related to other games of infinite duration, such as mean payoff games, discounted payoff games, and stochastic games [10]. Solving parity games is linear-time equivalent to the model checking problem for the modal $\mu$-calculus [19]. Hence, any parity game solver is also a model checker for the $\mu$-calculus (and vice versa).

Many algorithms have been suggested for solving parity games [4, 12, 20, 21, yet none of them is known to run in polynomial time. McNaughton [15] showed that the winner determination problem belongs to the class $N P \cap \operatorname{coNP}$, and Jurdziński [10] strengthened this to UP $\cap$ coUP. It is a long-standing open question whether parity games can be solved in polynomial time. The fastest known deterministic algorithm is due to Jurdziński, Paterson, and Zwick [12] and it has a run time of $n^{O(\sqrt{n})}$ for general parity games and of $n^{O(\sqrt{n / \log n})}$ for parity games in which every node has out-degree at most two. The fastest known randomized algorithm for general parity games is due to Björklund et al. 4] and it has a run time of $n^{O(\sqrt{n / \log n})}$.

As a polynomial-time algorithm for solving parity games has remained elusive, researchers have started to consider which restrictions on the game allow for polynomial-time algorithms. One such well-studied restriction is the treewidth $t$ of the underlying undirected graph $G$ of the game. Obdržálek [16] found an

[^0]algorithm solving parity games on $n$ nodes in time $n^{O\left(t^{2}\right)}$. Later, Fearnley and Lachish [6] gave an algorithm solving parity games in time $n^{O(t \log n)}$. Another well-studied parameter for parity games is the number $p$ of distinct priorities by which the nodes of the game are labeled. The progress-measure lifting algorithm by Jurdziński [11] solves parity games in time $O\left(p m(2 n / p)^{p / 2}\right)$, where $m$ denotes the number of edges of $G$. This run time has been improved by Schewe 18 to $O\left(m\left((2 e)^{3 / 2} n / p\right)^{p / 3}\right)$. Fearnley and Schewe [7] presented an algorithm for solving parity games with run time $O\left(n(t+1)^{t+5}(p+1)^{3 t+5}\right)$, assuming that a tree decomposition of $G$ with width $t$ is given.

For a given parameter $\kappa$, one usually aims for fixed-parameter algorithm algorithms, i.e., algorithms that run in time $f(\kappa) \cdot n^{c}$ for some computable function $f$ and some constant $c$ that is independent of $\kappa$. Such an algorithm can be practical for large instances if $f$ grows moderately and $c$ is small. From the previously mentioned algorithms only the algorithm by Fearnley and Schewe [7] is a fixed-parameter algorithm for the combined parameter $(t, p)$. It is not known if fixed-parameter algorithms exist for the parameter $t$ or the parameter $p$ alone.

Further parameters for which polynomial-time algorithms for parity games have been suggested include DAG-width [1], clique-width [17], and entanglement [3]; none of these are fixed-parameter algorithms.

### 1.1 Our Contributions

We study as parameter the number $k$ of nodes that belong to the player who controls the smaller number of nodes in the parity game. Our first result is a subexponential fixed-parameter algorithm for solving general parity games for parameters $p$ and $k$ and for parameter only $k$ for bipartite parity games (where players alternate between their moves).
Theorem 1. There is a deterministic algorithm that solves any parity game $G$ on $n$ nodes and $m$ edges in time $(p+k)^{O(\sqrt{k})} \cdot O(p n m)$, where $k$ denotes the minimum number of nodes owned by one of the players and $p$ the number of distinct priorities. If $G$ is bipartite, the algorithm runs in time $k^{O(\sqrt{k})} \cdot O\left(n^{3}\right)$.

Thus, our algorithm is particularly efficient if the game is unbalanced, in the sense that one player owns only $k$ nodes and the other player owns the remaining $n-k \gg k$ nodes.

Let us remark that it is not very hard to show fixed-parameter tractability for parameter $p+k$; indeed McNaughton's algorithm [15] can be shown to run in time $p^{k} \cdot n^{O(1)}$, and this was improved to $p^{\log k} \cdot 4^{k} \cdot n^{O(1)}$ by Gajarský et al. [8]. Our key contribution here is to reduce the dependence of $k$ to a subexponential function. Indeed, this improvement allows us to derive the following immediate corollary of Theorem 1 to expedite the run time for solving general parity games.
Corollary 1. There is a deterministic algorithm that solves parity games in time $n^{O(\sqrt{k})}$.
Our algorithm is asymptotically always at least as fast as the fastest known deterministic parity game solver by Jurdziński, Paterson, and Zwick [12], which runs in time $n^{O(\sqrt{n})}$. For the case $k=o(n)$, our algorithm is asymptotically faster than theirs and constitutes the fastest known deterministic solver for such games.

We also prove the existence of a small kernel, as our second result. For a parameterized problem, a kernelization algorithm takes as input an instance $x$ with parameter $\kappa$ and computes in time $(|x|+\kappa)^{O(1)}$ an equivalent instance $x^{\prime}$ with parameter $\kappa^{\prime}$ (a kernel) with size $\left|x^{\prime}\right| \leq g(\kappa)$, for some computable function $g$; here, equivalent means that an optimal solution for $x$ can be derived in polynomial time from an optimal solution of $x^{\prime}$.

Theorem 2. Parity games can be kernelized in time $O(p m n)$ to at most $(p+1)^{k}+(p+1) k$ nodes, and bipartite parity games can be kernelized in time $O\left(n^{3}\right)$ to at most $k+2^{k} \cdot \min \{k, p\}$ nodes and at most $k 2^{k} \cdot \min \{k, p\}$ edges.

This kernelization result is not only interesting for its own sake, but it is also an important ingredient in the proof of Theorem 1 .

As our third result, we generalize the algorithm by Jurdziński, Paterson, and Zwick 12 for parity games with maximum out-degree 2 to arbitrary out-degree $\Delta$.

Theorem 3. There is a deterministic algorithm that solves parity games on nodes out of which $s_{j}$ nodes have out-degree at most $j$ in time

$$
n^{O\left(\min _{1 \leq j \leq n}\left\{\sqrt{n-s_{j}}+\sqrt{\frac{s_{j}}{\log _{j} s_{j}}}\right\}\right)}
$$

Corollary 2. There is a deterministic algorithm that solves parity games on $n$ nodes with maximum outdegree $\Delta$ in time $n^{O(\sqrt{\log (\Delta) \cdot n / \log (n)})}$ and parity games on $n$ nodes with average out-degree $\Delta$ in time $n^{O(\sqrt{\log (\log (n) \Delta) \cdot n / \log (n)})}$.

### 1.2 Detailed Comparison with Previous Work

Let us discuss in detail how our results compare to previous work. It is well-known (cf. [14, Lemma 3.2]) and easy to prove that the treewidth of a complete bipartite graph equals the size of the smaller side. Since the treewidth of a graph can only decrease when deleting edges, the graph underlying a bipartite parity game in which one player owns $k$ nodes has a treewidth of at most $k$. However, as it is not known if there exists a fixed-parameter algorithm for parameter treewidth, the result in Theorem 1 for the bipartite case does not follow from previous work about parity games with bounded treewidth. As a parity game in which one player owns $k$ nodes can have up to $n$ different priorities, also the fixed-parameter algorithm for the combined parameter $(t, p)$ by Fearnley and Schewe [7] does not imply our result.

The algorithm of Jurdziński, Paterson, and Zwick [12] for parity games with maximum out-degree two with run time $n^{O(\sqrt{n / \log n})}$ can easily be generalized to arbitrary parity games at the expense of its run time. For this, one only needs to observe that every parity game can be transformed into a game with maximum out-degree two by replacing each node with a higher out-degree by an appropriate binary tree. This transformation increases the number of nodes from $n$ to $\Theta(m)$ where $m$ denotes the number of edges in the original parity game. Hence, the run time becomes $m^{O(\sqrt{m / \log m})}=n^{O(\sqrt{m / \log n})}$. For graphs with average out-degree $\Delta=\omega(\log \log n)$ the resulting run time of $n^{O(\sqrt{\Delta n / \log n})}$ is asymptotically worse than the run time we obtain in Corollary 2

For graphs in which the variance of the out-degrees is large, our algorithm can even be better than stated in Corollary 2. If, for example, there are $n^{1-\varepsilon}$ nodes with an arbitrary out-degree for some $\varepsilon>0$ and all remaining nodes have constant out-degree at most $c$ then our algorithm has a run time of $n^{O\left(\sqrt{\frac{n}{\log n}}\right)}$ (the minimum in Theorem 3 is assumed for $j=c$ ). This matches the best known bound for randomized algorithms.

Gajarský et al. 8] present an algorithm that solves parity games in time $w^{O(\sqrt{w})} \cdot n^{O(1)}$, where $w$ denotes the modular width of $G$. Since the modular width of a bipartite graph can be exponential in the size of the smaller side, Theorem 1 does not follow from this result.

## 2 Fundamental Properties of Parity Games

A parity game $G=\left(V_{0} \uplus V_{1}, E, p\right)$ consists of a directed graph $\left(V_{0} \uplus V_{1}, E\right)$, where $V_{0}$ is the set of even nodes and $V_{1}$ is the set of odd nodes, and a priority function $p: V_{0} \cup V_{1} \rightarrow \mathbb{N}_{0}$. We often abuse notation and also refer to $\left(V_{0} \uplus V_{1}, E\right)$ as the graph $G$. For each node $v \in V(G)$, we denote by $N_{G}^{+}(v)$ and $N_{G}^{-}(v)$ the set of out-neighbors and in-neighbors of $v$ in $G$, respectively.

Two standard assumptions about parity games are (1) that $G$ is bipartite with $E \subseteq\left(V_{0} \times V_{1}\right) \cup\left(V_{1} \times V_{0}\right)$, and (2) that each node $u \in V$ has at least one outgoing edge $(u, v) \in E$. The first assumption is often made because it is easy to transform a non-bipartite instance into a bipartite instance. However, the usual transformation increases the number of nodes in $V_{i}$ by an amount of $\left|\left\{v \in V_{1-i} \mid N_{G}^{-}(v) \cap V_{1-i} \neq \emptyset\right\}\right|$, and can therefore increase the parameter $k=\min \left\{\left|V_{0}\right|,\left|V_{1}\right|\right\}$ significantly. We therefore consider bipartite and non-bipartite instances separately in Theorem 1 .

We write $n=|V(G)|, m=|E|$ and $p=|\{p(v) \mid v \in V(G)\}|$. The game is played by two players, the even player (or player 0 ) and the odd player (or player 1). The game starts at some node $v_{0} \in V(G)$. The players construct an infinite path (a play) as follows. Let $u$ be the last node added so far to the path. If $u \in V_{0}$, then player 0 chooses an edge $(u, v) \in E$. Otherwise, if $u \in V_{1}$, then player 1 chooses an edge $(u, v) \in E$. In either case, node $v$ is added to the path and a new edge is then chosen by either player 0 or player 1 . As each node has at least one outgoing edge, the path constructed can always be continued. Let $v_{0}, v_{1}, v_{2}, \ldots$ be the infinite path constructed by the two players and let $p\left(v_{0}\right), p\left(v_{1}\right), p\left(v_{2}\right), \ldots$ be the sequence of the priorities of the nodes on the path. Player 0 wins the game if the largest priority seen infinitely often is even, and player 1 wins if the largest priority seen infinitely often is odd.

We will define $p_{1}(v)$ as $p(v)$ if $p(v)$ is odd and as $-p(v)$ if $p(v)$ is even. This allows us to say that, in case $p_{1}(v)>p_{1}(u)$ for some $v, u \in V$, player 1 prefers $p(v)$ over $p(u)$. Observe that removing an arbitrary finite prefix of a play in a parity game does not change the winner; we refer to this property of parity games as prefix independence. A strategy for player $i \in\{0,1\}$ in a game $G$ specifies, for every finite path $v_{0}, v_{1}, \ldots, v_{k}$ in $G$ that ends in a node $v_{k} \in V_{i}$, an edge $\left(v_{k}, v_{k+1}\right) \in E$. A strategy is positional if the edge $\left(v_{k}, v_{k+1}\right) \in E$ chosen depends only on the last node $v_{k}$ visited and is independent of the prefix path $v_{0}, v_{1}, \ldots, v_{k-1}$. A strategy for player $i \in\{0,1\}$ is winning (for player $i$ ) from a start node $v_{0}$ if following this strategy ensures that player $i$ wins the game, regardless of which strategy is used by the other player.

The fundamental determinacy theorem for parity games [5, 9] says that for every parity game $G$ and every start node $v_{0}$, either player 0 has a winning strategy or player 1 has a winning strategy. Furthermore, if a player has a winning strategy from some node in a parity game, then she also has a winning positional strategy from this node. From now on we will therefore, unless stated differently, assume every strategy to be positional. Given positional strategies $s_{0}$ on $V_{0}$ and $s_{1}$ on $V_{1}$ and a start node $v_{0} \in V$ the infinite path starting in $v_{0}$ corresponding to these strategies consists of a finite prefix and an infinite recurrence of a cycle $C=C\left(s_{0}, s_{1}, v_{0}\right)$. We call $C$ the cycle corresponding to $s_{0}, s_{1}, v_{0}$ and say that $s_{0}$ and $s_{1}$ create $C$. The parity of the highest priority $p(u)$ of all nodes $u \in V(C)$ in cycle $C$ then determines the winner of the game. The winning set of player $i \in\{0,1\}$ is the set $\operatorname{win}_{i}(G) \subseteq V$ of nodes of the game $G$ from which player $i$ has a winning strategy.

For $i \in\{0,1\}$, an $i$-dominion is a set of nodes $D \subseteq V$ so that player $i$ can win from every node of $D$, without leaving $D$ and without allowing the other player to leave $D$. An example of an $i$-dominion is the set $\operatorname{win}_{i}(G)$, but there may be smaller subsets of $\operatorname{win}_{i}(G)$ that are $i$-dominions as well. Although finding $i$-dominions can be just as hard as finding $\operatorname{win}_{i}(G)$, searching only for dominions with certain properties (e.g. small dominions) can be easier. In our algorithm we will use the fact that once an $i$-dominion is found, it can easily be removed from the graph, leaving a smaller game to be solved.

Next, we recall some well-known results about parity games that form the basis of the algorithms for solving parity games by McNaughton [15] and Zielonka [21. We include them here as our algorithm relies on them as well; for a detailed exposition we refer to Grädel et al. [9]. Fix a parity game $G=\left(V_{0} \uplus V_{1}, E, p\right)$.

For $i \in\{0,1\}$, a set $B \subseteq V(G)$ is $i$-closed if for every $u \in B$ the following holds (we use the notation $\neg i$ for the element $1-i \in\{0,1\})$ :

- If $u \in V_{i}$, then there exists some $(u, v) \in E$ such that $v \in B$; and
- if $u \in V_{\neg i}$, then for every $(u, v) \in E$, we have $v \in B$.

In other words, a set $B$ is $i$-closed if player $i$ can always choose to stay in $B$ while simultaneously player $\neg i$ cannot escape from it, i.e., $B$ is a "trap" for player $\neg i$.

Lemma 1. For each $i \in\{0,1\}$, the set $\operatorname{win}_{i}(G)$ is $i$-closed.
Let $A \subseteq V(G)$ be a set of nodes and let $i \in\{0,1\}$. The $i$-reachability set of $A$ is the set $\operatorname{reach}_{i}(A)$ of nodes in $A$ together with all nodes in $V(G) \backslash A$ from which player $i$ has a strategy $\sigma$ to enter $A$ at least once (regardless of the strategy of the other player); we call such a strategy $\sigma$ an $i$-reachability strategy to $A$.

Lemma 2. For $A \subseteq V(G)$ and $i \in\{0,1\}$, the set $V(G) \backslash \operatorname{reach}_{i}(A)$ is $(\neg i)$-closed.

We will from now on assume that the graph of the parity game we operate on is encoded as an adjacency list.

Lemma 3. For every set $A \subseteq V(G)$ and $i \in\{0,1\}$, the set reach ${ }_{i}(A)$ can be computed in $O(m)$ time, where $m=|E|$ is the number of edges in the game.

If $B \subseteq V(G)$ is such that for each node $u \in V(G) \backslash B$ there is an edge $(u, v)$ with $v \in V(G) \backslash B$, then the sub-game $G-B$ is the game obtained from $G$ by removing the nodes of $B$. We will only be using $B$ 's for which $V(G) \backslash B$ is an $i$-closed set for some $i$. In this case every node in $v \in V(G) \backslash B$ has at least one out-going edge $(v, w)$ with $w \in V(G) \backslash B$ and $G-B$ will therefore be well-defined. The next lemmas show some useful properties of sub-games.

Lemma 4. Let $G^{\prime}$ be a sub-game of $G$ and let $i \in\{0,1\}$. If the node set of $G^{\prime}$ is $i$-closed in $G$, then $\operatorname{win}_{i}\left(G^{\prime}\right) \subseteq \operatorname{win}_{i}(G)$.

The next lemma shows that if we know some non-empty subset $U$ of the winning set of some player $\neg i$ in a game $G$, then computing the winning sets of both players in $G$ can be reduced to computing their winning sets in the smaller game $G-\operatorname{reach}_{\neg i}(U)$.

Lemma 5. For any parity game $G$ and $i \in\{0,1\}$, if $U \subseteq \operatorname{win}_{\neg i}(G)$ and $U^{*}=\operatorname{reach}_{\neg i}(U)$, then $\operatorname{win}_{\neg i}(G)=$ $U^{*} \cup \operatorname{win}_{\neg i}\left(G-U^{*}\right)$ and $\operatorname{win}_{i}(G)=\operatorname{win}_{i}\left(G-U^{*}\right)$.

The next lemma complements Lemma 5 by providing a way to find a non-empty subset of the winning set of player $\neg i$ in a parity game $G$ or to conclude that player $i$ can win from every node in $G$.

Lemma 6. Let $G$ be a parity game with largest priority $p_{\max }$ and let $V_{p_{\max }} \subseteq V(G)$ be the set of nodes with priority $p_{\max }$. Let $i=p_{\max }(\bmod 2)$ and let $G^{\prime}=G-\operatorname{reach}_{i}\left(V_{p_{\max }}\right)$. Then $\operatorname{win}_{\neg i}\left(G^{\prime}\right) \subseteq \operatorname{win}_{\neg i}(G)$. Also, if $\operatorname{win}_{\neg i}\left(G^{\prime}\right)=\emptyset$, then $\operatorname{win}_{i}(G)=V$, i.e., player $i$ wins from every node of $G$.

## 3 Kernelization of Parity Games

In this section, we describe some reduction rules for parity games. Theses rules are such that we can efficiently compute the winning sets of the original parity game once we know the winning sets of the reduced game.

### 3.1 General Parity Games

Lemma 7. Any parity game $G=\left(V_{0} \uplus V_{1}, E, p\right)$ can be transformed in time $O(p m n)$ to a parity game $G^{\prime}=\left(V_{0}^{\prime} \uplus V_{1}^{\prime}, E^{\prime}, p^{\prime}\right)$ with $V_{1}^{\prime} \subseteq V_{1}$ such that

- there are no edges inside $V_{1}^{\prime}$, and
- for each node $v \in V_{0}^{\prime}$ either $N_{G}^{+}(v) \subseteq V_{1}^{\prime}$ or $N_{G}^{-}(v) \subseteq V_{1}^{\prime}$, and
- $\left|V_{0}^{\prime}\right| \leq \min \left\{n+p k,(p+1)^{k}+p k\right\}$, where $k=\left|V_{1}\right|$.

Moreover, $G$ and $G^{\prime}$ have the same winning sets on $V_{1}^{\prime}$ and the winner of the remaining nodes of $G$ can be computed either during the transformation or from the winning sets of $G^{\prime}$ in linear time.

Proof. We will modify $G$ in multiple steps. We will slightly abuse notation and refer in every step to the parity game that we obtained in the step before as $G=\left(V_{0} \uplus V_{1}, E, p\right)$. First we eliminate all edges inside $V_{1}$. This can easily be achieved by adding for each edge $e=(v, w) \in E$ with $v, w \in V_{1}$ a new node $v_{e}$ with $p^{\prime}\left(v_{e}\right)=p(w)$ to $V_{0}$ and by replacing the edge $e$ by the two edges $\left(v, v_{e}\right)$ and $\left(v_{e}, w\right)$. Since the new node $v_{e}$ has only a single outgoing edge, this transformation does neither change the winning sets nor the winning strategies.

Next, we remove certain cycles inside $V_{0}$ from the game. Let $W_{0} \subseteq V_{0}$ denote all nodes in $V_{0}$ that are part of at least one cycle that lies completely inside $V_{0}$ and whose highest priority is even. Clearly player 0
can win from all nodes in reach ${ }_{0}\left(W_{0}\right)$ by enforcing that such a cycle is entered and never left again. Hence, we can remove reach $h_{0}\left(W_{0}\right)$ from the game according to Lemma 5 Let $W_{1} \subseteq V_{0}$ denote all nodes that are left in $V_{0}$ and from which player 0 cannot reach $V_{1}$. Then all paths that start in some node $u \in W_{1}$ must end in some cycle that is completely contained in $V_{0}$. Since we have removed all cycles whose highest priority is even, the maximum priority of this cycle must be odd. Thus, player 1 wins from all nodes in reach $\left(W_{1}\right)$. Hence, we can also remove reach $\left(W_{1}\right)$ from the game according to Lemma 5 ,

We use again the notation $G=\left(V_{0} \uplus V_{1}, E, p\right)$ to refer to the parity game obtained after the previously discussed steps. Since we have removed all cycles from $V_{0}$ whose highest priority is even, player 0 loses for sure if she does not leave $V_{0}$. Hence, we can assume without loss of generality that the play leaves $V_{0}$ from every starting node if player 0 plays an optimal strategy. Then for every node $v \in V_{0}$ player 0 uses a (possibly empty) path inside $V_{0}$ followed by an edge that leads to some node $w \in V_{1}$. To determine the winning sets of a strategy of player 0 it is not important to know the exact paths player 0 chooses. Rather, it suffices to know for each $v \in V_{0}$ which node $w \in V_{1}$ will be reached and what the highest priority on the chosen $v$-w-path is. To get rid of long paths, we add $p \cdot\left|V_{1}\right|$ new nodes to $V_{0}$, one node $v\left(p^{\prime}, w\right)$ for each pair of a priority $p^{\prime}$ and a node $w \in V_{1}$. Node $v\left(p^{\prime}, w\right)$ has a priority $p^{\prime}$ and its only out-neighbor is $w$. The winner does not change if player 0 goes from $v \in V_{0}$ directly to $v\left(p^{\prime}, w\right)$ and from there directly to $w \in V_{1}$ instead of taking some other path from $v$ inside $V_{0}$ with maximum priority $p^{\prime}$, followed by an edge that leads to $w$. For all such paths we add the corresponding edge $\left(v, v\left(p^{\prime}, w\right)\right)$ and can therefore delete all edges inside $V_{0}$ that do not end in one of the new nodes $v\left(p^{\prime}, w\right)$ without changing the winning sets of the game. Observe that this ensures that all out-neighbors of the new nodes $v\left(p^{\prime}, w\right)$ belong to $V_{1}$ while all in-neighbors of the old nodes $v \in V_{0}$ belong to $V_{1}$.

It can be the case that for some pair $(v, w) \in V_{0} \times V_{1}$ there are multiple nodes $v\left(p^{\prime}, w\right)$ that can be reached from $v$. We can assume without loss of generality that if player 0 decides to go from $v$ to $w$ via one of these nodes then she chooses the one that is best for her, i.e., the one with lowest $p_{1}$-value. All edges from $v$ to other nodes $v\left(p^{\prime}, w\right)$ can be removed.

The inequality $\left|V_{0}^{\prime}\right| \leq n+\min \left\{m, k^{2}\right\}+p k$ follows directly from the previously discussed construction: initially $V_{0}$ consists of $n-k \leq n$ nodes, there are at most $\min \left\{m, k^{2}\right\}$ edges inside $V_{1}$ for which we create a new node $v_{e}$, and there are only $p k$ new nodes $v\left(p^{\prime}, w\right)$. To get rid of the term $\min \left\{m, k^{2}\right\}$ we can identify each node $v_{e}$, which derived from an edge $e=(v, w)$ inside $V_{1}$, with the node $v(p(w), w)$. This ensures that there are only $p k$ new nodes. To show that $\left|V_{0}^{\prime}\right| \leq(p+1)^{k}+p k$ we can reduce the number of old nodes in $V_{1}$ to ensure that at most $(p+1)^{k}$ remain. At first we remove all nodes $v \in V_{0}$ with $N_{G}^{-}(v)=\emptyset$, because they obviously cannot be part of a cycle and we can compute in linear time to which winning set they belong, once we know to which winning set their out-neighbors belong. Now let $v$ and $v^{\prime}$ be two such nodes in $V_{0}$ with $N_{G}^{+}(v)=N_{G}^{+}\left(v^{\prime}\right)$. We then identify $v$ and $v^{\prime}$ without changing the winning sets in $V_{1}$, since all nodes in $N_{G}^{+}(v)$ must have a priority at least as high as $\max \left\{p(v), p\left(v^{\prime}\right)\right\}$. This is because the priority of any node in $N_{G}^{+}(v)\left(N_{G}^{+}\left(v^{\prime}\right)\right)$ corresponds to the highest priority on a path that starts in $v\left(v^{\prime}\right)$ and therefore must be at least $p(v)\left(p\left(v^{\prime}\right)\right)$. Afterwards there remains at most one node $v \in V_{0}$ for each possible set $N_{G}^{+}(v)$.

Since $N_{G}^{+}(v)$ can contain at most one new node corresponding to $w$ for each $w \in V_{1}$ and there are $p$ different ones to choose from there are at most $(p+1)^{k}$ different possibilities for $N_{G}^{+}(v)$.

It remains to analyze the run time of the transformation. We consider the different steps of the reduction separately. The first step of removing all edges inside $V_{1}$ can be performed in $O(m)$ because we only need to check for every edge $e=(v, w) \in E$ if $v, w \in V_{1}$ and then remove one edge and add two edges and a node. The second step of removing dominions completely inside $V_{0}$ can be executed in time $O(\log (p) \cdot m)$ as follows. First, we solve the solitary game on $V_{0} \backslash \operatorname{reach}_{1}\left(V_{1}\right)$ and remove the 0 -reachability-set of its 0 -winning set; this can be done in time $O(\log p \cdot m)[2]$. Thereafter, we compute the 1-reachability set of $V_{0} \backslash \operatorname{reach}_{0}\left(V_{1}\right)$ and remove it; this can be done in time $O(m)$ [12. The third step of removing long paths inside $V_{0}$ can be performed as follows. The algorithm computes the best priority for player 0 that a path in $V_{0}$ from a node $v \in V_{0}$ to a node $w \in V_{1}$ can have. To determine which nodes in $v \in V_{0}$ can reach which nodes $w \in V_{1}$ via a path in $V_{0}$ whose highest priority has fixed value of $p^{\prime}$, consider the subgraph $G \leq p^{\prime}$ of $G$ that is induced by the set $V \leq p^{\prime}$ of nodes of priority at most $p^{\prime}$ and remove from it those edges that start in $V_{1}$. We then consider the set of nodes with priority exactly $p^{\prime}$ and compute by DFS in time $O(m)$ all nodes in $V_{1}$ reachable from
them. Then we compute with DFS for each node in $V_{0}$ which of the nodes with priority $p^{\prime}$ they can reach. This takes a total of at most $2 n$ applications of DFS for each priority and therefore in total $O(p m n)$ time. In the last step where we remove and contract some of the nodes in $V_{0}$ we can find all nodes without incoming edges in time $O(m)$ and we can order all remaining nodes by their outgoing edges in time $O\left(\left|V_{1}\right| \cdot(n+p+1)\right)$ using a version of radix-sort, where we view the set of out-neighbors as an $(p+1)$-adic number with $\left|V_{1}\right|$ digits. Thereafter, in linear time we identify sets of nodes with the same outgoing neighbors and identify them in total time $O(n+m)$.

### 3.2 Bipartite Parity Games

In this section we give some reduction rules that efficiently reduce any bipartite game $G=\left(V_{0} \uplus V_{1}, E, p\right)$ to a structurally simpler bipartite game $G^{\prime}=\left(V_{0}^{\prime} \uplus V_{1}^{\prime}, E^{\prime}, p^{\prime}\right)$, such that the winning sets of $G$ can be efficiently recovered from the winning sets of $G^{\prime}$. After exhaustive application of the reduction rules, the reduced game $G^{\prime}$ will have size bounded by some function of $k$ and $p$ only, independent of the size of $G$.

The digraphs of our underlying parity game may have self-loops and bidirected edges, but (without loss of generality) no parallel edges between the same two nodes. Thus, whenever parallel edges arise during the application of one of the reduction rules, we remove one of them without explicit mention.

Lemma 8. Let $G=\left(V_{0} \uplus V_{1}, E, p\right)$ be a bipartite parity game, and let $u, v \in V_{0}$ be such that $N_{G}^{+}(v) \subseteq N_{G}^{+}(u)$ and $p_{1}(v) \geq p_{1}(u)$. Let $G^{\prime}$ be the parity game obtained from $G$ by deleting the edges $\{(w, u) \in E \mid(w, v) \in E\}$. Then the winning sets of $G$ and $G^{\prime}$ are equal.

Proof. We show that an edge $\left(w^{\prime}, u\right) \in\{(w, u) \in E \mid(w, v) \in E\}$ can only be part of a winning strategy for player 1 on node $w^{\prime}$ if the edge $\left(w^{\prime}, v\right)$ is part of a winning strategy for player 1 on $w^{\prime}$ as well. Therefore, after deleting $\left(w^{\prime}, u\right)$, player 1 wins from $w^{\prime}$ in $G^{\prime}$ if and only if he wins from $w^{\prime}$ in $G$. Deleting the edges in $\{(w, u) \in E \mid(w, v) \in E\}$ does therefore not change the winning sets.

Assume that player 1 has a winning strategy $s_{1}: V_{1} \rightarrow V_{0}$ for $w^{\prime}$ with $s_{1}\left(w^{\prime}\right)=u$. Let $s_{1}^{\prime}: V_{1} \rightarrow V_{0}$ be defined by $s_{1}^{\prime}\left(w^{\prime}\right)=v$ and $s_{1}^{\prime}(w)=s_{1}(w)$ for all $w \in V_{1} \backslash\left\{w^{\prime}\right\}$. We claim that $s_{1}^{\prime}$ is a winning strategy for player 1 on $w^{\prime}$ as well. Assume that there exists a counter strategy $s_{0}^{\prime}$ for $s_{1}^{\prime}$ such that player 0 wins the game with starting node $w^{\prime}$. We will define a strategy $s_{0}$ for player 0 and show that $s_{0}$ is a counter strategy for $s_{1}$. Note that $s_{0}$ will not necessarily be a positional strategy. For all $w \in V_{0} \backslash\{u\}, s_{0}$ chooses the same successor as $s_{0}^{\prime}$, but on $u$ it might change its behavior. Each time the play encounters $u$ directly after encountering $w^{\prime}$, strategy $s_{0}$ chooses $s_{0}^{\prime}(v)$ as the successor of $u$. Every other time the play encounters $u$, strategy $s_{0}$ chooses $s_{0}^{\prime}(u)$ as the successor of $u$.

The play defined by $s_{0}^{\prime}$ and $s_{1}^{\prime}$ can then be transformed into the play defined by $s_{0}$ and $s_{1}$ by replacing every appearance of the sequence $w^{\prime}, v, s_{0}^{\prime}(v)$ with the sequence $w^{\prime}, u, s_{0}^{\prime}(v)$. Let $C^{\prime}$ be the cycle created by $s_{0}^{\prime}$ and $s_{1}^{\prime}$; then $C^{\prime}$ is a winning cycle for player 0 . Then $s_{0}$ and $s_{1}$ will also create the cycle $C^{\prime}$, if $C^{\prime}$ does not contain the sequence $w^{\prime}, v, s_{0}^{\prime}(v)$. Let us therefore assume that $C^{\prime}$ contains the sequence $w^{\prime}, v, s_{0}^{\prime}(v)$. Let $C$ be the closed walk obtained, when replacing $v$ with $u$ in $C^{\prime}$. After a finite prefix the play defined by $s_{0}$ and $s_{1}$ will consist of an infinite recurrence of $C$. Since we have $p_{1}(v) \geq p_{1}(u)$ player 0 is wining the play defined by $s_{0}$ and $s_{1}$. This contradicts that $s_{1}$ is a winning strategy for player 1 .

Lemma 9. Let $G=\left(V_{0} \uplus V_{1}, E, p\right)$ be a bipartite parity game, and let $u, v \in V_{0}$ be nodes with $N_{G}^{+}(u)=N_{G}^{+}(v)$ and $p(v)=p(u)$. Let $G^{\prime}$ be the parity game obtained from $G$ by contracting $u$ and $v$ into a new node $v^{\prime}$ with priority $p(v)$. Then $u$ and $v$ belong to the same winning set $\operatorname{win}_{i}(G)$ in $G$ and $v^{\prime}$ belongs to the winning set $\operatorname{win}_{i}\left(G^{\prime}\right)$ of the same player in $G^{\prime}$. For all other nodes the winning sets of $G$ and $G^{\prime}$ coincide.

Proof. Note that $u$ and $v$ belong to the winning set of the same player $i$ in $G$. We can assume that player 0 chooses the same successor for $u$ and $v$ in her optimal strategy. Then no cycle created by optimal strategies contains both $u$ and $v$ and, after the contraction, each simple cycle that does not contain both $u$ and $v$ is again a simple cycle with the same priorities. Also, each cycle in the contracted game either exists in the original game (i.e., it does not contain $v^{\prime}$ ) or an equivalent cycle, which can be created by replacing $v^{\prime}$ with $v$ or $u$, exists in the original game. We can also map winning strategies in the original game, where $u$ and $v$
have the same successor into winning strategies in the resulting game and vice versa by simply identifying the successors of $v^{\prime}$ with the successor of $u$ and $v$ and vice versa, while keeping the rest of the strategy. We again assume that in the winning strategies in the original game, $v$ and $u$ have the same successor $w$. We then set the successor of $v^{\prime}$ to $w$ and set the successor of any node $w^{\prime}$ with successor $v$ or $u$ to $v^{\prime}$. The other way around $v$ and $u$ get the same successor as $v^{\prime}$ and any node $w^{\prime}$ with successor $v^{\prime}$ gets either $v$ or $u$ as its successor, depending on which of the edges $\left(w^{\prime}, v\right)$ and $\left(w^{\prime}, u\right)$ exists in the original game. A pair of strategies and the pair of strategies, to which they are mapped to, then create corresponding cycles and must therefore either both be winning for player 1 or both be winning for player 0 .

Lemma 10. Let $G=\left(V_{0} \uplus V_{1}, E, p\right)$ be a bipartite parity game, and let $v \in V(G)$ be such that $N_{G}^{-}(v)=\emptyset$. Then for the parity game $G^{\prime}=G-v$ and for $i \in\{0,1\}$, any node $v^{\prime} \neq v$ is winning for player $i$ in $G$ if and only if it is winning for player $i$ in $G^{\prime}$.

Proof. The condition $N_{G}^{-}(v)=\emptyset$ implies that $v$ cannot be part of any cycle. Let $v \in V_{j}$; then $v \in \operatorname{win}_{j}(G)$ is equivalent to the existence of some node $w \in \operatorname{win}_{j}(G) \cap N_{G}^{+}(v)$. Since all possible strategies for all nodes except $v$ are also possible strategies in $G-\{v\}$, all nodes in $V \backslash\{v\}$ belong to the same winning set in $G$ and in $G-v$. (Notice that $G-\{v\}$ is again a parity game.) Once we computed the winning sets for $G-\{v\}$, we can check in time $O(n)$ whether $v \in \operatorname{win}_{j}(G)$ or $v \in \operatorname{win}_{\neg j}(G)$.

Lemma 11. Let $G=\left(V_{0} \uplus V_{1}, E, p\right)$ be a parity game with largest priority $p_{\max }=\max \{p(v) \mid v \in V(G)\}$. If $p^{-1}(z)=\emptyset$ for some $z \in\left\{1, \ldots, p_{\max }\right\}$ then let $G^{\prime}=\left(V_{0} \uplus V_{1}, E, p^{\prime}\right)$ be the parity game obtained from $G$ by setting $p^{\prime}(v)=p(v)-2$ for all $v \in V$ with $p(v)>z$ and $p^{\prime}(v)=p(v)$ for all $v \in V$ with $p(v)<z$. Then the winning sets of the games $G$ and $G^{\prime}$ coincide.

Proof. Let $s_{0}$ and $s_{1}$ be strategies for player 0 and player 1, respectively, and let $C=\left(v_{0}, v_{1}, \ldots, v_{\ell}\right)$ be the cycle created by these strategies when the game starts at some node $v$. The parity $j$ of the largest element in the set $Q=\left\{p\left(v_{0}\right), \ldots, p\left(v_{\ell}\right)\right\}$ determines which player wins in $G$ and the parity $j^{\prime}$ of the largest element in the set $Q^{\prime}=\left\{p^{\prime}\left(v_{0}\right), \ldots, p^{\prime}\left(v_{\ell}\right)\right\}$ determines which player wins in $G^{\prime}$. It is easy to see that our reduction ensures that $j=j^{\prime}$. Since this is true for any cycle, the lemma follows.

Corollary 3. In any parity game with maximum priority $p_{\max }$ to which the reduction rule described in Lemma 11 cannot be applied anymore, the set of priorities is either $\left\{0,1, \ldots, p_{\max }\right\}$ or $\left\{1, \ldots, p_{\max }\right\}$.

Lemma 12. Let $G=\left(V_{0} \uplus V_{1}, E, p\right)$ be a bipartite parity game with $\left|V_{1}\right|=k$ that is reduced according to Lemmas 810. Then $\left|V_{0}\right| \leq 2^{k} \cdot \min \{k, p\}$.

Proof. For each node $v \in V_{0}$ there are $2^{k}$ possible choices for $N_{G}^{+}(v)$. Lemma 8 yields that for two nodes $v \neq u \in V_{0}$ with $N_{G}^{+}(v)=N_{G}^{+}(u)$ we must have $N_{G}^{-}(v) \cap N_{G}^{-}(u)=\emptyset$. Lemma 10 then yields that there can be at most $k$ nodes in $V_{0}$ for every possible choice of $N_{G}^{+}(v)$. Also Lemma 9 yields that for each possible choice of $\left(N_{G}^{+}(v), p(v)\right)$ there exists at most one node in $V_{0}$.

Lemma 13. There exists a sequence of applications of the reduction rules described in Lemmas 811 with a total run time of $O\left(n^{3}\right)$ that leads to a game in which none of these rules applies anymore.

Proof. We show for each reduction rule separately how to apply it exhaustively in time $O\left(n^{3}\right)$. Although it can happen, that some reductions corresponding to one of the rules could lead to allowing some other reductions which were not allowed before. Therefore we cannot only apply all reductions corresponding to one rule after all reductions corresponding to another rule.

Most of the run time will be necessary to test if Lemma 8 or Lemma 9 applies to an ordered pair of nodes. We will argue how to apply the reductions such that we do not have to test the same ordered pair of nodes more than once, yielding a total run time of $O\left(n^{3}\right)$.

To apply all reductions of Lemma 11, we first sort the nodes in increasing order of their priorities and create an order of subsets each containing all nodes with the same priority; this can be done in $O(n \log (n))$ time. We then save for each of the subsets if its corresponding priority is odd or even and unite consecutive sets with the same parity. If the parity of the priority in the first subset is even, all nodes in the $i$-th subset
get priority $i-1$; otherwise all nodes in the $i$-th subset get priority $i$. Uniting sets can be done in linear time, and we cannot unite more than $n$ times. The time for applying Lemma 11 is thus $O\left(n^{2}\right)$.

To apply all reductions for Lemma 8, we need to check for each pair of nodes $\{u, v\} \subseteq V_{0}$ with $p_{1}(v) \geq$ $p_{1}(u)$ whether $N_{G}^{+}(v) \subseteq N_{G}^{+}(u)$, and find all nodes $w$ with $(w, u) \in E$ and $(w, v) \in E$. There are $O\left(n^{2}\right)$ node pairs $\{u, v\} \in V_{0}$ with $p_{1}(v) \geq p_{1}(u)$, which can easily be found using the order of subsets created for Lemma [11. Checking if $N_{G}^{+}(v) \subseteq N_{G}^{+}(u)$ and finding all nodes $w$ with $(w, u) \in E$ and $(w, v) \in E$ can be done in time $O(n)$. The total run time for Lemma 8 therefore is $O\left(n^{3}\right)$.

To apply all reductions for Lemma 9 we need to check for each pair of nodes $\{u, v\} \subseteq V_{0}$ with $p(v)=p(u)$ whether $N_{G}^{+}(v)=N_{G}^{+}(u)$. There are $O\left(n^{2}\right)$ such pairs $\{u, v\}$ with $p(v)=p(u)$, which can easily be found using the order of subsets created for Lemma 11. Testing whether $N_{G}^{+}(v)=N_{G}^{+}(u)$ and identifying $u$ and $v$ can be done in time $O(n)$. The total run time for Lemma 9 therefore is $O\left(n^{3}\right)$.

To apply all reductions for Lemma 10, we only need to check for each node if it has incoming edges and possibly delete it. Testing a node can be done in constant time, and deleting a node takes at most linear time. The time for applying Lemma 10 is thus $O\left(n^{2}\right)$.

We will first apply all feasible reductions for Lemma 11, then all feasible reductions for Lemmas 8, 9 and 10. Any reduction that is now possible was not feasible in the beginning.

Observe that some reductions can result in other reductions becoming feasible. Since we do not change the out-neighborhood of any node in $V_{0}$, reductions corresponding to Lemmas 8 and 9 for a pair of nodes $\{u, v\} \subseteq V_{0}$ can only become feasible when we combine the two subsets containing $v$ and $u$ in a reduction corresponding to Lemma 11. For each node pair $\{u, v\} \subseteq V_{0}$ this can happen at most once. The total run time for all reductions corresponding to Lemma 8 and 9 therefore is in $O\left(n^{3}\right)$. Reductions corresponding to Lemma 10 and a node $v \in V_{0}$ can only become feasible when we remove incoming edges of $v$. This can happen at most $n$ times for each node $v \in V_{0}$, before we remove it. The total run time for all reductions corresponding to Lemma 10 therefore is in $O\left(n^{2}\right)$. Reductions corresponding to Lemma 11 can only become feasible when all nodes of one subset have been removed. This can happen at most $n$ times; hence any node will be moved to another subset at most $n$ times. The total run time for all reductions corresponding to Lemma 11 therefore is in $O\left(n^{2}\right)$.

We can now prove our main kernelization result.
Theorem 2. The part of the theorem for general instances follows directly from Lemma 7 . The part for bipartite instances follows from Lemma 12 and Lemma 13 because the reduced bipartite parity game $G^{\prime}=$ $\left(V_{0}^{\prime} \uplus V_{1}^{\prime}, E^{\prime}, p^{\prime}\right)$ satisfied $\left|V_{0}^{\prime}\right| \leq 2^{k} \cdot \min \{k, p\}$ and $\left|V_{1}^{\prime}\right| \leq k$. Since $G^{\prime}$ is bipartite, this implies that it contains at most $k 2^{k} \cdot \min \{k, p\}$ edges.

## 4 A Simple Exponential-Time Algorithm

A simple algorithm with run time $O\left(2^{n}\right)$ for the solution of parity games originates from the work of McNaughton [15] and was first presented for parity games by Zielonka 21; see also Grädel et al. 9]. Algorithm $\operatorname{win}(G)$ receives a parity game $G$ and returns the pair of winning sets $\left(\operatorname{win}_{0}(G)=W_{0}, \operatorname{win}_{1}(G)=W_{1}\right)$.

Algorithm $\operatorname{win}(G)$ is based on Lemmas 5 and 6. Let $p_{\max }$ be the largest priority in $G$ and let $V_{p_{\max }}$ be the set of nodes with priority $p_{\max }$. Let $i=p_{\max }(\bmod 2)$ be the player who owns the highest priority. The algorithm first finds the winning sets $\left(W_{0}^{\prime}, W_{1}^{\prime}\right)$ of the smaller game $G^{\prime}=G-\operatorname{reach}_{i}\left(V_{p_{\max }}\right)$ in a first recursive call. If $W_{\neg i}^{\prime}=\emptyset$, then by Lemma 6 player $i$ wins from all nodes of $G$ and we are done. Otherwise, again by Lemma 6 we know that $W_{\neg i}^{\prime} \subseteq \operatorname{win}_{\neg i}(G)$. The algorithm then finds the winning sets $\left(W_{0}^{\prime \prime}, W_{1}^{\prime \prime}\right)$ of the smaller game $G^{\prime \prime}=G-\operatorname{reach}_{\neg i}\left(W_{\neg i}^{\prime}\right)$ by a second recursive call. By Lemma 5 , $\operatorname{win}_{i}(G)=W_{i}^{\prime \prime}$ and $\operatorname{win}_{\neg i}(G)=\operatorname{reach}_{\neg i}\left(W_{\neg i}^{\prime}\right) \cup W_{\neg i}^{\prime \prime}=V(G) \backslash W_{i}^{\prime \prime}$.

Theorem 4. Algorithm $\operatorname{win}(G)$ finds the winning sets of any parity game on nodes in time $O\left(2^{n}\right)$.
Proof. The correctness of the algorithm follows from Lemmas 5 and 6, as argued above. Let $T^{\prime}(n)$ be the number of steps needed by algorithm $\operatorname{win}(G)$ to solve a game $G$ on $n$ nodes. Algorithm win $(G)$ makes two

```
Algorithm 1 win( \(G\) )
Input: A parity game \(G=\left(V_{0} \uplus V_{1}, E, p\right)\) with maximum priority \(p_{\max }\).
Output: \(\left(W_{0}, W_{1}\right)\), where \(W_{i}\) is the winning set of player \(i \in\{0,1\}\).
    if \(V=\emptyset\) then
        return \((\emptyset, \emptyset)\)
    \(i \leftarrow p_{\text {max }}(\bmod 2) ; j \leftarrow \neg i\)
    \(\left(W_{0}^{\prime}, W_{1}^{\prime}\right) \leftarrow \operatorname{win}\left(G-\operatorname{reach}_{i}\left(V_{p_{\max }}\right)\right)\)
    if \(W_{j}^{\prime}=\emptyset\) then
        \(\left(W_{i}, W_{j}\right) \leftarrow(V, \emptyset)\)
    else
        \(\left(W_{0}^{\prime \prime}, W_{1}^{\prime \prime}\right) \leftarrow \operatorname{win}\left(G-\operatorname{reach}_{j}\left(W_{j}^{\prime}\right)\right)\)
        \(\left(W_{i}, W_{j}\right) \leftarrow\left(W_{i}^{\prime \prime}, V \backslash W_{i}^{\prime \prime}\right)\)
    return \(\left(W_{0}, W_{1}\right)\)
```

recursive calls $\boldsymbol{\operatorname { w i n }}\left(G^{\prime}\right)$ and $\boldsymbol{\operatorname { w i n }}\left(G^{\prime \prime}\right)$ on games with at most $n-1$ nodes. Other than that, it performs only $O\left(n^{2}\right)$ operations. (The most time-consuming operations are the computations of the sets reach ${ }_{i}\left(V_{p_{\max }}\right)$ and $\operatorname{reach}_{j}\left(W_{j}^{\prime}\right)$.) Therefore, $T^{\prime}(n) \leq 2 T^{\prime}(n-1)+O\left(n^{2}\right)$, which implies $T^{\prime}(n)=O\left(2^{n}\right)$.

## 5 Overview of the New Algorithms

Before we describe our new algorithms that lead to Theorems 1 and 3 in detail in Sect. 7 and Sect. [8 we present an overview of the main ideas. The algorithm new-win $(G)$ by Jurdziński, Paterson, and Zwick [12] with run time $n^{O(\sqrt{n})}$ is a slight modification of the just described algorithm $\operatorname{win}(G)$. At the beginning of each recursive call it tests in time $O\left(n^{\ell}\right)$ if the parity game contains a dominion $D$ of size at most $\ell=\lceil\sqrt{2 n}\rceil$. If this is the case then $D$ is removed and the remaining game is solved recursively. Else, the parity game is solved by the algorithm $\boldsymbol{\operatorname { w i n }}(G)$, except that the recursive calls in lines 4 and 8 are made to new-win $(G)$. Since this happens only when $G$ does not contain a dominion of size at most $\ell$, the dominion reach $h_{j}\left(W_{j}^{\prime}\right)$ that is removed in line 8 has size greater than $\ell$ and hence, the second recursive call is to a substantially smaller game. Overall, this leads to the improved run time of $n^{O(\sqrt{n})}$.

Our new algorithms are based on a similar idea. Instead of simply searching for a dominion of size at most $\ell$, our algorithm new- $\operatorname{win}_{1}(G)$ that leads to Theorem $\mathbb{1}$ searches for a dominion that contains at most $\ell=\lfloor\sqrt{2 k}\rfloor$ nodes of the odd player, assuming without loss of generality that the odd player controls fewer nodes, i.e., $k=\left|V_{1}\right|$. If such a dominion is found then we remove it from the game and solve the remaining game recursively. Otherwise, we use the algorithm $\operatorname{win}(G)$ to solve the parity game, except that the recursive calls in lines 4 and 8 are made to new-win $(G)$. It can happen that in the game to which the first recursive call in line 4 is made, the odd player controls again $k$ nodes. We will show that in bipartite instances this cannot happen in two consecutive calls. For general instances we use that the observation that at least the number of different priorities decreases by one in the recursive call. Searching efficiently for a dominion that contains at most $\ell=\lfloor\sqrt{2 k}\rfloor$ nodes of the odd player is more involved than simply searching for dominions whose total size is at most $\ell$. We use multiple recursive calls of new-win n $_{1}$ to test if such a dominion exists, which makes the recursion of our algorithm and its analysis more complicated.

Our second algorithm leading to Theorem 3 is based on the same approach and inspired by the algorithm of Jurdziński, Paterson, and Zwick [12. In this case we let $s_{j}$, for some $j \in \mathbb{N}$, equal the number of nodes with out-degree at most $j$. We separate the nodes into $s_{j}$ nodes with out-degree at most $j$ and $n-s_{j}$ nodes with out-degree larger than $j$ and, at the beginning of each iteration, search for and remove dominions that contain at most $\ell=\left\lceil\sqrt{2\left(n-s_{j}\right)}\right\rceil$ nodes with out-degree larger than $j$ and at most $s=\left\lceil\sqrt{s_{j} \cdot \log _{j} s_{j}}\right\rceil$ nodes


## 6 Finding Small Dominions

We now describe how dominions with the previously discussed properties can be found. Let $G=\left(V_{0} \uplus V_{1}, E, p\right)$ be a parity game. Recall that for $i \in\{0,1\}$, a set $D \subseteq V$ is an $i$-dominion if player $i$ can win from every node of $D$ without ever leaving $D$, regardless of the strategy of player $\neg i$. Note that any $i$-dominion must be $i$-closed. A set $D \subseteq V$ is a dominion if it is either a 0 -dominion or a 1 -dominion. By prefix independence of parity games, the winning set $\operatorname{win}_{i}(G)$ of player $i$ is an $i$-dominion.

For $k, p \in \mathbb{N}$, let $T(k)$ denote the maximum number of steps needed to solve a bipartite parity game $G=\left(V_{0} \uplus V_{1}, E, p\right)$ and let $T(k, p)$ denote the maximum number of steps needed to solve a general parity game $G=\left(V_{0} \uplus V_{1}, E, p\right)$ with $\left|V_{1}\right|=k$ and $p=|\{p(v) \mid v \in V\}|$ using some fixed algorithm. For $k, p, \ell \in \mathbb{N}$, let $\operatorname{dom}_{k}(\ell)$ denote the maximum number of steps required to find a dominion $D$ with $\left|V_{1} \cap D\right| \leq \ell$ in a bipartite parity game $G=\left(V_{0} \uplus V_{1}, E, p\right)$ with $\left|V_{1}\right|=k$ and let dom ${ }_{k, p}(\ell)$ denote the maximum number of steps required to find a dominion $D$ with $\left|V_{1} \cap D\right| \leq \ell$ in a general parity game $G=\left(V_{0} \uplus V_{1}, E, p\right)$ with $\left|V_{1}\right|=k$ and $p=|\{p(v) \mid v \in V\}|$, or to determine that no such dominion exists.

We will in the analysis of run times make the assumption that computation and removal of reachability sets as well as kernelization are elementary operation and can therefore be performed in time $O(1)$. To obtain the actual run times of our algorithms we will in the end multiply the computed run times by a factor corresponding to the time needed for these operations.

Lemma 14. For $k \geq 4, \operatorname{dom}_{k}(\ell)=O\left(k^{\ell} \cdot T(\ell)\right)$ and $\operatorname{dom}_{k, p}(\ell)=O\left(k^{\ell} \cdot T(\ell, p)\right)$.
Proof. There are $O\left(k^{\ell}\right)$ sets $V_{D} \subseteq V_{1}$ with $\left|V_{D}\right|=\ell$. We argue below that for each such set $V_{D}$, one can determine whether or not there exists a dominion $D$ with $D \cap V_{1} \subseteq V_{D}$ by solving two parity games that are sub-games of $G$, i.e., these games arise from $G$ by removing some of the nodes. This implies the lemma because each of these sub-games can be solved in time $T(\ell)$ or $T(\ell, p)$ for bipartite or general parity games, respectively.

Let $V_{D} \subseteq V_{1}$ be a set with $\left|V_{D}\right|=\ell$. We will now show how to check if there exists an $i$-dominion $D$ with $D \cap V_{1} \subseteq V_{D}$. If such an $i$-dominion $D$ exists, then it is $i$-closed. Therefore, it does not contain any node $v \in V$ from which player $\neg i$ can reach a node in $V_{1} \backslash V_{D}$. Let $V^{\prime}=V(G) \backslash$ reach $_{\neg i}\left(V_{1} \backslash V_{D}\right)$ be the set of nodes from which player $\neg i$ cannot force to reach a node in $V_{1} \backslash V_{D}$; the set $V^{\prime}$ can therefore be computed by computing and removing a reachability set, which as we assumed is an elementary operations. We then have $D \subseteq V^{\prime}$, and since no node in $V_{1} \backslash V_{D}$ can be part of $V^{\prime}$, it holds that $V^{\prime} \cap V_{1} \subseteq V_{D}$. Since $V^{\prime}$ is an $i$-closed set, the game $G-$ reach $_{\neg i}\left(V_{1} \backslash V_{D}\right)$ is well defined. Let $\operatorname{win}_{i}\left(V^{\prime}\right)$ be the winning set of player 1 in the game $G-\operatorname{reach}_{\neg i}\left(V_{1} \backslash V_{D}\right)$. Then $\operatorname{win}_{i}\left(V^{\prime}\right)$ is an $i$-dominion that contains $D$.

This shows that for each set $V_{D} \subseteq V_{1}$ with $\left|V_{D}\right|=\ell$ we only need to compute for $i \in\{0,1\}$ the sets $V_{i}^{\prime}=V \backslash \operatorname{reach}_{\neg i}\left(V_{1} \backslash V_{D}\right)$ of nodes from which player $\neg i$ cannot force to enter $V_{1} \backslash V_{D}$ and compute the winning sets of the game $G-\operatorname{reach}_{\neg i}\left(V_{1} \backslash V_{D}\right)$ to determine whether or not there exists a dominion $D$ with $D \cap V_{1} \subseteq V_{D}$.

With the algorithm described in Lemma 14 we can find a dominion $D$ such that $\left|D \cap V_{1}\right| \leq \ell$ if such a dominion exists. We denote this algorithm by $\operatorname{dominion}_{1}(G, \ell)$ and assume that it returns either the pair ( $D, i$ ) if an $i$-dominion $D$ is found, or $(\emptyset,-1)$ if not.

We will give the pseudocode for $\operatorname{algorithm} \operatorname{dominion}(G, \ell, s)$. In the pseudocode, let $U_{m}$ denote the set of marked nodes from $U$ and let $\operatorname{king}\left(U\right.$, strategy $\left._{i}\right)$ denote an execution of the algorithm by King et al. [13] that determines the winners of the sub-game $G$ restricted to $U$ with a given strategy for player $i$.

## 7 New Algorithms for Solving Parity Games

We present the algorithm new-win ${ }_{1}(G)$ discussed in Sect. 5 in detail. Let $G=\left(V_{0} \uplus V_{1}, E, p\right)$ with $\left|V_{1}\right|=k$ be a parity game with $p$ distinct priorities.

The algorithm new-win ${ }_{1}$ starts by trying to find a "small" dominion $D$, where small means $\left|D \cap V_{1}\right| \leq \ell$, where $\ell=\lfloor\sqrt{2 k}\rfloor$ is a parameter chosen to minimize the run time of the algorithm. If such an $i$-dominion is

```
Algorithm 2 dominion \((G, \ell, s)\)
Input: A parity game \(G=\left(V_{0} \uplus V_{1}, E, p\right)\) and \(\ell, s \in\{0, \ldots,|V(G)|\}\).
Output: An \(i\)-dominion \((D, i)\) for \(i \in\{0,1\}\) or \((\emptyset,-1)\) if no dominion is found.
    Fix a total order \(\prec\) on the nodes of \(G\).
    For each \(u \in V(G)\) sort the edges emanating from \(u\) by \(\prec\) on their respective endpoint.
    for \(i \in\{0,1\}\) do
        for \(v \in V_{i}\) do
            for \(\left\langle a_{1}, \ldots, a_{\ell}, b_{1}, \ldots, b_{s}\right\rangle \in\{1, \ldots,|V(G)|\}^{\ell} \times\{1, \ldots, j\}^{s}\) do
                \(r_{1}=1, r_{2}=1, U=\{v\}, U_{m}=\emptyset\)
                while \(U \neq \emptyset, r_{1} \leq \ell\) and \(r_{2} \leq s\) do
                Choose \(u=\min (U, \prec)\).
                \(U=U \backslash\{u\}, U_{m}=U_{m} \cup\{u\}\).
                    if \(u \in V_{i}\) then
                    if \(\left|\delta^{+}(u)\right|>j\) then
                        if \(\left|\delta^{+}(u)\right| \leq r_{1}\) then
                        Let \(e=(u, w)\) be the \(a_{r_{1}}\)-th outgoing edge of \(u\).
                                    \(U=U \cup\left(\{w\} \backslash U_{m}\right), r_{1}=r_{1}+1, \operatorname{strategy}_{i}(u)=w\).
                    else
                        \(U=\emptyset, U_{m}=\emptyset\).
                    else
                        if \(\left|\delta^{+}(u)\right| \leq r_{2}\) then
                        Let \(e=(u, w)\) be the \(b_{r_{2}}\)-th outgoing edge of \(u\).
                                    \(U=U \cup\left(\{w\} \backslash U_{m}\right), r_{2}=r_{2}+1, \operatorname{strategy}_{i}(u)=w\).
                                    else
                                    \(U=\emptyset, U_{m}=\emptyset\).
                else
                    \(U=U \cup\left(N^{+}(u) \backslash U_{m}\right)\)
                if \(U_{m} \neq \emptyset\) contains at most \(\ell\) high out-degree nodes and at most \(s\) low out-degree nodes then
                \(\left(W_{0}, W_{1}\right)=\boldsymbol{\operatorname { k i n g }}\left(U\right.\), strategy \(\left._{i}\right)\).
                if \(W_{i}=U\) then
                    return \((U, i)\)
    return \((\emptyset,-1)\).
```

found, then we remove it together with its $i$-reachability set from the game and solve the remaining game recursively. If no small dominion is found, then new-win simply calls algorithm old-win $_{1}$, which is almost identical to algorithm win. The only difference between old-win ${ }_{1}$ and win is that its recursive calls are made to new-win ${ }_{1}$ and not to itself.

The recursion stops once the number of odd nodes is at most 4 , in which case we will test each of the at most $\left((p+1)^{4}\right)^{4}$ (due to the size of our kernel) different strategies for player 1 in constant time. We will call this brute force method solve $(G)$. We will also kernelize using the reduction rules described in Sect. 3. We will call the kernelization subroutine $\operatorname{kernel}(G)$. The pseudocode of new-win $\boldsymbol{m}_{1}(G)$ can be found in Sect. 9 ,

The correctness of the algorithm follows analogously to the correctness of $\operatorname{win}(G)$. We analyze the run time of new-win $\operatorname{win}_{1}(G)$ and prove Theorem 1 in section 10.

## 8 Out-degree based Algorithm

We now describe our second algorithm new-win $\boldsymbol{w}_{2}(G, j)$. In order to describe it, let $j \in \mathbb{N}$ and let $s_{j}$ denote the number of nodes of out-degree at most $j$. new-win $\boldsymbol{w i n}_{2}(G, j)$ is then almost identical to new-win $(G)$, but instead of dominions that contains at most $\ell^{\prime}=\lfloor\sqrt{2 k}\rfloor$ nodes of the odd player, we search for and delete
dominions that contain at most $\ell=\left\lceil\sqrt{2\left(n-s_{j}\right)}\right\rceil$ nodes with out-degree larger than $j$ and at most $s=$
 which implies Theorem 3.

In the following let us assume $j=\arg \min _{1 \leq j^{\prime} \leq n}\left\{\sqrt{n-s_{j^{\prime}}}+\sqrt{\frac{s_{j^{\prime}}}{\log _{j^{\prime}} s_{j^{\prime}}}}\right\}$. We say that a node $v$ has high out-degree if $\left|\delta^{+}(v)\right|>j$ and low out-degree otherwise. For $n, z, \ell, s \in \mathbb{N}$, let dom $\operatorname{dom}_{n, z}(\ell, s)$ denote the maximum number of steps required to find a dominion $D$ with at most $\ell$ high out-degree and at most $s$ low out-degree nodes in a parity game $G=\left(V_{0} \uplus V_{1}, E, p\right)$ with $n$ nodes out of which $z$ are high out-degree nodes, or to determine that no such dominion exists.

Lemma 15. For all values of $\ell, s \in\{0, \ldots, n\}$, it holds

$$
\operatorname{dom}_{n, z}(\ell, s)=O\left(n^{\ell+1} j^{s}(\ell+s)^{2} \cdot \max \{1, \log (\ell+s)\}\right)=O\left(n^{\ell+4} j^{s}\right)
$$

Proof. Fix an arbitrary total order $\prec$ on $V(G)$. Let $u \in V(G)$ be a node of $G$ and let $\left(u, v_{1}\right), \ldots,\left(u, v_{\left|\delta^{+}(u)\right|}\right)$ be the edges emanating from $u$, where $v_{i} \prec v_{i+1}$ for all $i \in\left\{1, \ldots\left|\delta^{+}(v)\right|-1\right\}$; we call $\left(u, v_{i}\right)$ the $i$-th outgoing edge of $u$. The algorithm generates at most $O\left(n \cdot n^{\ell} j^{s}\right) 0$-closed sets of nodes that contain at most $\ell$ nodes with an out-degree greater than $j$ and at most $s$ nodes with an out-degree at most $j$, which are candidates for being 0 -dominions. For every node $v \in V$ and every sequence $\left\langle a_{1}, \ldots, a_{\ell}, b_{1}, \ldots, b_{s}\right\rangle \in\{1, \ldots, n\}^{\ell} \times\{1, \ldots, j\}^{s}$ construct a set $U \subseteq V$ as follows. Start with $U=\{v\}$ and $r_{1}=1, r_{2}=1$. Nodes added to $U$ are initially unmarked. As long as there is still an unmarked node in $U$, pick the smallest such node $u \in U$ with respect to $\prec$ and mark it.

- If $u \in V_{0}$ and $u$ has high out-degree then add the endpoint of the $a_{r_{1}}$-th outgoing edge of $u$ to $U$ (if it is not already present in $U$ ) and increment $r_{1}$.
- If $u \in V_{0}$ and $u$ has low out-degree then add the endpoint of the $b_{r_{2}}$-th outgoing edge of $u$ to $U$ (if it is not already present in $U$ ) and increment $r_{2}$.
- If $u \in V_{1}$ then add the endpoints of all outgoing edges of $u$ that are not yet part of $U$ to $U$.

If at some stage $U$ contains either more than $\ell$ nodes with high out-degree or more than $s$ nodes with low out-degree, or the endpoint of the $i$-th outgoing edge of some node $v$ with out-degree $\left|\delta^{+}(v)\right|<i$ should be added to $U$, then discard the set $U$ and restart the construction with the next sequence. If the process above ends without discarding $U$, then a 0 -closed set containing at most $\ell$ high out-degree and at most $s$ low out-degree nodes has been found. Furthermore, for every node $u \in U \cap V_{0}$, one of the outgoing edges of $u$ was selected. This corresponds to a suggested strategy for player 0 in the game $G$ restricted to the set $U$.

Our algorithm therefore considers by exhaustive search all 0 -closed sets containing at most $\ell$ high outdegree and at most $s$ low out-degree nodes, and for each set considers all possible positional strategies for player 0. Using an algorithm of King et al. 13 we can check in time $O\left((\ell+s)^{2} \log (\ell+s)\right)$ time whether a given pair of set $U$ and proposed strategy is indeed a winning strategy for player 0 from all nodes of $U$. Thus, if there is a 0 -dominion containing at most $\ell$ high out-degree and at most $s$ low out-degree nodes, then the algorithm will find one. Finding 1-dominions can be done in an analogous manner.

With the just described algorithm, we can find a dominion $D$ with at most $\ell$ nodes with high outdegree and at most $s$ nodes with low out-degree if such a dominion exists. We denote this algorithm by dominion $_{2}(G, \ell, s)$, and suppose that it returns either the pair $(D, i)$ if such an $i$-dominion $D$ is found, or $(\emptyset,-1)$ if not.

The algorithm new-win starts $^{2}$ by trying to find a "small" dominion $D$, where small means that $D$ contains at most $\ell$ nodes with out-degree greater than $j$ and at most $s$ nodes with out-degree at most $j$, where $\ell=\left\lceil\sqrt{2\left(n-s_{j}\right)}\right\rceil$ and $s=\left\lceil\sqrt{s_{j} \cdot \log _{j} s_{j}}\right\rceil$ are parameters chosen to minimize the run time of the whole algorithm. If such an $i$-dominion is found, then we remove it together with its $i$-reachability set from the game and solve the remaining game recursively. If no small dominion is found, then new-win $\boldsymbol{m}_{2}$ simply
calls algorithm old-win , $_{2}$, which is almost identical to algorithm win. The only difference between old-win ${ }_{2}$ and win is that its recursive calls are made to new-win ${ }_{2}$ and not to itself.

The recursion stops once the number of nodes with out-degree at most $j$ and the number of nodes with out-degree greater than $j$ are both at most 3 , in which case we will test all of the at most constant different strategies for the two players in constant time. We will call this brute force method solve $(G)$. The pseudocode of new-win ${ }_{2}$ can be found in Sect. 9 ,

The correctness of new-win n $_{2}$ follows analogously to the correctness of the simple algorithm win. We analyze the run time of new-win $\boldsymbol{w}_{2}$ and prove Theorem 3 in Sect. 10 ,

We can now proove Corollary 2 .
Corollary 园, First consider a parity game on $n$ nodes played on a graph with maximum out-degree $\Delta$. Then $s_{\Delta}=n$ and

$$
\sqrt{n-s_{\Delta}}+\sqrt{\frac{s_{\Delta}}{\log _{\Delta} s_{\Delta}}}=\sqrt{\frac{n}{\log _{\Delta} n}}=\sqrt{\frac{\log (\Delta) n}{\log n}} .
$$

Now the first part of the corollary follows immediately from Theorem 3,
Let us now consider the case that the average out-degree is $\Delta$ and let $z=\log (n) \Delta$. Then Markov's inequality implies $s_{z} \geq(1-1 / \log (n)) n$. Hence,

$$
\sqrt{n-s_{z}}+\sqrt{\frac{s_{z}}{\log _{z} s_{z}}} \leq \sqrt{\frac{n}{\log (n)}}+\sqrt{\frac{n}{\log _{z} n}}=\sqrt{\frac{n}{\log (n)}}+\sqrt{\frac{\log (\log (n) \Delta) n}{\log n}} .
$$

Now the second part of the corollary follows immediately from Theorem 3,

## 9 Pseudocode for Algorithms new-win

We will now give the pseudocode for the algorithms new-win ${ }_{1}(G)$ and new-win ${ }_{2}(G, j)$ together with their subroutines old-win $\boldsymbol{w i n}_{1}(G)$ and old-win $\boldsymbol{w i n}_{2}(G, j)$. In the pseudocodes, we call a function solve $(G)$. This function denotes a bruteforce method to solve parity games and is only used on very small games.

```
Algorithm 3 new-win \({ }_{1}(G)\)
Input: A parity game \(G=\left(V_{0} \uplus V_{1}, E, p\right)\).
Output: A partition \(\left(W_{0}, W_{1}\right)\) of \(V\), where \(W_{i}\) is the winning set of player \(i \in\{0,1\}\).
    \(k \leftarrow\left|V_{1}\right| ; \ell \leftarrow\lfloor\sqrt{2 k} \mid ; G=\operatorname{kernel}(G)\)
    if \(k \leq 4\) then return solve \((G)\)
    \((D, i) \leftarrow \operatorname{dominion}_{1}(G, \ell)\)
    if \(D=\emptyset\) then
        \(\left(W_{0}, W_{1}\right) \leftarrow\) old-win \((G)\)
    else
        \(\left(W_{0}^{\prime}, W_{1}^{\prime}\right) \leftarrow\) new-win \(\operatorname{win}_{1}\left(G-\operatorname{reach}_{i}(D)\right)\)
        \(\left(W_{\neg i}, W_{i}\right) \leftarrow\left(W_{\neg i}^{\prime}, V \backslash W_{\neg i}^{\prime}\right)\)
    return \(\left(W_{0}, W_{1}\right)\)
```

```
Algorithm 4 old-win \({ }_{1}(G)\)
Input: A parity game \(G=\left(V_{0} \uplus V_{1}, E, p\right)\).
Output: A partition \(\left(W_{0}, W_{1}\right)\) of \(V\), where \(W_{i}\) is the winning set of player \(i \in\{0,1\}\).
    \(G=\operatorname{kernel}(G)\)
    \(i \leftarrow p_{\text {max }}(\bmod 2)\)
    \(\left(W_{0}^{\prime}, W_{1}^{\prime}\right) \leftarrow\) new-win \(\left(G-\operatorname{reach}_{i}\left(V_{p_{\text {max }}}\right)\right)\)
    if \(W_{-i}^{\prime}=\emptyset\) then
        \(\left(W_{i}, W_{\neg i}\right) \leftarrow(V, \emptyset)\)
    else
        \(\left(W_{0}^{\prime \prime}, W_{1}^{\prime \prime}\right) \leftarrow\) new- \(\boldsymbol{w i n}_{1}\left(G-\operatorname{reach}_{\neg i}\left(W_{\neg i}^{\prime}\right)\right)\)
        \(\left(W_{i}, W_{\neg i}\right) \leftarrow\left(W_{i}^{\prime \prime}, V \backslash W_{i}^{\prime \prime}\right)\)
    return \(\left(W_{0}, W_{1}\right)\)
```

```
Algorithm 5 new-win \(2(G, j)\)
Input: A parity game \(G=\left(V_{0} \uplus V_{1}, E, p\right)\) and \(j \in\{1, \ldots|V|\}\).
Output: A partition \(\left(W_{0}, W_{1}\right)\) of \(V\), where \(W_{i}\) is the winning set of player \(i \in\{0,1\}\).
    \(s_{j} \leftarrow\left|\left\{v \in V| | \delta^{+}(v) \mid \leq j\right\}\right| ; \ell \leftarrow\left\lceil\sqrt{2\left(n-s_{j}\right)}\right] ; s \leftarrow\left\lceil\sqrt{s_{j} \cdot \log _{j} s_{j}}\right\rceil\)
    if \(s_{j} \leq 3\) and \(n-s_{j} \leq 3\) then return solve \((G)\)
    \((D, i) \leftarrow \operatorname{dominion}_{2}(G, \ell, s)\)
    if \(D=\emptyset\) then
        \(\left(W_{0}, W_{1}\right) \leftarrow\) old-win \({ }_{2}(G, j)\)
    else
        \(\left(W_{0}^{\prime}, W_{1}^{\prime}\right) \leftarrow\) new-win \(\operatorname{win}_{2}\left(G-\operatorname{reach}_{i}(D), j\right)\)
        \(\left(W_{\neg i}, W_{i}\right) \leftarrow\left(W_{\neg i}^{\prime}, V \backslash W_{\neg i}^{\prime}\right)\)
    return \(\left(W_{0}, W_{1}\right)\)
```

```
Algorithm 6 old-win \({ }_{2}(G, j)\)
Input: A parity game \(G=\left(V_{0} \uplus V_{1}, E, p\right)\).
Output: A partition \(\left(W_{0}, W_{1}\right)\) of \(V\), where \(W_{i}\) is the winning set of player \(i \in\{0,1\}\).
    \(i \leftarrow p_{\max }(\bmod 2)\)
    \(\left(W_{0}^{\prime}, W_{1}^{\prime}\right) \leftarrow\) new-win \({ }_{2}\left(G-\operatorname{reach}_{i}\left(V_{p_{\text {max }}}\right), j\right)\)
    if \(W_{\neg i}^{\prime}=\emptyset\) then
        \(\left(W_{i}, W_{\neg i}\right) \leftarrow(V, \emptyset)\)
    else
        \(\left(W_{0}^{\prime \prime}, W_{1}^{\prime \prime}\right) \leftarrow\) new- \(\boldsymbol{w i n}_{2}\left(G-\operatorname{reach}_{\neg i}\left(W_{\neg i}^{\prime}\right), j\right)\)
        \(\left(W_{i}, W_{\neg i}\right) \leftarrow\left(W_{i}^{\prime \prime}, V \backslash W_{i}^{\prime \prime}\right)\)
    return \(\left(W_{0}, W_{1}\right)\)
```


## 10 Analysis of the Run Time

We will show that algorithm new-win $(G)$ has a run time of $O(p \cdot m \cdot n) \cdot(p+k)^{O(\sqrt{k})}$ on general instances and in time $O\left(n^{3}\right) \cdot k^{O(\sqrt{k})}$ on bipartite instances. We will also show that algorithm new-win $(G, j)$ has a run time of $n=0\left(\sqrt{n-s_{j}}+\sqrt{\frac{s_{j}}{\operatorname{Tg}_{j_{j}}}}\right)$, where $s_{j}$ the number of nodes in $G$ with out-degree at most $j$.

Note that the part $O(p m n)$ of the run time comes from the reduction of the instance and the computation and removal of reachability sets of found dominions. Since we do both of these often, we assume them to be elementary computations with computation time $O(1)$ and show that the total run time remaining is $(p+k)^{O(\sqrt{k})}$. In bipartite instances we need $O\left(n^{3}\right)$ time to reduce the instance and to compute and remove reachability sets. We will show that the run time on bipartite instances is $k^{O(\sqrt{k})}$, when computation and removal of reachability sets and the reductions are viewed as an elementary operation. Let $T(k, p)$ denote the time required by algorithm new-win on a game $G=\left(V_{0} \uplus V_{1}, E, p\right)$ with $\left|V_{1}\right|=k$ and $p=|\{p(v) \mid v \in V\}|$, when reduction of an instance and computation and removal of reachability sets of found dominions are viewed as elementary computations and have run time $O(1)$.

Lemma 16. The following recurrence relation holds:
(a) $T(k, p) \leq \max \{T(k-1, p), T(k, p-1)+T(k-\ell, p)\}+\operatorname{dom}_{k, p}(\ell)+O(1)$.

Proof. Algorithm new-win $(G)$ tries to find dominions $D$ with $\left|D \cap V_{1}\right| \leq \ell=\lfloor\sqrt{2 k}\rfloor$. By definition this takes at most $\operatorname{dom}_{k, p}(\ell)$ time on general instances. If a (non-empty) dominion is found, then the algorithm simply proceeds on the remaining game, which has at most $k-1$ odd nodes, and thus it solves this game in time bounded by $T(k-1, p)$. Otherwise, a call to old-win $(G)$ is made. This results in a call to new-win $(G-$ $\left.\operatorname{reach}_{i}\left(V_{p_{\text {max }}}\right)\right)$. In this case the call takes at most $T(k, p-1)$ time because we removed all nodes with the highest priority. If the set $W_{j}^{\prime}$ returned by the call is empty, then we are done. Otherwise, $W_{j}^{\prime}=$ $\operatorname{win}_{j}\left(G-\operatorname{reach}_{i}\left(V_{p_{\text {max }}}\right)\right)$ and $W_{j}^{\prime} \subseteq \operatorname{win}_{j}(G)$ by Lemma 4. Therefore, $W_{j}^{\prime}$ is a $j$-dominion of $G$. We are in the case that there is no dominion $D$ with $\left|D \cap V_{1}\right| \leq \ell$ in $G$. Thus, $\left|W_{j}^{\prime} \cap V_{1}\right|>\ell$, and hence the second recursive call new-win $\left(G-\operatorname{reach}_{j}\left(W_{j}^{\prime}\right)\right)$ takes time at most $T(k-\lceil\ell\rceil, p)$. Consequently,

$$
T(k, p) \leq \max \{T(k-1, p), T(k, p-1)+T(k-\lceil\ell\rceil, p)\}+\operatorname{dom}_{k, p}(\ell)+O(1)
$$

Let $T(k$, even) and $T(k$, odd $)$ denote the time required by algorithm new-win on a bipartite game $G=$ $\left(V_{0} \uplus V_{1}, E, p\right)$ with $\left|V_{1}\right|=k$ when the largest priority is even respectively odd, when computation and removal of reachability sets of found dominions and the reductions are viewed as elementary computations. We denote by $T(k)$ the time required by algorithm new-win on any bipartite game with $\left|V_{1}\right|=k$; thus $T(k) \leq \max \{T(k$, even $), T(k$, odd $)\}$.

Lemma 17. The following recurrence relations hold:

$$
\left.\begin{array}{rl}
\left(b_{1}\right) & T(k, \text { odd })
\end{array}\right) \leq T(k-1)+T(k-\ell)+\operatorname{dom}_{k}(\ell)+O(1), ~\left(b_{2}\right) T(k, \text { even }) \leq \max \{T(k, \text { odd }), T(k-1)\}+T(k-\ell)+\operatorname{dom}_{k}(\ell)+O(1),
$$

Proof. From the definition it follows directly that $T(k) \leq \max \{T(k$, even $), T(k$, odd $)\}$. Showing $\left(b_{1}\right)$ and $\left(b_{2}\right)$ therefore yields $\left(b_{3}\right)$. Algorithm new-win $(G)$ tries to find dominions $D$ with $\left|D \cap V_{1}\right| \leq \ell=\lceil\sqrt{2 k}\rceil$. By definition this takes at most $\operatorname{dom}_{k}(\ell)$ time on bipartite instances. If a (non-empty) dominion is found, then the algorithm simply proceeds on the remaining game, which has at most $k-1$ odd nodes, and the remaining run time is therefore at most $T(k-1)$. Otherwise, a call to old-win $(G)$ is made. This results in a call to new-win $\left(G-\operatorname{reach}_{i}\left(V_{p_{\max }}\right)\right)$. Here we have to distinguish whether the highest priority is odd or even.

If the highest priority is odd then, by Lemma 10 the set reach ${ }_{1}\left(V_{p_{\max }}\right) \cap V_{1}$ is non-empty and the call takes at most $T(k-1)$ time.

In case the highest priority is even, we either have reach ${ }_{0}\left(V_{p_{\max }}\right) \cap V_{1} \neq \emptyset$ or reach $\left(V_{p_{\max }}\right)=V_{p_{\text {max }}}$ in which case Lemma 11 yields that in $G-\operatorname{reach}_{i}\left(V_{p_{\text {max }}}\right)$ the highest priority has to be odd. Therefore, this call needs time at most $\max \{T(k$, odd),$T(k-1)\}$.

If the set $W_{j}^{\prime}$ returned by the call is empty, then we are done. Otherwise, $W_{j}^{\prime}=\operatorname{win}_{j}\left(G-\operatorname{reach}_{i}\left(V_{p_{\text {max }}}\right)\right)$ and this is part of $\operatorname{win}_{j}(G)$ by Lemma 4. Therefore, $W_{j}^{\prime}$ is a $j$-dominion of $G$. We are in the case that there
is no dominion $D$ with $\left|D \cap V_{1}\right|$ at most $\ell$ in $G$, so we know that $\left|W_{j}^{\prime} \cap V_{1}\right|>\ell$, and therefore the second recursive call new-win $\left(G-\operatorname{reach}_{j}\left(W_{j}^{\prime}\right)\right)$ takes at most $T(k-\ell)$ time. Thus, we obtain

$$
T(k, \text { odd }) \leq T(k-1)+T(k-\ell)+\operatorname{dom}_{k}(\ell)
$$

and

$$
\begin{aligned}
T(k, \text { even }) & \leq \max \{T(k, \text { odd }), T(k-1)\}+T(k-\ell)+\operatorname{dom}_{k}(\ell) \\
& \leq T(k-1)+2 T(k-\ell)+2 \operatorname{dom}_{k}(\ell)
\end{aligned}
$$

which yields $T(k) \leq T(k-1)+2 T(k-\ell)+2 \operatorname{dom}_{k}(\ell)$.
For $j \in \mathbb{N}_{0}$, let $T^{\prime}\left(s_{j}, n-s_{j}\right)$ denote the time required by algorithm new-win on a game $G$ on $n$ nodes of which $s_{j}$ nodes have out-degree at most $j$.
Lemma 18. The following recurrence relation holds:

$$
\begin{aligned}
(c) \quad T^{\prime}\left(s_{j}, n-s_{j}\right) \leq & \max \left\{T^{\prime}\left(s_{j}-1, n-s_{j}\right), T^{\prime}\left(s_{j}, n-s_{j}-1\right)\right\} \\
& +\max \left\{T^{\prime}\left(s_{j}-s, n-s_{j}\right), T^{\prime}\left(s_{j}, n-s_{j}-\ell\right)\right\}+\operatorname{dom}_{n, n-s_{j}}(\ell, s)+O(1) .
\end{aligned}
$$

Proof. Algorithm new-win $(G, j)$ tries to find dominions $D$ containing at most $s$ nodes with out-degree at most $j$ and at most $\ell$ nodes with out-degree greater than $j$. By definition this takes at most dom $n, n-s_{j}(\ell, s)$ time. If a (non-empty) dominion is found, then the algorithm simply proceeds on the remaining game, which has at most $n-1$ nodes, and the remaining time is therefore at most $\max \left\{T^{\prime}\left(s_{j}-1, n-s_{j}\right), T^{\prime}\left(s_{j}, n-s_{j}-1\right)\right\}$. Otherwise, a call to old-win $(G, j)$ is made. This results in a call to new-win $\left(G-\operatorname{reach}_{i}\left(V_{p_{\max }}\right)\right.$, $\left.j\right)$, this call is to a game with fewer nodes and can be solved in time bounded by

$$
\max \left\{T^{\prime}\left(s_{j}-1, n-s_{j}\right), T^{\prime}\left(s_{j}, n-s_{j}-1\right)\right\}
$$

If the set $W_{k}^{\prime}$ returned by the call is empty, then we are done. Otherwise, $W_{k}^{\prime}=\operatorname{win}_{k}\left(G-\operatorname{reach}_{i}\left(V_{p_{\text {max }}}\right)\right)$, and $W_{k}^{\prime} \subseteq \operatorname{win}_{k}(G)$ by Lemma 4. Therefore, $W_{k}^{\prime}$ is a $k$-dominion of $G$. We are in the case that there is no dominion $D$ containing at most $s$ nodes with out-degree at most $j$ and at most $\ell$ nodes with out-degree greater than $j$, so $W_{k}^{\prime}$ either contains more than $s$ nodes with out-degree at most $j$ or more than $\ell$ nodes with out-degree greater than $j$, and therefore the second recursive call new-win $\left(G-\operatorname{reach}_{k}\left(W_{k}^{\prime}\right)\right)$ takes time bounded by $\max \left\{T^{\prime}\left(s_{j}-s, n-s_{j}\right), T^{\prime}\left(s_{j}, n-s_{j}-\ell\right)\right\}$.

All other computations can be done in constant time. Thus, we obtain

$$
\begin{aligned}
T^{\prime}\left(s_{j}, n-s_{j}\right) & \leq \max \left\{T^{\prime}\left(s_{j}-1, n-s_{j}\right), T^{\prime}\left(s_{j}, n-s_{j}-1\right)\right\} \\
& +\max \left\{T^{\prime}\left(s_{j}-s, n-s_{j}\right), T^{\prime}\left(s_{j}, n-s_{j}-\ell\right)\right\}+\operatorname{dom}_{n, n-s_{j}}(\ell, s)+O(1)
\end{aligned}
$$

We analyze recurrences $(a)$ and $(b)$ with $\ell=\lfloor\sqrt{2 k}\rfloor$ in Theorem 5 in Sect. 11 which eventually shows that $T(k, p) \leq(p+k)^{O(\sqrt{k})}$ and $T(k) \leq k^{O(\sqrt{k})}$, and recurrence $(c)$ with $\ell=\left\lceil\sqrt{2\left(n-s_{j}\right)}\right\rceil$ and $s=\left\lceil\sqrt{\frac{s_{j}}{\log _{j} s_{j}}}\right\rceil$ in
 the analysis of the run time of new-win $(G)$ and new-win $(G, j)$, and it proves Theorem 1 and Theorem3.

## 11 Recurrence Relation Computations

In this section we analyze the recurrence relations used to bound the run time of new-win.
Theorem 5. For $k \in \mathbb{N}$ and $\ell=\lfloor\sqrt{2 k}\rfloor$, we obtain

$$
\begin{aligned}
T(k, p) & =(p+k)^{O(\sqrt{k})} \\
T(k) & =k^{O(\sqrt{k})}
\end{aligned}
$$

To prove Theorem [5] we first establish some lemmas.
Lemma 19. For $k, p \in \mathbb{N}$ and $\ell=\lfloor\sqrt{2 k}\rfloor$, it holds

$$
\begin{aligned}
T(k, p) & \leq 2(k+p)^{\lfloor\sqrt{2 k}\rfloor} \cdot \operatorname{dom}_{k, p}(\lfloor\sqrt{2 k}\rfloor) \text { and } \\
T(k) & \leq 2(2 k)\lfloor\sqrt{2 k}\rfloor \cdot \operatorname{dom}_{k}([\sqrt{2 k}\rceil)
\end{aligned}
$$

Proof. For every pair of integers $k$ and $p$ we construct binary trees $T_{k, p}$ and $T_{k}$ in the following way. The root of $T_{k, p}$ is labeled by $k$ and $k+p$ and the root of $T_{k}$ is labeled by $k$. A node labeled by a number $k>4$ has two children: in $T_{k, p}$ a left child labeled by $k$ and $k+p-1$ and a right child labeled by $k-\lceil\sqrt{2 k}\rceil$ and $p+k-\lceil\sqrt{2 k}\rceil$. In $T_{k}$ a left child labeled by $k^{\prime}$ and a right child labeled by $k-\lceil\sqrt{2 k}\rceil$. A node labeled by $k^{\prime}$ in $T_{k}$ has two children: a left child labeled by $k-1$ and a right child labeled by $k-\lceil\sqrt{2 k}\rceil$. Nodes labeled by a number $k \leq 4$ are leaves. A node labeled by $k$ and $k+p$ has a cost of $\operatorname{dom}_{k, p}(\lfloor\sqrt{2 k} \mid)$ associated with it and a node labeled by $k$ or $k^{\prime}$ has a cost of $\left.\operatorname{dom}_{k}([\sqrt{2 k}])\right)$ associated with it. It follows from Lemma 16 and Lemma 17 that the sum of the costs of the nodes of $T_{k, p}$ and $T_{k}$, is an upper bound on $T(k, p)$ and $T(k)$, respectively. Clearly, the length of every path from the root to a leaf is at most $p+k+1$ in $T_{k, p}$ and $2 k$ in $T_{k}$. We say that such a path makes a right turn when it descends from a node to its right child. We next claim that each such path makes at most $\lfloor\sqrt{2 k}\rfloor$ right turns. This follows immediately from the observation that the function $f(n)=n-\lceil\sqrt{2 n}\rceil$ can be iterated on $2 k$ at most $\lfloor\sqrt{2 k}\rfloor$ times before reaching the value of 4 or less. This observation can be proved by induction, based on the fact that if $\frac{1}{2} j^{2}<n \leq \frac{1}{2}(j+1)^{2}$, then $n-\lceil\sqrt{2 n}\rceil \leq \frac{1}{2} j^{2}$. (Initially we have $j=\lfloor\sqrt{2 k}\rfloor$ and finally, with $1 \leq n \leq 4$, we have $j \geq 1$.) As each leaf of $T_{k, p}$ and $T_{k}$ is determined by the positions of the right turns on the path leading to it from the root, we get that the number of leaves is at most $\binom{k+p}{\lfloor\sqrt{2 k}\rfloor}$ in $T_{k, p}$ and at most $\binom{2 k}{2 k}$ in $T_{k}$. The total number of nodes is therefore at most at most $2\binom{k+p}{\lfloor\sqrt{2 k}\rfloor} \leq 2(k+p)\lfloor\sqrt{2 k}\rfloor$ in $T_{k, p}$ and at most $2(\lfloor\sqrt{2 k}\rfloor) \leq 2(2 k)\lfloor\sqrt{2 k}\rfloor$ in $T_{k}$. As the cost of each node is at most $\operatorname{dom}_{k, p}(\lfloor\sqrt{2 k}\rfloor)$ in $T_{k, p}$ and at most dom $\left(\lceil\sqrt{2 k} \mid)\right.$ in $T_{k}$, we immediately get

$$
\begin{aligned}
T(k, p) & \leq 2(k+p)^{\lfloor\sqrt{2 k}\rfloor} \cdot \operatorname{dom}_{k, p}(\lfloor\sqrt{2 k}\rfloor) \text { and } \\
T(k) & \leq 2(2 k)^{\lfloor\sqrt{2 k}\rfloor} \cdot \operatorname{dom}_{k}(\lceil\sqrt{2 k}\rceil)
\end{aligned}
$$

We obtain together with Lemma 14 that

$$
T(k, p) \leq 2(k+p)^{\lfloor\sqrt{2 k}\rfloor} \cdot O\left(k^{\lfloor\sqrt{2 k}\rfloor} \cdot T(\lfloor\sqrt{2 k}\rfloor, p)\right)
$$

as well as

$$
T(k) \leq 2(2 k)^{\lfloor\sqrt{2 k}\rfloor} \cdot O\left(k^{\lceil\sqrt{2 k}\rceil} T(\lceil\sqrt{2 k}\rceil)\right)
$$

Lemma 20. Suppose that

$$
T(k, p) \leq 2(k+p)^{\lfloor\sqrt{2 k}\rfloor} \cdot O\left(k^{\lfloor\sqrt{2 k}\rfloor} \cdot T(\lfloor\sqrt{2 k}\rfloor, p)\right)
$$

and that $T(\ell, q) \leq c^{\prime} \cdot(q+\ell)^{8\lfloor\sqrt{2 \ell}\rfloor}$ for some constant $c^{\prime} \in \mathbb{R}$ and for all pairs $(\ell, q) \in\{1, \ldots, 4\} \times \mathbb{N}$. Then there exist constants $c_{1} \geq c^{\prime}, c_{2} \geq 8$ such that for all $k \in \mathbb{N}$,

$$
T(k, p) \leq c_{1} \cdot(p+k)^{c_{2}\lfloor\sqrt{2 k}\rfloor}
$$

Proof. Since we have $T(k, p) \leq 2(k+p)^{\lfloor\sqrt{2 k}\rfloor} \cdot O\left(k^{\lfloor\sqrt{2 k}\rfloor} \cdot T(\lfloor\sqrt{2 k}\rfloor, p)\right)$, there exists a constant $c_{1}^{\prime}>0$ such that $T(k, p) \leq 2(k+p)^{\lfloor\sqrt{2 k}\rfloor} \cdot c_{1}^{\prime} k^{\lfloor\sqrt{2 k}\rfloor} \cdot T(\lfloor\sqrt{2 k}\rfloor, p)$. Let $\alpha_{k}=\frac{\lfloor\sqrt{2 k}\rfloor}{\lfloor\sqrt{2\lfloor\sqrt{2 k}\rfloor}}$. Then for $k \geq 5$, it holds $\alpha_{k} \geq 1.5>1$. Let $c_{1}=\max \left\{c_{1}^{\prime}, c^{\prime}\right\}$, and let $c_{2}=6+3 \log \left(2 c_{1}\right)$. Suppose, for sake of contradiction, that the statement of the lemma does not hold for this choice of $\left(c_{1}, c_{2}\right)$. Then there exists a pair $\left(k^{\prime}, p^{\prime}\right) \in \mathbb{N} \times \mathbb{N}$ for which $T\left(k^{\prime}, p^{\prime}\right)>c_{1} \cdot\left(p^{\prime}+k^{\prime}\right)^{c_{2}\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor}$. Let $k^{\prime}$ be the smallest integer for which such a pair exists, and
 $T(\ell, q) \leq c_{1} \cdot q^{3} \cdot(q+\ell)^{c_{2} \sqrt{\ell}}$ for all pairs $(\ell, q)$ with $\ell \leq k^{\prime}, q \leq p^{\prime}$ and $\ell+q<k^{\prime}+p^{\prime}$. Further, it holds $c_{2} \geq \frac{c_{2}}{\alpha_{k}}+2+\log \left(2 c_{1}^{\prime}\right)$ for all $k \geq 4$. For $k \geq 4$ we also have $[\sqrt{2 k}]<k$. This implies that

$$
\begin{aligned}
& \left.T\left(k^{\prime}, p^{\prime}\right) \leq 2\left(k^{\prime}+p^{\prime}\right)^{\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor} \cdot c_{1}^{\prime} \cdot k^{\prime\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor} \cdot T\left(\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor, p^{\prime}\right)\right) \\
& \leq 2\left(k^{\prime}+p^{\prime}\right)^{\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor} \cdot c_{1}^{\prime} \cdot k^{\prime\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor} \cdot c_{1} \cdot\left(p^{\prime}+\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor\right)^{c_{2}\left\lfloor\sqrt{2\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor}\right\rfloor} \\
& \leq 2\left(k^{\prime}+p^{\prime}\right)^{\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor} \cdot c_{1}^{\prime} \cdot k^{\prime\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor} \cdot c_{1} \cdot\left(p^{\prime}+\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor\right)^{c_{2}\left\lfloor\sqrt{2\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor}\right\rfloor} \\
& \leq\left(2 c_{1}^{\prime} c_{1}\right)\left(k^{\prime}+p^{\prime}\right)^{2\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor+c_{2}\left\lfloor\sqrt{2\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor}\right\rfloor} \\
& \leq c_{1}\left(k^{\prime}+p^{\prime}\right)^{2\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor+\frac{c_{2}}{a_{k^{\prime}}}\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor+\log \left(2 c_{1}^{\prime}\right)} \\
& \leq c_{1}\left(k^{\prime}+p^{\prime}\right)^{\left(2+\frac{c_{2}}{a_{k^{\prime}}}+\log \left(2 c_{1}^{\prime}\right)\right)\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor} \\
& \leq c_{1}\left(k^{\prime}+p^{\prime}\right)^{c_{2}\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor}
\end{aligned}
$$

This contradicts the existence of $k^{\prime}$, and therefore concludes the proof.
Lemma 21. Suppose that

$$
T(k) \leq 2(2 k)^{\lfloor\sqrt{2 k}\rfloor} \cdot O\left(k^{\lfloor\sqrt{2 k}\rfloor} T(\lfloor\sqrt{2 k}\rfloor)\right)
$$

and that $T(\ell) \leq c^{\prime} \ell\lfloor\sqrt{2 \ell}\rfloor$ for some constant $c^{\prime} \in \mathbb{R}$ and for all $\ell \leq 4$. Then there exist constants $c_{1} \geq c^{\prime}, c_{2} \geq 1$ such that for all $k \in \mathbb{N}$,

$$
T(k) \leq c_{1} k^{c_{2}\lfloor\sqrt{2 k}\rfloor} .
$$

Proof. Since $T(k) \leq 2(2 k)^{\lfloor\sqrt{2 k}\rfloor} \cdot O\left(k^{\lfloor\sqrt{2 k}\rfloor} T(\lfloor\sqrt{2 k}\rfloor)\right)$, there exists a constant $c_{1}^{\prime}>0$ such that $T(k) \leq$ $2(2 k)^{\lfloor\sqrt{2 k}\rfloor} \cdot c_{1}^{\prime}\left(k^{\lfloor\sqrt{2 k}\rfloor} T(\lfloor\sqrt{2 k} \mid))\right.$. Let $\alpha_{k}=\frac{\lfloor\sqrt{2 k}\rfloor}{\lfloor\sqrt{2\lfloor\sqrt{2 k}\rfloor}}$. Then for $k \geq 5$ it holds that $\alpha_{k} \geq 1.5>1$. Let $c_{1}=\max \left\{c_{1}^{\prime}, c^{\prime}\right\}$ and let $c_{2}=12+3 \log c_{1}$. Suppose, for sake of contradiction, that the statement of the lemma does not holds for this choice of $\left(c_{1}, c_{2}\right)$. Then exists a $k^{\prime} \in \mathbb{N}$ such that $T\left(k^{\prime}\right)>c_{1} k^{\prime c_{2}\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor \text {. Let } k^{\prime}, ~(l)}$ be the smallest such integer. Note that we must have $k^{\prime} \geq 4$ and $T(\ell) \leq c_{1} \ell^{c_{2}\lfloor\sqrt{2 \ell}\rfloor}$ for all $\ell<k^{\prime}$. Further,
it holds that $c_{2} \geq \frac{c_{2}}{\alpha_{k}}+4+\log c_{1}$ for all $k \geq 5$. For $k \geq 4$ we also have $\lfloor\sqrt{2 k}\rfloor<k$. This implies that

$$
\begin{aligned}
T\left(k^{\prime}\right) & \left.\leq 2\left(2 k^{\prime}\right)\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor \cdot c_{1}^{\prime}\left(k^{\prime}\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor T\left(\sqrt{2 k^{\prime}}\right\rfloor\right)\right) \\
& \leq 2\left(2 k^{\prime}\right)\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor \cdot c_{1}\left(2 c_{1} k^{\prime}\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor k^{\prime c_{2}}\left\lfloor\sqrt{2\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor}\right\rfloor\right) \\
& \leq 2 c_{1}\left(2 k^{\prime}\right)^{\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor} k^{\prime\left(\frac{c_{2}}{\alpha_{k}}+1\right)\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor+\log c_{1}} \\
& \leq 2 c_{1} k^{\prime 2\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor} h^{\prime\left(\frac{c_{2}}{\alpha_{k}}+1\right)\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor+\log c_{1}} \\
& \leq c_{1} k^{\prime\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor\left(\frac{c_{2}}{\alpha_{k}}+3\right)+\log c_{1}+\log 2} \\
& \leq c_{1} k^{\prime\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor\left(\frac{c_{2}}{\alpha_{k}}+4+\log c_{1}\right)} \\
& \leq c_{1} k^{\prime c_{2}\left\lfloor\sqrt{2 k^{\prime}}\right\rfloor} .
\end{aligned}
$$

This contradicts the existence of $k^{\prime}$, and therefore concludes the proof.
Since $T(k, p) \in O\left(p^{\left(k^{2}\right)}\right)$, it holds that $O\left(p^{8}\lfloor\sqrt{2 k}\rfloor\right.$. Moreover, as $T(k)=O(1)$ for $k \leq 4$, we conclude that $T(k, p)=(p+k)^{O(\sqrt{k})}$ and $T(k)=k^{O(\sqrt{k})}$. This completes the proof of Theorem 5

Next, we will prove the following.
Lemma 22. For $s_{j}, j, n \in \mathbb{N}, s=\left\lceil\sqrt{\frac{s_{j}}{\log _{j_{j}}}}\right\rceil$ and $\ell=\left\lceil\sqrt{2\left(n-s_{j}\right)}\right\rceil$ we obtain

$$
T\left(s_{j}, n-s_{j}\right)=n^{o\left(\sqrt{n-s_{j}}+\sqrt{\frac{s}{j}^{\log _{j} s_{j}}}\right)} .
$$

To prove Lemma 22, we first establish another lemma.
Lemma 23. For $s_{j}, j, n \in \mathbb{N}, s=\left\lceil\sqrt{\frac{s_{j}}{\log _{j} s_{j}}}\right\rceil$ and $\ell=\left\lceil\sqrt{2\left(n-s_{j}\right)}\right\rceil$, it holds

$$
\left.T\left(s_{j}, n-s_{j}\right) \leq n^{o\left(\sqrt{n-s_{j}}+\sqrt{\frac{s_{j}}{\log _{j}^{s_{j}}}}\right.}\right) \cdot\left(\operatorname{dom}_{n, n-s_{j}}(\ell, s)+O(1)\right) .
$$

Proof. For each parity game $G$ on $n$ nodes and $s_{j}=s_{j}(G)$ we construct a binary tree $T_{G}$ in the following way. The root of $T_{G}$ is labeled by $\left(s_{j}, n-s_{j}\right)$. A node labeled by $(a, b)$ with $a>3$ and $b>3$ has up to two children: a left child labeled by $(a-1, b)$ or ( $a, b-1$ ), and possibly a right child labeled by ( $a-\sqrt{a \cdot \log _{j} a}, b$ ) or ( $a, b-\sqrt{b}$ ). Each child of a node corresponds to one of the recursive calls. The choice on how we label the children depends on the behavior of the algorithm. We label the children of a node by ( $a^{\prime}, b^{\prime}$ ) and ( $a^{\prime \prime}, b^{\prime \prime}$ ) such that the recursive calls of the algorithm are to games containing at most $a^{\prime}$, respectively $a^{\prime \prime}$ nodes of outdegree at most $j$ and at most $b^{\prime}$, respectively $b^{\prime \prime}$, nodes of out-degree greater than $j$. Nodes labeled by $(a, b)$ with $a, b \in\{0,1,2,3\}$ are leaves. A node labeled by $(a, b)$ has a cost of $\left(\operatorname{dom}_{a+b, b}\left(\sqrt{b}, \sqrt{a \cdot \log _{j} a}\right)+O(1)\right)$ associated with it.

It follows from Lemma 18 that the sum of the costs of the nodes of $T_{G}$ is an upper bound on the run time of new-win $(G, j)$. The worst possible sum of the costs of the nodes of $T_{G}$ we can obtain for some instance $G$ with $s_{j}=s_{j}(G)$ and $n=|V|$ therefore is an upper bound of $T\left(s_{j}, n-s_{j}\right)$. Clearly, the length of every path in $T_{G}$ from the root to a leaf is at most $n$. We say that such a path makes a right turn when it descends from a node to its right child. We next claim that each such path makes at most $O\left(\sqrt{n-s_{j}}+\sqrt{\frac{s_{j}}{\log _{j} s_{j}}}\right)$ right turns. This follows immediately from the observation that the function $f(x)=x-\lceil\sqrt{2 x}\rceil$ can be iterated on $n-s_{j}$ at most $O\left(\sqrt{n-s_{j}}\right)$ times before reaching the value of 3 or less and the function $g(x)=x-\left\lceil\sqrt{x \cdot \log _{j} x}\right\rceil$
can be iterated on $s_{j}$ at most $O\left(\sqrt{\frac{s_{j}}{\log _{j} s_{j}}}\right)$ times before reaching the value of 3 or less. As each leaf of $T_{G}$ is determined by the positions of the right turns on the path leading to it from the root, we get that the number of leaves in $T_{G}$ is at most $n=O\left(\sqrt{n-s_{j}}+\sqrt{\frac{s_{j}}{\log _{j} s_{j}}}\right)$. The total number of nodes in $T_{G}$ is therefore at
 have that

$$
\left.T\left(s_{j}, n-s_{j}\right) \leq n^{O\left(\sqrt{n-s_{j}}+\sqrt{\frac{s_{j}}{\log _{j} s_{j}}}\right.}\right) \cdot\left(\operatorname{dom}_{n, n-s_{j}}(\ell, s)+O(1)\right) .
$$

Together with Lemma 15, we obtain

$$
T\left(s_{j}, n-s_{j}\right)=n^{O\left(\sqrt{n-s_{j}}+\sqrt{\frac{s_{j}}{\log _{j} s_{j}}}\right) .}
$$

This completes the proof of Lemma 22.
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