Crossing Numbers of Meshes

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Abstract. We prove that the crossing number of the cartesian product of 2 cycles, $C_m \times C_n$, $m \leq n$, is of order $\Omega(mn)$, improving the best known lower bound. In particular we show that the crossing number of $C_m \times C_n$ is at least $mn/90$, and for $n = m, m+1$ we reduce the constant 90 to 6. This partially answers a 20-years old question of Harary, Kainen and Schwenk [3] who gave the lower bound m and the upper bound $(m-2)n$ and conjectured that the upper bound is the actual value of the crossing number for $C_m \times C_n$. Moreover, we extend this result to $k \geq 3$ cycles and paths, and obtain such lower and upper bounds on the crossing numbers of the corresponding meshes, which differ by a small constant only.

1 Introduction

The crossing number of a graph G, denoted by $cr(G)$, is the minimum number of crossings of its edges over all drawings of G in the plane, such that no more than two edges intersect in any point and no edge passes through a vertex. Computing $cr(G)$ is NP-hard and there have been only few results concerning the exact value of crossing number for very special and restricted classes of graphs. Besides Kleitman's exact result [6] on the crossing number of $K_{m,n}$, for $m \leq 6$, most effort has been devoted to crossing numbers of some cartesian product graphs [4, 7, 8, 22]. For a detailed exposition of the crossing number problem see our survey [19].

For $G_1 = (V_1, V_2)$ and $G_2 = (V_2, E_2)$, let $G_1 \times G_2$ denote the cartesian product of G_1 and G_2. Thus $G_1 \times G_2$ is a graph with the vertex set $V_1 \times V_2$ in which $(i,j)(r,s)$ are adjacent iff either $i = r$ and $js \in E_2$ or $j = s$ and $ir \in E_1$. Let P_n and C_n denote the n-vertex path and the n-vertex cycle , respectively. For $2 \leq n_1 \leq n_2 \leq ... \leq n_k$, let $M_k = \prod_{i=1}^{k} P_{n_i}$ and $TM_k = \prod_{i=1}^{k} C_{n_i}$. We will call M_k and TM_k the k-dimensional mesh and the k-dimensional toroidal mesh, respectively.

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Clearly, M_2 is planar and has crossing number 0, but estimating the crossing number of TM_2 has been an open problem. Harary et al. [3] provided a simple drawing of $C_m \times C_n$, $m \le n$ with $(m-2)n$ crossings. They also derived a weak lower bound m on $cr(C_m \times C_n)$ and conjectured that $cr(C_m \times C_n) = (m-2)n$, for $3 \le m \le n$. Beineke and Ringeisen [1, 16] proved the conjecture for $m \le 4$. Richter and Thomassen [15] proved the conjecture for $m = n = 5$. Finally, very recently Klešč [9] and Richter and Stobert [14] announced their proof for $m = 5$ and arbitrary $n \ge 5$. It is instructive to mention that all existing standard methods for estimating lower bounds on the crossing number fail to give good lower bounds for $cr(C_m \times C_n)$. In particular two very powerful methods developed by VLSI community [11] – the bisection method and the embedding method – give a weak lower bound for $cr(C_m \times C_n)$. We suspect that the reason is that $C_m \times C_n$ has genus 1 and very much resembles the planar 2-dimensional mesh.

In this paper we take a major step to prove the conjecture and show that for $6 \le m \le n$, $cr(C_m \times C_n) \ge mn/90$. For $n = m, m+1$ we improve the constant 90 to 6. It is worth mentioning that the method used here to prove our main result employs Dilworth' chain decomposition theorem which was shown to be very effective when dealing with problems in combinatorial geometry [10, 13].

Moreover for $k \ge 3$, we derive upper and lower bounds within a constant multiplicative factor for the crossing number of M_k and TM_k. We indicate that, since bisection width of M_k and TM_k are known, see for instance [12], one can use the relationship between the crossing number and the bisection and derive lower bounds for $cr(M_k)$ and $cr(TM_k)$, $k \ge 3$ which are of the same order magnitude as the lower bounds we have derived here. Nevertheless, such an approach does not provide constants, whereas, our method identifies relatively large constants associated with our lower bounds. Moreover, for $k \ge 3$ we have been able to provide (new) drawings of M_k and TM_k, with number of crossings which are within relatively small constant factors from the lower bounds, under reasonable conditions.

2 Crossing Number of $C_m \times C_n$

One finds in $C_m \times C_n$ in a natural way n vertex disjoint row cycles and m vertex disjoint column cycles. We will call them r-cycles and c-cycles. Deleting all edges of any r-cycle (c-cycle) yields a graph which is a subdivision of $C_m \times C_{n-1}$ (of $C_{m-1} \times C_n$) and therefore has a crossing number less than or equal to that of $C_m \times C_n$. It immediately follows, that $cr(C_m \times C_n)$ is monotone nondecreasing in both parameters. We will use this fact implicitly throughout our proof many times.

Theorem 2.1 *For* $6 \le m \le n$,

$$cr(C_m \times C_n) \ge \frac{mn}{90}.$$

Proof. If $6 \leq m \leq n \leq 180$ then by monotonicity and a result of Beineke and Ringeisen [1]: $cr(C_4 \times C_n) = 2n$ we have

$$cr(C_m \times C_n) \geq cr(C_4 \times C_n) = 2n \geq \frac{mn}{90}.$$

Assume $n \geq 181$. For a closed curve C in the plane R^2, define its body $B(C)$ as the closure of the union of the bounded components of $R^2 \setminus C$. See Fig. 1. Define the exterior of C as the complement of $B(C)$ (the white part of Fig. 1).

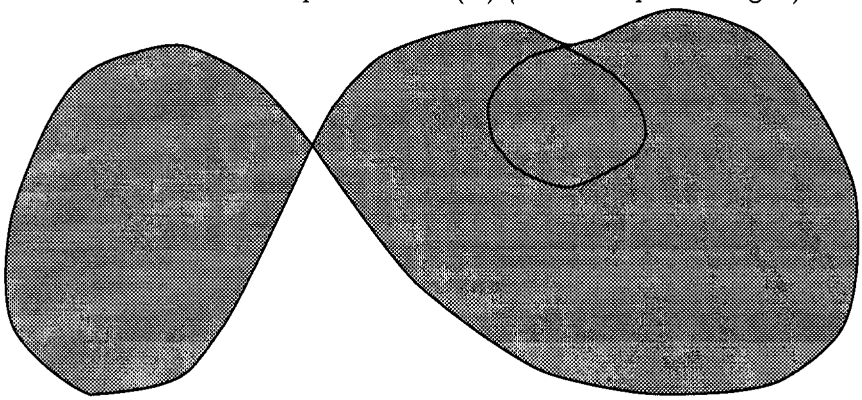

Fig. 1 : The body of a closed curve

Let us be given a drawing D of $C_m \times C_n$ in the plane. We may assume without loss of generality that (1) crossing edges really cross, i.e. no touching situation occurs, (2) the number of crossings in D is finite. Define a partial order on the set of its r-cycles by $C < Z$ if C lies in one of the bounded connected components of $R^2 \setminus Z$. Dilworth' Theorem [2] p. 62 applies to this poset. If k is the size of the largest antichain, then the poset may be decomposed into chains of length $a_1, a_2, ..., a_k$ such that $a_1 + a_2 + ... + a_k = n$. In such a chain each member contains the next member of the chain in one of the bounded connected components of the plane that it defines. Each c-cycle has exactly one vertex on every member cycle in the i-th chain. Hence, by Jordan's Curve Theorem, every c-cycle has at least $a_i - 2$ crossings with the r-cycles of the i-th chain. Therefore, $cr(D) \geq m(a_1 + a_2 + ... + a_k - 2k) = m(n - 2k) \geq mn/50$, if $k < 49n/100$.

Assume that $k \geq 49n/100$. We have an antichain of size at least $49n/100$, i.e. at least $49n/100$ r-cycles such that any two cross or they lie in the exterior of each other. Ignoring the edges of the other r-cycles, we consider our drawing as a drawing of a $C_m \times C_{\lceil 49n/100 \rceil}$. For any r-cycle C let $a(C)$ denote the number of c-cycles whose bodies lie in the body of C. Observe that for two r-cycles C, Z taken from an antichain, $a(C) > 0$ implies 2 crossing of C and Z, since every c-cycle was supposed to have a common vertex with every r-cycle and if two r-cycles cross then they must cross at least twice. Either we have at least $(1/24)(49n/100)$ r-cycles C with $a(C) > 0$, and the total number of crossings is $2(49n/2400) \cdot (49n/100 - 1) > mn/90$ and we are at home, or we have at least $(23/24)(49n/100)$ r-cycles C with $a(C) = 0$. Reduce our drawing further to the drawing of the corresponding $C_m \times C_{\lceil 1127n/2400 \rceil}$.

Now every c-cycle either crosses an r-cycle or is in the exterior of it. Either we have at least $(1/3)(1127n/2400)$ r-cycles, such that each of them is crossed by the c-cycles at least $m/14$ times, or we have at least $(2/3)(1127n/2400)$ r-cycles, such that the c-cycles cross them at most $m/14$ times. In the first case $cr(D) \geq mn/90$, and the Theorem holds. In the second case reduce our interest to a drawing D of $G = C_m \times C_{\lceil 1127n/3600 \rceil}$ so that for every r-cycle C, $a(C) = 0$ and at most $m/14$ c-cycles cross C. Let $cr(D)$ denote the number of crossings in D. The proof is completed employing the following claim.

Claim: D can be extended to a drawing D' of a graph G' with $cr(G') \geq mn/15$ and $cr(D') \leq 6cr(D)$.

Proof of the Claim. We construct D' by adding edges to each r-cycle C. In particular we add at least $3m/14$ new edges to each r-cycle C.

Let S be the set of all c-cycles that cross C. By deleting all edges of C that are crossed by c-cycles from S, we divide C to at most $\lfloor m/14 \rfloor$ vertex-disjoint paths. On these paths, there are at most $\lfloor m/14 \rfloor$ vertices, in which c-cycles from S have a vertex in common with C, since each r-cycle and each c-cycle have exactly one vertex in common. These vertices divide the $\lfloor m/14 \rfloor$ vertex disjoint paths into at most $2\lfloor m/14 \rfloor$ edge disjoint paths of lengths $d_1, d_2, d_3, \ldots, d_{2\lfloor m/14 \rfloor}$, where $\sum_{i=1}^{2\lfloor m/14 \rfloor} d_i \geq m - \lfloor m/14 \rfloor$. To each path of length d_i, we can add at least $\lfloor d_i/3 \rfloor$ new edges so that

(i) in each path above we join vertices of distance 3 in a greedy way, such that the new "long edges" do not overlap
(ii) the new edges are drawn very close to C and inside $B(C)$, as far as it is possible
(iii) the new edges do not cross each other unless the corresponding part of the drawing of C is self-intersecting
(iv) the new edges do not cross c-cycles.

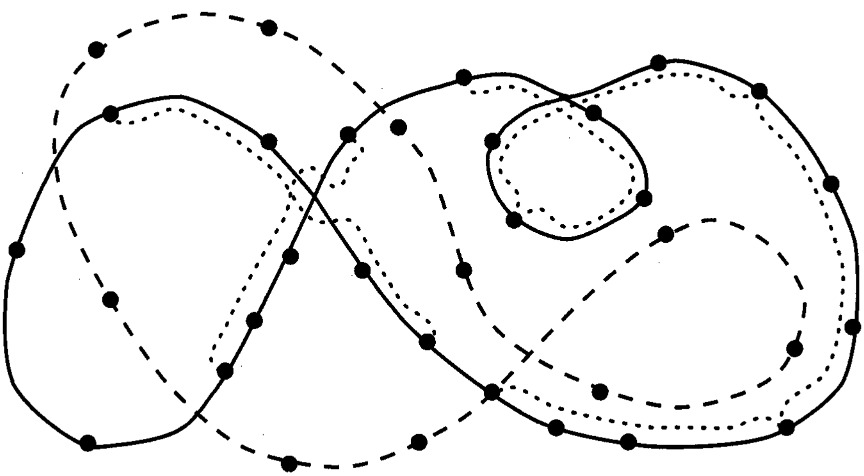

Fig. 2

Fig. 2 shows details of a drawing of $C_{23} \times C_{11}$ with one r-cycle (solid line) and one c-cycle (dashed line). The new edges are drawn by dotted lines. Hence the total number of added edges is

$$\sum_{i=1}^{2\lfloor \frac{m}{14} \rfloor} \left\lfloor \frac{d_i}{3} \right\rfloor \geq \sum_{i=1}^{2\lfloor \frac{m}{14} \rfloor} \frac{d_i - 2}{3} \geq \frac{1}{3}\left(m - \left\lfloor \frac{m}{14} \right\rfloor - 4\left\lfloor \frac{m}{14} \right\rfloor\right) \geq \frac{3m}{14}.$$

The drawing D' obtained in this way corresponds to a graph G' which has $m\lceil 1127n/3600 \rceil$ vertices, at least $2m\lceil 1127n/3600 \rceil + 161mn/2400$ edges, and of girth 4. Due to the construction $cr(D') \leq 6cr(D)$, since a crossing of two distinct r-cycles in D may result in at most 4 crossings in D', a self-crossing of an r-cycle in D may result in 6 crossings in D' (see Fig. 2). Crossings of r-cycles and c-cycles do not multiply. Finally, Kainen's lower bound [5] applies to G':

$$cr(G') \geq |E(G')| - \frac{girth(G')}{girth(G') - 2}(|V(G')| - 2).$$

This formula immediately finishes the proof. □

In special cases, we can improve the lower bound to the following:

Theorem 2.2 *For $3 \leq m \leq n$, where $n = m$ or $m + 1$*

$$cr(C_m \times C_n) \geq \frac{(m-2)n}{6}.$$

Proof. We inductively show that if the claim holds for $cr(C_m \times C_m)$, then it also holds for $cr(C_m \times C_{m+1})$, and use this to show that the claim holds for $cr(C_{m+1} \times C_{m+1})$. We call the crossing of two different r-cycles as rr-crossing. Similarly define the cc-crossing and the rc-crossing. The claim is true for $C_m \times C_m$ when $m \leq 10$, as $cr(C_m \times C_m) = (m-2)m$, for $m \leq 5$, [1, 16, 15] and $cr(C_m \times C_m) \geq cr(C_5 \times C_5) = 15 \geq (m-2)m/6$, for $m = 6, 7, ..., 10$. Let the claim hold for $C_m \times C_m$, $m \geq 10$. We first show it holds for $C_m \times C_{m+1}$. Consider any drawing of $C_m \times C_{m+1}$. Suppose that there exists an r-cycle in $C_m \times C_{m+1}$ containing at least $m/6$ crossings. Deleting the edges of this r-cycle we get a subdivision of $C_m \times C_m$. Hence

$$cr(C_m \times C_{m+1}) \geq cr(C_m \times C_m) + \frac{m}{6} \geq \frac{(m-2)m}{6} + \frac{m}{6} \geq \frac{(m-2)(m+1)}{6}.$$

Suppose that each r-cycle in $C_m \times C_{m+1}$ contains at most $\lfloor m/6 \rfloor$ crossings. We may assume that there exist 3 distinct r-cycles, so that no two of them have a crossing. Otherwise each triplet of r-cycles would determine at least 2 rr-crossings and a simple counting argument shows that

$$cr(C_m \times C_{m+1}) \geq \frac{2\binom{m+1}{3}}{m-1} > \frac{(m-2)(m+1)}{6}.$$

Consider the 3 distinct r-cycles. Since for each of them, there exist at least $m - \lfloor m/6 \rfloor$ c-cycles, which do not cross it, there are at least $m - 3\lfloor m/6 \rfloor$ c-cycles, none of them crosses any of the 3 distinct r-cycles. Now every triplet of these $m - 3\lfloor m/6 \rfloor$ c-cycles together with the 3 distinct r-cycles determine a subdivision of $C_3 \times C_3$. If the subdivision contains a selfcrossing of a cycle we can redraw it without the selfcrossing and without producing a new crossing. Hence we may assume that the subdivision is without selfcrossings. It has 3 cc-crossings, since $cr(C_3 \times C_3) = 3$, but it must contain an even number of cc-crossings, since c-cycles are vertex disjoint and if they cross, they must cross an even number of times. Thus, this subdivision contains at least 4 cc-crossing. Further, a counting argument shows that,

$$cr(C_m \times C_{m+1}) \geq \frac{4\binom{m-3\lfloor m/6 \rfloor}{3}}{m - 3\lfloor m/6 \rfloor - 2} \geq \frac{(m-2)(m+1)}{6}.$$

We may conclude that if the claim holds for $C_m \times C_m$ it also holds for $C_m \times C_{m+1}$. Now we use this fact to prove a lower bound for $C_{m+1} \times C_{m+1}$. Suppose that there exists an r-cycle in $C_{m+1} \times C_{m+1}$ containing at least $(m+1)/6$ crossings. Deleting the edges of the r-cycle we get a subdivision of $C_{m+1} \times C_m$. Hence

$$\begin{aligned} cr(C_{m+1} \times C_{m+1}) &\geq cr(C_{m+1} \times C_m) + \frac{m+1}{6} \geq \frac{(m-2)(m+1)}{6} + \frac{m+1}{6} \\ &\geq \frac{(m-1)(m+1)}{6}. \end{aligned}$$

Suppose that each r-cycle in $C_{m+1} \times C_{m+1}$ contains at most $(m+1)/6$ crossings. The rest of the proof is the same as the proof of the lower bound for $C_m \times C_{m+1}$. □

3 Crossing Number of M_k and TM_k

It is straightforward to show, using Theorem 2.1 that, for $k \geq 3$, $cr(TM_k) = \Omega(\Pi_{i=1}^k n_i)$ which is a weak lower bound in some cases. In this section we prove near-optimal lower and upper bounds on the crossing number of TM_k and M_k, for $k \geq 3$. In particular, when $n_i = n, i = 1, 2, .., k$, the upper and lower bounds differ by a multiplicative factor only. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that $|V_1| \leq |V_2|$.

An embedding ω of G_1 in G_2 is a pair of injections (ϕ, ψ)

$$\phi : V_1 \rightarrow V_2, \qquad \psi : E_1 \rightarrow \{\text{all paths in } G_2\},$$

such that if $uv \in E_1$ then $\psi(uv)$ is a path between $\phi(u)$ and $\phi(v)$. Define the congestion of ω

$$\mu_\omega = \max_{e \in E_2}\{|\{f \in E_1 \ : \ e \in \psi(f)\}|\}.$$

Leighton [11] invented a lower bound technique for crossing numbers, based on an embedding of the complete graph in the given graph. Several authors [18, 20, 22] realized that the method can be generalized for arbitrary graphs in the following form:

Lemma 3.1 *Let ω be an embedding of $G_1 = (V_1, E_1)$ into $G_2 = (V_2, E_2)$, $|V_1| \leq |V_2|$. Let D_2 be a drawing of G_2 with $cr(D_2)$ crossings, then there is a drawing D_1 of G_1 with $cr(D_1)$ crossings so that*

$$cr(D_2) \geq \frac{cr(D_1)}{\mu_\omega^2} - \frac{|V_2|\Delta_2^2}{2},$$

where Δ_2 is the maximum degree of G_2. □

For $2 \leq n_1 \leq n_2 ..... \leq n_k$, let $N_i = n_1 n_2 ... n_i$, for $i = 1, 2, ..., k$.

Theorem 3.1 *For $k \geq 3$*

$$\frac{N_{k-1}^2}{5} - \frac{(5k^2+8)N_k}{2} \leq cr(M_k) \leq 4N_{k-2}N_k,$$

$$\frac{4N_{k-1}^2}{5} - 2(k^2+2)N_k \leq cr(TM_k) \leq 16N_{k-2}N_k + 8k^2 N_k.$$

Proof. We first prove the lower bound for $cr(TM_k)$ and construct an upper bound for $cr(M_k)$. Set $G_2 = TM_k$ and $G_1 = 2K_{N_k}$, where $2K_{N_k}$ denote the complete multigraph on N_k vertices, obtained from K_{N_k} by replacing every edge by two new edges. Shahrokhi and Székely [17] constructed an embedding ω of G_1 into G_2 with

$$\mu_\omega \leq \frac{N_k n_k}{4}.$$

Substituting this into Lemma 3.1, recalling from [21] the following formula:

$$cr(K_{N_k}) \geq \frac{N_k^4 - 7N_k^3}{80},$$

and noting that $cr(2K_{N_k}) = 4cr(K_{N_k})$ we get the claimed lower bound for TM_k.

Now we prove an upper bound for $cr(M_k)$. We construct a recursive drawing L_k for M_k in which all vertices are placed along a straight line in the plane. For $M_1 = P_{n_1}$ we place successively the vertices of P_{n_1} on a line and obtain a drawing with no crossings. Assume that we have constructed a drawing L_{k-1} of M_{k-1} with $cr(L_{k-1})$ crossings. The drawing L_k of $M_k = M_{k-1} \times P_{n_k}$ is constructed in the following way. Place n_k copies of the drawings L_{k-1} successively on a line such that 2 neighboring copies are symmetric according to a perpendicular line between the 2 copies. Join the corresponding vertices of the first and the second copy by edges drawn as half-circles above the line. Similarly, join the corresponding vertices of the second and the third copy by edges drawn as half-circles below the line and continue in this fashion until a drawing L_k of M_k is obtained. Let us call the inserted edges the edges of the dimension k and denote the number of crossings in L_k by $cr(L_k)$. Clearly

$$cr(L_k) \leq n_k cr(L_{k-1}) + l_k,$$

where l_k denotes the number of crossings of the edges of the k-th dimension with edges of n_k copies of the drawing L_{k-1}, i.e. the number of crossings of edges of the k-th dimension with edges of smaller dimensions. A counting analysis shows that there are at most $N_k N_{i-1}$ crossings of the edges of the k-th dimension with edges of the i-th dimension, where $N_0 = 1$. Hence

$$l_k \leq N_k \sum_{i=1}^{k-1} N_{i-1}$$

and

$$cr(L_k) \leq n_k cr(L_{k-1}) + N_k \sum_{i=0}^{k-2} N_i.$$

The solution is

$$\begin{aligned} cr(L_k) &\leq N_k \sum_{i=0}^{k-2} (k-i-1) N_i \\ &\leq N_k N_{k-2} \left(1 + \frac{2}{n_{k-2}} + \frac{3}{n_{k-2} n_{k-3}} + \ldots + \frac{k-1}{n_{k-2} \ldots n_1}\right) \leq 4 N_k N_{k-2}. \end{aligned}$$

Finally, we use Lemma 3.1 with $G_1 = TM_k$ and $G_2 = M_k$ and note that there is an embedding ω of G_1 into G_2 with $\mu_\omega = 2$. To get the upper bound for $cr(TM_k)$, we take D_2 to be L_k, then D_1 is a desirable drawing of TM_k. To get the lower bound for $cr(M_k)$, we substitute the term $cr(D_1)$ in the Lemma, by our lower bound for $cr(TM_k)$. □

Corollary 3.1 *If $n_k = O(n_{k-1})$ then the bounds in Theorem 3.1 are optimal within a constant multiplicative factor and in addition, if $n_1 = n_2 = \ldots = n_k = n$ then*

$$\frac{n^{2k-2}}{5} - \frac{(5k^2+8)n^k}{2} \leq cr(M_k) \leq 4n^{2k-2},$$

$$\frac{4n^{2k-2}}{5} - 2(k^2+2)n^k \leq cr(TM_k) \leq 16n^{2k-2} + 8k^2 n^k.$$

□

Corollary 3.2 *Consider a general k-dimensional mesh $GM_k = \prod_{i=1}^{k} A_{n_i}$, where A_{n_i} equals either P_{n_i} or C_{n_i}. Then*

$$cr(M_k) \leq cr(GM_k) \leq cr(TM_k).$$

□

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