Int J Game Theory (1998) 27:443–450

International Journal of Game Theory

Note

Computing the nucleolus of min-cost spanning tree games is *NP*-hard

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Received October 1997/Final version May 1998

Abstract. We prove that computing the nucleolus of minimum cost spanning tree games is in general *NP*-hard. The proof uses a reduction from minimum cover problems.

Key words: Nucleolus, *N*-person game, spanning tree, *NP*-hard

1. Introduction

Minimum cost spanning tree problems have been widely studied in the literature. After their introduction by Bird [1976], various results about the core and nucleolus were established (see, *e.g.* Granot and Huberman [1981], [1984]). Megiddo [1987] describes an $O(n^3)$ algorithm for computing the nucleolus in the special case where the underlying graph is a tree. Galil [1980] subsequently reduced the number of operations to $O(n \log n)$ and Granot and Granot [1992] consider an extended model in which they compute the nucleolus in strongly polynomial time.

An alternative approach for computing the nucleolus efficiently in the tree case is discussed by Granot *et al.* [1996]. The general case, however, has remained unsolved so far. (Kuipers *et al.* [1995] present an algorithm whose complexity is n cubed times the number of "essential" coalitions). The purpose of the present note is to show that the problem of computing the nucleolus for general minimum cost spanning tree games is *NP*-hard. So it is unlikely that it can be computed efficiently. More precisely, we show that computing the nucleolus for a special class of graphs introduced in Faigle *et al.* [1997] is already *NP*-hard.

A *minimum cost spanning tree* game (MCST-game, for short) is defined by a set N of players, a *supply* node $s \notin N$ and a complete graph on $V = N \cup \{s\}$

with a non-negative *distance* or *length* function $l \geq 0$ defined on the edge set of the complete graph. The *cost* $c(S)$ of a coalition $S \subseteq N$ is, by definition, the length of a minimum spanning tree in the subgraph induced by $S \cup \{s\}$.

The concept of *nucleolus* has been introduced by Schmeidler [1969]. For our purposes, the following well-known algorithmic definition is the most convenient one:

Consider a sequence of linear programs defined inductively as follows. Let $\mathscr{S}_0 := 2^N \backslash \{\emptyset, N\}$ and $\mathscr{T}_0 := \emptyset$. Solve the linear program

$$\begin{aligned} (P_0) \quad & \max \quad \varepsilon \\ & s.t. \quad x(S) \leq c(S) - \varepsilon \quad \forall S \in \mathscr{S}_0 \\ & \qquad\;\; x(N) = c(N) \end{aligned}$$

(Throughout the paper, we use the shorthand notation $x(S) = \sum_{i \in S} x_i$.)

Let ε_1 be the optimal value of (P_0) and let $\mathscr{T}_1 \subseteq \mathscr{S}_0$ be the set of coalitions S that are forced to be tight at an optimum solution, *i.e.*, satisfy

$$x(S) = c(S) - \varepsilon_1$$

for each optimal solution (x, ε_1) of (P_0). Let $\mathscr{S}_1 := \mathscr{S}_0 \backslash \mathscr{T}_1$ and consider

$$\begin{aligned} (P_1) \quad & \max \quad \varepsilon \\ & s.t. \quad x(S) = c(S) - \varepsilon_1 \quad \forall S \in \mathscr{T}_1 \\ & \qquad\;\; x(S) \leq c(S) - \varepsilon \quad \forall S \in \mathscr{S}_1 \\ & \qquad\;\; x(N) = c(N) \end{aligned}$$

Now let ε_2 be the optimum value of (P_2) and let $\mathscr{T}_2 \subseteq \mathscr{S}_1$ be the coalitions that are "forced to be tight" at ε_2, and so on. One can show (cf., *e.g.*, Maschler *et al.* [1979]) that, after at most $n - 1$ iterations, one arrives this way at a problem

$$\begin{aligned} (P_k) \quad & \max \quad \varepsilon \\ & s.t. \quad x(S) = c(S) - \varepsilon_1 \quad \forall S \in \mathscr{T}_1 \\ & \qquad\qquad \vdots \\ & \qquad\;\; x(S) = c(S) - \varepsilon_k \quad \forall S \in \mathscr{T}_k \\ & \qquad\;\; x(S) \leq c(S) - \varepsilon \quad \forall S \in \mathscr{S}_k \\ & \qquad\;\; x(N) = c(N) \end{aligned}$$

which has a unique optimal solution (x, ε_{k+1}). This solution x is called the *nucleolus* of the MCST-game.

2. Exact cover graphs

Let $q \in \mathbb{N}$, and let U be a set of $k \geq q$ elements and W be a set of $3q$ elements.

Consider a bipartite graph with node set $U \cup W$ (partitioned into U and W) such that each node $u \in U$ is adjacent to exactly three nodes in W and such that each node $w \in W$ has at least 2 neighbors in U. We say that the node $u \in U$ *covers* its three neighbors in W.

A set $C \subseteq U$ is called a *cover* if each $w \in W$ is incident with some $u \in C$. A *minimum cover* is a cover that minimizes $|C|$. Finding a minimum cover is a well-known *NP*-hard problem even restricted to the class of bipartite graphs as above. It includes the *NP*-complete problem known as EXACT 3-COVER ("X3C") (cf. Garey and Johnson [1979]).

We construct an MCST-game from a minimum cover problem as follows (cf. Faigle *et al.* [1997]). Define the graph $G = (V, E)$ such that the node set of G consists of $U \cup W$ and three additional nodes: The *Steiner node St*, the *guardian g*, and the *supply s*. The edge set E of G comprises the following:

- all edges e from the bipartite graph on $U \cup V$, each of them having length $l(e) = q + 1$;
- for each $u \in U$, an edge (u, St) between u and St of length $l(u, St) = q$ and an edge (u, g) between u and g of length $l(u, g) = q + 1$;
- an edge (St, g) between St and g of length $l(St, g) = q + 1$;
- an edge (g, s) between g and s of length $l(g, s) = 2q - 1$.

We extend G to the complete graph $\bar{G}$ on V with distances induced from G, *i.e.*, if $e = (i, j)$ is an edge in $\bar{G}$, then $l(i, j)$ is the length of a shortest path from i to j in G.

A minimum spanning tree ("MST") in $\bar{G}$ is obtained by connecting each $w \in W$ to some $u \in U$ by which it is covered. Such a $u \in U$ exists because each node $w \in W$ has a neighbor in U (indeed, it has at least 2 neighbors in U). Then one connects each $u \in U$ to St, and finally connects St to g and g to s.

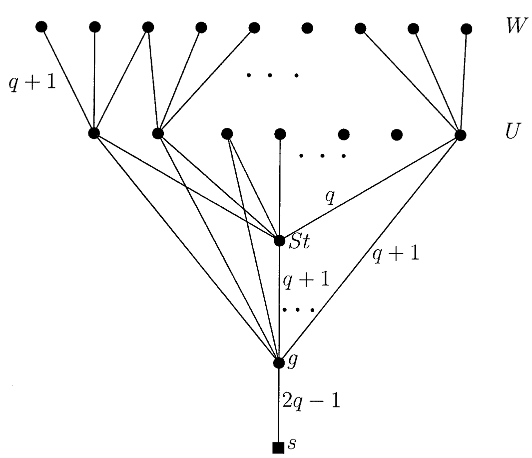

Fig. 2.1.

The resulting MST has a total length of

$$c(N) = 3q(q+1) + kq + 3q.$$

Furthermore note that each $w \in W$ is covered by at least *two* vertices in U (because each node of the bipartite graph on $U \cup W$ has at least 2 neighbors). Hence it is straightforward to see that the following property holds for $\overline{G}$:

(**L**) For each $v \in U \cup W$, there exists a MST T in the graph $\overline{G}$ such that v is a leaf of T.

3. The nucleolus of minimum cover graphs

Consider a graph $G = (V, E)$ and its completion $\overline{G}$ as described in the previous section. The first step in computing the nucleolus of the corresponding MCST-game is to solve

$$\begin{aligned} (P_0) \quad & \max \quad \varepsilon \\ & s.t. \quad x(S) \le c(S) - \varepsilon \quad \forall S \subset N \\ & \qquad\quad x(N) = c(N), \end{aligned}$$

where $N = V \backslash \{s\}$ and $c(S)$ is the length of a MST in $\overline{G}$ connecting S to the supply s.

A basic observation is now the following. If a node $v \in N$ occurs as a leaf in some MST T for $\overline{G}$ and if e is the unique edge in T incident with v, then $T \backslash e$ is a MST for $V \backslash \{v\}$. Thus $c(N \backslash \{v\}) = c(N) - l(e)$, where $l(e)$ is the length of e.

Hence, by property (**L**) of the previous section, the feasibility constraints of (P_0) imply the following inequalities

$$x(w) \ge q + 1 + \varepsilon \quad (w \in W)$$

$$x(u) \ge q + \varepsilon \quad (u \in U).$$

Furthermore, the coalition $S = N \backslash \{g\}$ can be connected to the supply node s at a total cost of $c(N)$. Hence, the feasibility constraints of (P_0) also imply

$$x(g) \ge \varepsilon.$$

This motivates the following definition.

For $\varepsilon > 0$, let $x^\varepsilon \in \mathbb{R}^N$ be the vector defined by

$$x^\varepsilon(u) = q + \varepsilon \quad \forall u \in U$$

$$x^\varepsilon(w) = q + 1 + \varepsilon \quad \forall w \in W$$

$$x^\varepsilon(g) = \varepsilon$$

$$x^\varepsilon(St) = c(N) - 3q(q + 1 + \varepsilon) - k(q + \varepsilon) - \varepsilon.$$

Theorem 1 below and its proof will reveal the computation of the nucleolus to be at least as hard as solving problem (P_0) and the latter to be equivalent with determining the maximum value of $\varepsilon > 0$ such that x^ε is still a feasible solution for (P_0).

Note that for $\varepsilon = 0$, the node St pays exactly for its own connection to the source (which automatically connects g to the source) a total amount of $3q$. Increasing ε amounts to crediting St and at the same time "overcharging" all other nodes (relative to x^0). Intuitively, this is justified because the addition of St to a large coalition S (*i.e.*, $|S \cap U| > q$) results is a decrease of the total connection cost. (The latter property motivates the name "Steiner node"). Of course, when we increase ε, some coalition S will become tight at some point, *i.e.* S will satisfy

$$x^\varepsilon = c(S) - \varepsilon,$$

for the first time. The detailed analysis below shows that such a coalition S necessarily consists of all nodes in W together with a minimum cover $C \subseteq U$, and the node g.

Our main result can now be stated as follows

Theorem 1. *The nucleolus x^* of the MCST-game allocates $x^*(w) = q + 1 + \varepsilon^*$ to each node $w \in W$, where*

$$\varepsilon^* = \frac{|C| + 2q - 1}{|C| + 3q + 2}$$

and $C \subseteq U$ is a minimum cover.

Proof: We will prove the following three statements:

(i) The optimum value of (P_0) is at most ε^*
(ii) The optimum value of (P_0) is at least ε^*
(iii) Each optimal solution (x, ε^*) of (P_0) allocates precisely $x(w) = q + 1 + \varepsilon^*$ to each $w \in W$.

The claim of the Theorem then follows immediately from (iii).

Proof of (i): Suppose (x, ε) is a feasible solution of (P_0). As we have seen, this implies

$$x(w) \geq q + 1 + \varepsilon \quad (w \in W)$$

$$x(u) \geq q + \varepsilon \qquad (u \in U)$$

$$\text{and} \quad x(g) \geq \varepsilon.$$

Now let $C \subseteq U$ be a minimum cover. Consider the coalition $S = \{g\} \cup C \cup W$. Then

$$x(S) \geq \varepsilon + |C|(q + \varepsilon) + 3q(q + 1 + \varepsilon)$$

whereas, obviously,

$$c(S) \leq 3q(q+1) + |C|(q+1) + 2q - 1.$$

Since $x(S) \leq c(S) - \varepsilon$, we get

$$|C|(\varepsilon - 1) + 3q\varepsilon + 2\varepsilon - 2q + 1 \leq 0$$

or

$$\varepsilon \leq \frac{|C| + 2q - 1}{|C| + 3q + 2} = \varepsilon^*.$$

Proof of (ii): We show that $x := x^{\varepsilon^*}$ and ε^* are feasible for (P_0). Let $\emptyset \subset S \subset N$ maximize $\delta(S) := x(S) - c(S)$. We have to show that $\delta(S) \leq -\varepsilon^*$.

Case (1): $St \in S$.

If $|N \setminus S| = 1$, *i.e.*, $S = N \setminus v$ for some $v \in \{g\} \cup U \cup W$, then $\delta(S) \leq -\varepsilon^*$ by definition of x.

Hence we assume that $|N \setminus S| \geq 2$. If $u \in U \setminus S$, then $S' := S \cup \{u\} \subset N$. Adding u to S, however, increases $x(S)$ by $q + \varepsilon^*$ and increases $c(S)$ by only q (since u can be connected to $St \in S$). This contradicts the maximality of $\delta(S)$. Hence $U \subseteq S$. But then, similarly, adding some $w \in W \setminus S$ to S would increase $\delta(S)$ by ε^*, contradicting the maximality of $\delta(S)$. So $W \subseteq S$, which contradicts our assumption that $|N \setminus S| \geq 2$.

Case (2): $St \notin S$.

In this case, $g \in S$. (Adding g would not increase $c(S)$, but would increase $x(S)$ by ε^*.) Hence $S = \{g\} \cup U' \cup W'$ for some $U' \subseteq U$, $W' \subseteq W$.

We claim that U' covers just W' and $W' = W$.

Suppose first that some $w' \in W'$ is not covered by U'. Then in an MST for S, all edges incident with w' have length at least $2q + 2$. This shows that if we include any $u \in U \setminus U'$ which covers w', we may connect w' via u to g at a cost of $2(q+1)$, *i.e.*, $c(S)$ does not increase, whereas $x(S)$ increases by $q + \varepsilon^*$ if we add u to S. This contradicts the maximality of $\delta(S)$. Hence U' indeed covers W'.

If there were some $w \in W \setminus W'$ which is also covered by U', then addition of w to S would increase $x(S)$ by $q + 1 + \varepsilon^*$ while $c(S)$ increases only by $q + 1$. This would contradict the maximality of $\delta(S)$. Hence U' covers just W'.

Next assume that $W \setminus W' \neq \emptyset$, *i.e.*, some $w \in W$ is not covered by U'. Choose any $u \in U \setminus U'$ covering w. Adding w and u to S increases $c(S)$ by no more than $2(q+1)$ whereas x increases by $2q + 1 + 2\varepsilon^*$. Since $\varepsilon^* > 1/2$ (assuming $q > 1$), this again leads to a contradiction. Hence $W' = W$ holds indeed and $U' \subseteq U$ is a cover.

In this case now, obviously,

$$x(S) = 3q(q + 1 + \varepsilon^*) + |U'|(q + \varepsilon^*) + \varepsilon^* \quad \text{and}$$

$$c(S) = 3q(q+1) + |U'|(q+1) + 2q - 1$$

Hence

$$x(S) - c(S) = (3q + 1)\varepsilon^* + |U'|(\varepsilon^* - 1) - 2q + 1.$$

Since $\varepsilon^* < 1$, this is maximized if U' is a minimum cover, in which case $x(S) \le c(S) - \varepsilon^*$ follows from the definition of ε^*.

Proof of (iii): Let (x, ε^*) be an optimal solution of (P). As in the proof of (i), we conclude

$$x(w) \ge q + 1 + \varepsilon^* \quad (w \in W)$$

$$x(u) \ge q + \varepsilon^* \quad (u \in U)$$

$$x(g) \ge \varepsilon^*.$$

Furthermore, if $C \subseteq U$ is a minimum cover and $S = \{g\} \cup C \cup W$, then $x(S) \le c(S) - \varepsilon^*$ implies that the above inequalities must be tight for all nodes in S. Hence in particular, $x(w) = q + 1 + \varepsilon^*$ for all $w \in W$. $\diamondsuit$

Corollary 3.1. *Computing the nucleolus of MCST-games is NP-hard.*

Proof: Let $\mathscr{A}$ be any algorithm that computes the nucleolus x^* of a MCST-game. Then $\mathscr{A}$ allows us to compute $\varepsilon^* = x^*(w) - q - 1$ for some $w \in W$. By the Theorem, ε^* uniquely determines the size of a minimal cover C.

Given x^*, we can thus compute the size of a minimum cover C in polynomial time. Hence the computation of the nucleolus is at least as hard as the computation of the size of a minimum cover. $\diamondsuit$

We conjecture that also the recognition problem (*i.e.*, the decision problem: "given $x^* \in \mathbb{R}^N$, is x^* is the nucleolus?") is *NP*-hard. It is not clear to us whether the recognition problem is in *NP* or *co – NP*, but we guess it is in neither. Other interesting open problems are the complexity of approximating the nucleolus or computing alternative solution concepts like the nucleon (cf. Faigle *et al.* [1996]) or the per capita nucleolus.

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