On a spanning tree with specified leaves

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Abstract

Let $k \geq 2$ be an integer. We show that if G is a $(k+1)$-connected graph and each pair of nonadjacent vertices in G has degree sum at least $|G|+1$, then for each subset S of $V(G)$ with $|S| = k$, G has a spanning tree such that S is the set of endvertices. This result generalizes Ore's theorem which guarantees the existence of a Hamilton path connecting any two vertices.

Keywords: spanning tree; leaf connected; Hamilton path; Hamilton-connected

1 Introduction

Many results concerning conditions for the existence of a Hamilton path are known. We can regard a Hamilton path as a spanning tree with precisely two endvertices. Thus it is natural to look for conditions which ensure the existence of a spanning tree with the bounded number of endvertices or with a specified set of endvertices. This paper is mainly concerned with sufficient conditions for a graph to have a spannning tree with a specified set of endvertices.

We consider finite undirected graphs without loops nor multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The order of G is denoted by $|G|$. For a vertex $x \in V(G)$, we denote the degree of x in G by $d_G(x)$ and the set of vertices adjacent to x in G by $N_G(x)$; thus $d_G(x) = |N_G(x)|$. For a subset $S \subset V(G)$, let $N_G(S) = \bigcup_{x \in S} N_G(x)$, and let $G - S$ denote the subgraph induced by $V(G) \setminus S$. A *leaf* (or an *endvertex*) of a tree is a vertex of degree one, and a *branch vertex* of a tree is a vertex of degree strictly greater than two. For a tree T, let

$$L(T) = \{x \in V(T) \mid x \text{ is a leaf of } T\} \text{ and}$$
$$B(T) = \{x \in V(T) \mid x \text{ is a branch vertex of } T\}.$$

A graph G said to be *k-leaf-connected* if $|G| > k$ and for each subset S of $V(G)$ with $|S| = k$, G has a spanning tree T with $L(T) = S$.

We prove the following theorem, which gives an Ore-type condition for a graph to be k-leaf-connected.

Theorem 1 *Let $k \geq 2$ be an integer. Let G be a $(k+1)$-connected graph and suppose that $d_G(x) + d_G(y) \geq |G| + 1$ for any two nonajacent vertices $x, y \in V(G)$. Then G is k-leaf-connected.*

Theorem 1 is best possible in the following sense:

- We cannot replace the lower bound $|G| + 1$ in the degree condition by $|G|$.

  Consider the complete bipartite graph G with partite sets A and B such that $|A| = |B| = t$, where t is an integer with $t \geq k + 1$. Then G is $(k + 1)$-connected, $|G| = 2t$, and $d_G(x) + d_G(y) = |G|$ for any two nonadjacent vertices x and y of $V(G)$. Suppose that G is k-leaf-connected. Then G has a spanning tree T with $L(T) \subset B$. Consequently $d_T(x) \geq 2$ for all $x \in A$, and thus $|E(T)| \geq 2|A| = 2t$. However, this contradicts the fact $|E(T)| = |G| - 1 < 2t$. Hence G is not k-leaf-connected.

- For $k \geq 3$, the condition that G is $(k + 1)$-connected is necessary.

  Assume that $k \geq 3$. Let $r \geq 1$ be an integer and consider the graph $G := K_k + (K_1 \cup K_r)$. Then G is k-connected but not $(k + 1)$-connected, and for two vertices $x \in V(K_r)$ and $y \in V(K_1)$, we have $d_G(x)+d_G(y) = (|G|-2)+k \geq |G|+1$.

However, G has no spanning tree T with $L(T) = V(K_k)$. (For the case where $k = 2$, see Theorem 3 below and the first sentence in the paragraph following Theorem 3.)

As for the proof, we prove the following result, which is stronger than Theorem 1.

Theorem 2 *Let G be a graph, and let S be a subset of $V(G)$ such that $|S| \geq 2$, $|N_G(S) \setminus S| \geq 2$, $G - S$ is connected and $N_G(v) \setminus S \neq \emptyset$ for all $v \in S$. Suppose further that $d_G(x) + d_G(y) \geq |G| + 1$ for any two nonajacent vertices $x, y \in V(G) \setminus S$. Then G has a spanning tree T with $L(T) = S$.*

As in the case of Theorem 1, balanced complete bipartite graphs show that the lower bound in the degree condition in Theorem 2 is also sharp.

The following two results motivate our results. Since G has a Hamilton path connecting any two vertices if and only if it is 2-leaf-connected, Theorem 1 is a natural extension of the following famous result.

Theorem 3 (Ore [2]) *Let G be a graph. If $d_G(x)+d_G(y) \geq |G|+1$ for every two nonajacent vertices $x, y \in V(G)$, then G has a Hamilton path connecting any two vertices.*

Note that if $d_G(x)+d_G(y) \geq |G|+k-1$ for every two nonajacent vertices $x, y \in V(G)$ then G is $(k+1)$-connected. Thus the following result also follows from Theorem 1 (in [1], this result is derived from the assertion that the property of being k-leaf-connected is stable under a closure operation of Bondy-Chvátal type, i.e., if $x, y \in V(G)$ are nonadjacent vertices with $d_G(x)+d_G(y) \geq |G|+k-1$, then G is k-leaf-connected if and only if $G + xy$ is k-leaf-connected; see [1; Theorem 4]).

Theorem 4 (Gurgel and Wakabayashi [1; Corollary 6.1]) *Let G be a graph, and suppose that $d_G(x) + d_G(y) \geq |G| + k - 1$ for every two nonajacent vertices x, y of G. Then G is k-leaf-connected.*

2 Proof of Theorem 2

Let G and S be as in Theorem 2. Since $N_G(v) \setminus S \neq \emptyset$ for each $v \in S$, and $G - S$ is connected, G has a tree T with $L(T) = S$ and $V(T) \setminus S \neq \emptyset$. Choose such a tree T so that $|T|$ is as large as possible. If $V(G) = V(T)$, then we have nothing to prove. Thus we may assume that $G - V(T) \neq \emptyset$. Let H be a component of $G - V(T)$ and set $X = N_G(H) \cap V(T)$. Note that $X \setminus S \neq \emptyset$ because $V(T) \setminus S \neq \emptyset$ and $G - S$ is connected.

We assume that we have chosen H such that $|X|$ is as large as possible. We derive the proof into two cases according to the value of $|X|$.

Case 1. $|X| = 1$.

Set $X = \{x_0\}$. Since $(N_G(H') \cap V(T)) \setminus S \neq \emptyset$ for every component H' of $G - V(T)$, it follows from our choice of H that $N_G(G - V(T)) \cap S = \emptyset$, which implies $N_G(S) \subset V(T)$.

Since $|N_G(S) \setminus S| \geq 2$ by the assumption of the theorem, we can take $v_0 \in V(T) \setminus (S \cup \{x_0\})$. Now take $u_0 \in V(H)$. By the assumption of Case 1, $N_G(v_0) \cap N_G(u_0) \subset \{x_0\}$. Since $v_0u_0 \notin E(G)$, we also have $N_G(v_0) \cup N_G(u_0) \subset V(G) \setminus \{v_0, u_0\}$. Hence $d_G(v_0) + d_G(u_0) \leq |G| - 2 + 1 = |G| - 1$, which contradicts the degree condition of the theorem. This completes the proof for Case 1.

Case 2. $|X| \geq 2$.

By the maximality of T, we obtain the following fact.

Fact 1 *X is an indepedent set in T.*

We denote by $P_T(a, b)$ the unique path in T connecting two vertices a and b of T. We choose $x_1 \in X \setminus S$ and $x_2 \in X \setminus \{x_1\}$ so that $|P_T(x_1, x_2)|$ is as small as possible. By Fact 1, $x_1x_2 \notin E(T)$. We regard T as an outdirected tree with root x_1. For $U \subset V(T)$, define $U^+ = \bigcup_{u \in U} (N_T(u) \setminus V(P_T(x_1, u)))$ and $U^- = \bigcup_{u \in U} (N_T(u) \cap V(P_T(x_1, u)))$. For a vertex $u \in V(T) \setminus \{x_1\}$, having in mind the fact that $|\{u\}^-| = 1$, we let u^- denote the unique vertex in $\{u\}^-$. Recall that $B(T)$ denotes the set of branch vertices of T.

Claim 2 $B(T)^+ \cap X = \emptyset$.

Proof. Suppose that $x \in B(T)^+ \cap X$. Let $x' \in N_G(x) \cap V(H)$ and $x_1' \in N_G(x_1) \cap V(H)$, and let Q be a path in H connecting x' and x_1'. Then $T' := (T - xx^- + xx' + x_1x_1') \cup Q$ is a tree with $L(T') = S$ and $|T'| > |T|$. This contradicts the maximality of T. Hence $B(T)^+ \cap X = \emptyset$. $\square$

Set $W = B(T) \cup \{x_1\}$. Choose $y_1 \in (V(P_T(x_1, x_2)) \cap W) \setminus \{x_2\}$ so that $|P_T(y_1, x_2)|$ is as small as possible (possibly $y_1 = x_1$). By Claim 2, $y_1x_2 \notin E(T)$. Write $N_T(y_1) \cap V(P_T(y_1, x_2)) = \{v_1\}$ and $N_T(x_2) \cap V(P_T(y_1, x_2)) = \{v_2\}$ (possibly $v_1 = v_2$). Write $N_T(x_1) \cap V(P_T(x_1, x_2)) = \{w_1\}$ and define $T^* = T - V(P_T(w_1, v_2))$. We denote by $P_1, P_2, \ldots, P_m$ the components of $T^* - \{uv \in E(T) \mid u \in W, v \in \{u\}^+\}$. We may assume that $V(P_1) = \{x_1\}$ and $x_2 \in V(P_2)$. Note that P_i is a path for every $i = 1, \ldots, m$ and $|V(P_i) \cap W^+| = 1$ for each $i = 3, \ldots, m$. Write $V(P_i) \cap W^+ = \{a_i\}$ for each $i = 3, \ldots, m$. Then for each i, a_i is an endvertex of P_i.

For $j = 1, 2$, let $u_j \in N_G(x_j) \cap V(H)$ (possibly $u_1 = u_2$).

Claim 3 $|N_G(u_1) \cap V(T^*)| + |N_G(u_2) \cap V(T^*)| \leq |T^*| + 2$.

Proof. Since $|P_1| = |\{x_1\}| = 1$, $|N_G(u_1) \cap V(P_1)| + |N_G(u_2) \cap V(P_1)| \leq 2 = |P_1| + 1$.

By Fact 1, $(N_G(u_1) \cap V(P_i))^- \cap (N_G(u_2) \cap V(P_i)) = \emptyset$ for every $2 \leq i \leq m$. For the path P_2, we have $|N_G(u_1) \cap V(P_2)| = |(N_G(u_1) \cap V(P_2))^-|$ and $(N_G(u_1) \cap V(P_2))^- \cup (N_G(u_2) \cap V(P_2)) \subset V(P_2) \cup \{v_2\}$. Hence $|N_G(u_1) \cap V(P_2)| + |N_G(u_2) \cap V(P_2)| = |(N_G(u_1) \cap V(P_2))^-| + |N_G(u_2) \cap V(P_2)| \leq |P_2| + 1$. Let now $3 \leq i \leq m$. Then $a_i \notin N_G(u_1)$ by Fact 1 or Claim 2 according as $a_i \in \{x_1\}^+$ or $a_i \in B(T)^+$. Since $u^- \in V(P_i)$ for all $u \in V(P_i) \setminus \{a_i\}$, this implies $(N_G(u_1) \cap V(P_i))^- \cup (N_G(u_2) \cap V(P_i)) \subset V(P_i)$. Since $|(N_G(u_1) \cap V(P_i))^-| = |N_G(u_1) \cap V(P_i)|$, we obtain $|N_G(u_1) \cap V(P_i)| + |N_G(u_2) \cap V(P_i)| \leq$

$|(N_G(u_1) \cap V(P_i))^-| + |N_G(u_2) \cap V(P_i)| \leq |P_i|$. Thus $|(N_G(u_1) \cap V(P_i))| + |N_G(u_2) \cap V(P_i)| \leq |P_i|$ for every $3 \leq i \leq m$. Consequently

$$\begin{aligned} |N_G(u_1) \cap V(T^*)| + |N_G(u_2) \cap V(T^*)| &= \sum_{i=1}^{m} (|N_G(u_1) \cap V(P_i)| + |N_G(u_2) \cap V(P_i)|) \\ &\leq |P_1| + 1 + |P_2| + 1 + \sum_{i=3}^{m} |P_i| \\ &= |T^*| + 2. \end{aligned}$$

Hence the claim holds. □

Let R be a path in H connecting u_1 and u_2.

Claim 4 $|N_G(v_1) \cap V(T^*)| + |N_G(v_2) \cap V(T^*)| \leq |T^*| + 2$.

Proof. Note that $|N_G(v_1) \cap V(P_1)| + |N_G(v_2) \cap V(P_1)| \leq 2 = |P_1| + 1$. Note also that $(N_G(v_1) \cap V(P_2)) \cup (N_G(v_2) \cap V(P_2))^- \subset V(P_2) \cup \{v_2\}$. We now show $a_i \notin N_G(v_2)$ for every $i = 3, \ldots, m$. Suppose that $a_j \in N_G(v_2)$ for some j with $3 \leq j \leq m$. Then $T' := (T - a_j a_j^- - v_2 x_2 + a_j v_2 + x_1 u_1 + x_2 u_2) \cup R$ is a tree with $L(T') = S$ and $|T'| > |T|$. But this contradicts the maximality of T. Hence $a_i \notin N_G(v_2)$ for every $i = 3, \ldots, m$. Consequently, $(N_G(v_1) \cap V(P_i)) \cup (N_G(v_2) \cap V(P_i))^- \subset V(P_i)$ for each $i = 3, \ldots, m$.

Next, suppose that $(N_G(v_1) \cap V(P_j)) \cap (N_G(v_2) \cap V(P_j))^- \neq \emptyset$ for some j with $2 \leq j \leq m$. Then there exists $v \in V(P_j)$ such that $v \in N_G(v_2) \cap V(P_j)$ and $v^- \in N_G(v_1) \cap V(P_j)$. But then $T' := (T - vv^- - v_1 y_1 - v_2 x_2 + v_2 v + v_1 v^- + x_1 u_1 + x_2 u_2) \cup R$ is a tree with $L(T') = S$ and $|T'| > |T|$, which is a contradiction. Hence $(N_G(v_1) \cap V(P_i)) \cap (N_G(v_2) \cap V(P_i))^- = \emptyset$ for each $i = 2, \ldots, m$.

Since $|(N_G(v_2) \cap V(P_i))^-| = |N_G(v_2) \cap V(P_i)|$ for every $2 \leq i \leq m$, we obtain $|N_G(v_1) \cap V(P_2)| + |N_G(v_2) \cap V(P_2)| \leq |P_2| + 1$ and $|N_G(v_1) \cap V(P_i)| + |N_G(v_2) \cap V(P_i)| \leq |P_i|$ for every $3 \leq i \leq m$. Therefore $|N_G(v_1) \cap V(T^*)| + |N_G(v_2) \cap V(T^*)| \leq |T^*| + 2$. □

Now let $j \in \{1, 2\}$. By the minimality of $|P_T(x_1, x_2)|$, we have $u_j v_j \notin E(G)$. Note that $u_j, v_j \notin S$ because $u_j \notin V(T)$ and $v_j \in V(P_T(x_1, x_2)) \setminus \{x_1, x_2\}$. Thus by the degree condition, $d_G(u_j) + d_G(v_j) \geq |G| + 1$. Furthermore, by the choice of x_1 and x_2, $N_G(v_j) \cap V(H) = \emptyset$ and $N_G(u_j) \cap V(P_T(w_1, v_2)) = \emptyset$. Since we clearly have $N_G(u_j) \cap (V(G) \setminus (V(T) \cup V(H))) = \emptyset$, $N_G(u_j) \cap V(H) \subset V(H) \setminus \{u_j\}$ and $N_G(v_j) \cap V(P_T(w_1, w_2)) \subset V(P_T(w_1, v_2)) \setminus \{v_j\}$, this implies

$$\begin{aligned} &|N_G(u_j) \cap (V(G) \setminus V(T^*))| + |N_G(v_j) \cap (V(G) \setminus V(T^*))| \\ &\leq |G - (V(T) \cup V(H))| + (|H| - 1) + (|P_T(w_1, v_2)| - 1) = |G| - |T^*| - 2. \end{aligned}$$

Consequently

$$|N_G(u_j) \cap V(T^*)| + |N_G(v_j) \cap V(T^*)| \geq |G| + 1 - (|G| - |T^*| - 2) = |T^*| + 3.$$

Thus $|N_G(u_j) \cap V(T^*)| + |N_G(v_j) \cap V(T^*)| \geq |T^*| + 3$ for each $j = 1, 2$. This implies that we have $|N_G(u_1) \cap V(T^*)| + |N_G(u_2) \cap V(T^*)| \geq |T^*| + 3$ or $|N_G(v_1) \cap V(T^*)| + |N_G(v_2) \cap V(T^*)| \geq |T^*| + 3$, which contradicts Claim 3 or 4.

This completes the proof of Theorem 2. $\Box$

3 Application

As a consequence of Theorem 1, we prove the following result, which gurantees the existence of a spanning tree having the bounded number of leaves and containing specified vertices as leaves.

Corollary 5 *Let k and s be integers with $k \geq 2$ and $0 \leq s \leq k$. Suppose that G is an $(s+1)$-connected graph, and for any two nonadjacent vertices $x, y \in V(G)$,*

$$d_G(x) + d_G(y) \geq |G| - k + 1 + s.$$

Then for any subset $S \subset V(G)$ with $|S| = s$, G has a spanning tree T such that $S \subset L(T)$ and $|L(T)| \leq k$.

Proof. Construct a new graph H by joining two graphs G and K_{k-s}. Then H satisfies the conditions of Theorem 1, and hence H has a spanning tree T such that $L(T) = S \cup V(K_{k-s})$. Thus $T - V(K_{k-s})$ is a spanning tree of G with the desired properties. $\Box$

In Corollary 5, the lower bound in the degree condition is sharp. For example, let G be a complete bipartite graph with partite sets A and B such that $|A| = t + k$ and $|B| = t + s$, where $t \geq 1$. Then G is $(s+1)$-connected, $|G| = 2t + k + s$, and $d_G(x) + d_G(y) \geq 2|B| = 2t + 2s = |G| - k + s$ for any two nonadjacent vertices x and y of G. Suppose that G has a spanning tree T such that $|L(T)| \leq k$ and s specified vertices in B are contained in $L(T)$. Then the number of edges in T is at least $2|A| - (k-s) = 2t+k+s$. However, this is a contradiction because $2t + k + s > |G| - 1 = |E(T)|$. Thus G has no desired spanning tree.

Moreover, for $k \geq 3$ and $s \geq 1$, the condition that G is $(s+1)$-connected is necessary. Assume that $k \geq 3$. Let $r \geq 1$, and consider the graph $G := K_s + (K_1 \cup K_r)$. Then G is s-connected but not $(s+1)$-connected. For $x \in V(K_1)$ and any $y \in V(K_r)$, we have $d_G(x) + d_G(y) = (|G| - 2) + s \geq |G| - k + 1 + s$. However, G has no spanning tree T with $V(K_s) \subseteq L(T)$.

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