

# The intersection of two vertex coloring problems

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## Abstract

A *hole* is an induced cycle with at least four vertices. A hole is even if its number of vertices is even. Given a set  $L$  of graphs, a graph  $G$  is  $L$ -free if  $G$  does not contain any graph in  $L$  as an induced subgraph. Currently, the following two problems are unresolved: the complexity of coloring even hole-free graphs, and the complexity of coloring  $\{4K_1, C_4\}$ -free graphs. The intersection of these two problems is the problem of coloring  $\{4K_1, C_4, C_6\}$ -free graphs. In this paper we present partial results on this problem.

*Keywords:* Graph coloring, perfect graphs

## 1 Introduction

There has been recently keen interest in finding polynomial-time algorithms to optimally color graphs  $G$  that do not contain any graph in a list  $L$  as an induced subgraphs. Particular attention is focused on graphs whose forbidden list  $L$  contains graphs with four vertices, and recent papers of Lozin and Malyshev [15], and of Fraser, Hamel, Hoàng, Holmes and LaMantia [10] discuss the state of the art on this problem, identifying three outstanding classes:  $L = (4K_1, \text{claw})$ ,  $L = (4K_1, \text{claw, co-diamond})$ , and  $L = (4K_1, C_4)$ . As a resolution of these cases is likely challenging, a productive approach would be to consider increasing the number of graphs in  $L$ . But it makes sense to ask, which  $L$  is an interesting one to consider? We are particularly interested in cases that are slightly larger than the class  $L = (4K_1, C_4)$ , which is one of the unresolved cases. Before we introduce the problems we need some definitions:

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A *hole* is an induced cycle with at least four vertices. A hole is even if its number of vertices is even. The problem of coloring even-hole-free graphs has been much studied. A theorem of Addario-Berry, Chudnovsky, Havet, Reed, and Seymour [1] shows that for an even-hole-free graph  $G$ , the chromatic number of  $G$  is at most two times its clique number (the number of vertices in a largest clique of  $G$ ). It is currently not known whether even-hole-free graphs can be colored in polynomial time.

We offer the following four problems for consideration:

**Problem 1.1** *What is the complexity of coloring  $(4K_1, C_4)$ -free graphs?*

This is the original problem, and is likely the most challenging.

**Problem 1.2** *What is the complexity of coloring even-hole-free graphs?*

Problem 1.2 might even be NP-complete. Combining Problems 1.1 and 1.2, we get the following problem:

**Problem 1.3** *What is the complexity of coloring  $(4K_1, \text{even hole})$ -free graphs?*

This problem appears to be more tractable than the previous two. In this paper, we study Problem 1.3. Since a  $4K_1$ -free graph does not contain a hole of length at least 8, Problem 1.3 is equivalent to the following:

**Problem 1.4** *What is the complexity of coloring  $(4K_1, C_4, C_6)$ -free graphs?*

Even though we have not been able to solve Problem 1.4, we have succeeded, in some sense, in solving “half” of it, as follows. Consider a  $(4K_1, C_4, C_6)$ -free graph  $G$ . We may assume  $G$  is not perfect (there are known algorithms to color perfect graphs). Thus  $G$  has to contain a  $C_5$  or  $C_7$ . If  $G$  contains a  $C_7$ , then our result shows that  $G$  can be colored in polynomial time. The case where  $G$  contains a  $C_5$  but not a  $C_7$  is open. Investigation into this problem led us to a proof that there is a polynomial time algorithm to color a  $(4K_1, C_4, C_6, C_5\text{-twin})$ -free graph.

In Section 2, we discuss the background of the problem and state the main results. In Section 3, we study  $(4K_1, C_4, C_6)$ -free graphs that contain a  $C_7$  and show that such graphs can be colored in polynomial time. In Section 4, we study  $(4K_1, C_4, C_6)$ -free graphs that contain a  $C_5$ , but no  $C_7$ . In Section 5, we give a polynomial time algorithm to color a  $(4K_1, C_4, C_6, C_5\text{-twin})$ -free graph. Finally, in Section 6, we discuss open problems related to our work.

## 2 Background and results

Before discussing our results in more detail, we need introduce a few definitions. Let  $G$  be a graph. A *colouring* of a graph  $G = (V, E)$  is a mapping  $f : V \rightarrow \{1, \dots, k\}$  for

some nonnegative integer  $k$  such that  $f(u) \neq f(v)$  whenever  $uv \in E$ . The *chromatic number*, denoted  $\chi(G)$ , is the minimum number of colors needed to colour a graph  $G$ . *VERTEX COLORING* is the problem of determining the chromatic number of a graph.

Consider the following operations to build a graph.

- (i) Create a vertex  $u$  labeled by integer  $\ell$ .
- (ii) Disjoint union (i.e., co-join)
- (iii) Join between all vertices with label  $i$  and all vertices with label  $j$  for  $i \neq j$ , denoted by  $\eta_{i,j}$  (that is, add all edges between vertices of label  $i$  and label  $j$ ).
- (iv) Relabeling all vertices of label  $i$  by label  $j$ , denoted by  $\rho_{i \rightarrow j}$

The *clique width* of a graph  $G$ , denoted by  $cwd(G)$ , is the minimum number of labels needed to build the graph with the above four operations. It is well-known [8] that if the clique width of a graph is bounded then so is that of its complement. Clique widths have been intensively studied. In Rao [17], the following result is established.

**Theorem 2.1** *VERTEX COLORING is polynomial time solvable for graphs with bounded clique width.*  $\square$

We will need the following well known observation that is easy to establish (for example, see [6]).

**Observation 2.2** *Let  $G$  be a graph and  $G'$  be the graph obtained from  $G$  by removing a constant number of vertices. Then  $G$  has bounded clique width if and only if  $G'$  does.*  $\square$

The symbol  $\omega(G)$  denotes the number of vertices in a largest clique of  $G$ . A graph  $G$  is *perfect* if for each induced subgraph  $H$  of  $G$ , we have  $\chi(H) = \omega(H)$ . A *hole* is an induced cycle of length at least 4, i.e.  $C_k$  for  $k \geq 4$ . A hole is *even* or *odd* depending on the parity of the vertices in the hole. An *anti-hole* is the complement of a hole.

Two important results are known about perfect graphs. The Perfect Graph Theorem, proved by Lovász [14], states that a graph is perfect if and only if its complement is. The Strong Perfect Graph Theorem, proved by Chudnovsky, Robertson, Seymour, and Thomas [7], states that a graph is perfect if and only if it is odd-hole-free and odd-anti-hole-free. Both of the above results were long standing open problems proposed by Berge [3]. Grötschel, Lovász and Schrijver [12] designed a polynomial-time algorithm for finding a largest clique and a minimum coloring of a perfect graph.

Suppose we want to color a  $(4K_1, C_4, C_6)$ -free graph  $G$ . Note that  $G$  contains no  $C_\ell$  for  $\ell \geq 8$  because  $G$  is  $4K_1$ -free. By the result of Grötschel et al, we may assume

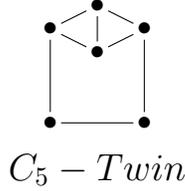


Figure 1: The  $C_5$ -twin

$G$  is not perfect. The result of Chudnovsky et al implies  $G$  contains a  $C_5$  or  $C_7$  as an induced subgraph (note that the anti-hole of length at least six contains a  $C_4$  and the  $C_5$  is self-complementary.) If  $G$  contains a  $C_7$ , then we will show that  $G$  has bounded clique-width.

**Theorem 2.3** *Let  $G$  be  $(4K_1, C_4, C_6)$ -free graph that contains a  $C_7$ . Then  $G$  has bounded clique width.*

We will use Theorem 2.3 to prove the following theorem that is the main result of this paper.

**Theorem 2.4** *VERTEX COLORING can be solved in polynomial time for the class of  $(4K_1, C_4, C_6)$ -free graphs that contain a  $C_7$ .*

Two adjacent vertices  $x, y$  of a graph  $G$  are *twins* if for any vertex  $z$  different from  $x$  and  $y$ ,  $xz$  is an edge if and only if  $yz$  is an edge. A *hole-twin* is the graph obtained from a hole by adding a vertex that form twins with some vertex of the hole. Figure 1 shows the  $C_5$ -twin. Hole-twins play an interesting role in graph theory. They are among the forbidden induced subgraphs for line-graphs (Beineke [2]).

By adding the  $C_5$ -twin to the list of forbidden induced subgraphs for our graph class, we obtain the following theorem.

**Theorem 2.5** *VERTEX COLORING can be solved in polynomial time for the class of  $(4K_1, C_4, C_6, C_5\text{-twin})$ -free graphs.*

We will rely on a theorem of Fraser et al [10]. To explain this theorem, we will need to introduce a few definitions. Given sets of vertices  $X, Y$ , we write  $X \textcircled{0} Y$  to mean there is no edge between any vertex in  $X$  and any vertex in  $Y$  (also called a *co-join*). Given sets of vertices  $X, Y$ , we write  $X \textcircled{1} Y$  to mean there are all edges between  $X$  and  $Y$  (also called a *join*).

Consider a partition  $\mathcal{P}$  of the vertices of  $G$  into sets  $S_1, S_2, \dots, S_k$  such that each  $S_i$  induces a clique. A set  $S_i$  is *uniform* to a set  $S_j$  if  $S_i \textcircled{0} S_j$  or  $S_i \textcircled{1} S_j$ . The

set  $S_i$  is *uniform in  $\mathcal{P}$*  if every set  $S_j, i \neq j$ , is uniform to it. The partition  $\mathcal{P}$  is *uniform* if every set  $S_i$  is uniform in  $\mathcal{P}$ . A set  $S_i$  is *near-uniform* if there is at most one set  $S_j, i \neq j$ , that is not uniform to  $S_i$ . The partition  $\mathcal{P}$  is *near-uniform* if every set  $S_i$  is near-uniform. Thus, a uniform partition is near-uniform. A *k-near-uniform* partition of  $G$  is a partition of the vertices of  $G$  into  $k$  near-uniform sets. In such a partition, the pair of sets  $S_i, S_j$  such that each set is not uniform to the other is call a *uniform-pair*. The following is proved in Fraser et al [10].

**Theorem 2.6** [10] *Let  $G$  be a graph admitting a  $k$ -near-uniform partition<sup>1</sup> such that the uniform pairs are  $C_4$ -free. Then we have  $\text{cwd}(G) \leq 2k$ .  $\square$*

We will establish the following theorem.

**Theorem 2.7** *Let  $G$  be  $(4K_1, C_4, C_6)$ -free graph that contains a  $C_7$ . Then  $G - C_7$  admits a  $k$ -uniform partition, for some constant  $k$ .*

By the above discussion, Theorem 2.7 implies Theorem 2.4. We will prove Theorem 2.7 in the next section.

### 3 When the graphs contain a $C_7$

Assume that the graph  $G$  is  $(4K_1, C_4, C_6)$ -free and contains a  $C_7$ . In this section, we examine the structure of the neighborhood of the  $C_7$ . Then we will prove Theorem 2.7.

We will need first to establish a number of preliminary results. Given a hole,  $H$ , and a vertex  $x$  not in  $H$ , we say  $x$  is a  $k$ -vertex (for  $H$ ) if  $x$  has exactly  $k$  neighbours in  $H$ .

For all the claims below, we shall now assume that  $G$  contains an induced  $C_7$  with vertices  $(1, 2, 3, 4, 5, 6, 7)$ . The vertex numbers of the  $C_7$  are taken modulo 7.

Let

- $X_i$  denote the set of 3-vertices adjacent to  $(i, i + 1, i + 2)$  in the  $C_7$ ,
- $Y_i$  denote the set of 3-vertices on  $(i, i + 1, i + 4)$  in the  $C_7$ ,
- $Z_i$  denote the set of 5-vertices on  $(i, i + 1, i + 2, i + 3, i + 4)$  in the  $C_7$ , and
- $W$  denote the set of 7-vertices in the  $C_7$ .

We will show that the sets  $X_i, Y_i, Z_i, W$  form a partition of  $V(G) - C_7$ .

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<sup>1</sup>We note that the definition of near-uniform partition in [10] is incomplete. The sets  $S_i$ 's must be cliques for the theorem to hold.

**Observation 3.1** *The  $C_7$  has no  $k$ -vertex in  $G$  for  $k = 0, 1, 2, 4, 6$ .*

*Proof.* Suppose that there is a 0-vertex  $v$  for  $C_7$ . This creates an induced  $4K_1$   $(v, i, i + 2, i + 4)$ , which is forbidden. Therefore, there is no 0-vertex. Now suppose that there is a 1-vertex  $v$  that is adjacent to  $i$ . This creates an induced  $4K_1$   $(v, i + 1, i + 3, i + 5)$ , which is forbidden. Therefore, there is no 1-vertex. Next, suppose that there is a 2-vertex  $v$  for  $C_7$ . It is easy to see that  $v$  and some three vertices in the  $C_7$  form a  $4K_1$ , a contradiction. Consequently, there is no 2-vertex. Next, suppose that there is a 4-vertex  $v$  for  $C_7$ . Then  $G$  contains a  $C_4$  or  $C_6$ , both of which are forbidden. Therefore, there is no 4-vertex. Finally, suppose that there exists a 6-vertex  $v$  for  $C_7$ . Then,  $G$  contains an induced  $C_4$ , which is forbidden. Therefore, there is no 6-vertex for  $C_7$ .  $\square$

It is a routine matter to verify the two observations below.

**Observation 3.2** *Let  $v$  be a 3-vertex for the  $C_7$ . Then  $v \in X_i \cup Y_i$  for some  $i$ .*  $\square$

**Observation 3.3** *Let  $v$  be a 5-vertex for the  $C_7$ . Then  $v \in Z_i$  for some  $i$ .*  $\square$

The above three observations imply the following observation.

**Observation 3.4** *Let  $G'$  be the graph obtained from  $G$  by removing the  $C_7$ . Then the sets  $X_i, Y_i, Z_i, W$  form a partition  $\mathcal{P}$  of  $G'$ .*

Our aim is to show that  $G'$  has bounded clique width. In fact, we will show that  $\mathcal{P}$  is an uniform partition of  $G'$ .

We now examine the adjacencies between the sets of the partition  $\mathcal{P}$ .

**Observation 3.5** *Each of the sets  $X_i, Y_i, Z_i, W$  of the partition  $\mathcal{P}$  is a clique.*

*Proof.* It is easy to see that if a set of  $\mathcal{P}$  is not a clique then there is a  $C_4$ .  $\square$

The next sequence of observations will imply that  $X_i$  is near-uniform in  $\mathcal{P}$ .

**Observation 3.6**  $X_i \overset{(1)}{\cup} X_{i+1} \cup X_{i+6}$

*Proof.* Suppose there are vertices  $x_1 \in X_i$  and  $x_2 \in X_{i+1}$  such that  $x_1 x_2 \notin E$ . Then there exists a  $4K_1$   $(x_1, x_2, i + 4, i + 6)$ , which is forbidden. Therefore,  $X_i \overset{(1)}{\cup} X_{i+1}$ . By symmetry, we have  $X_i \overset{(1)}{\cup} X_{i+6}$ .  $\square$

**Observation 3.7**  $X_i \overset{(0)}{\cup} X_{i+2} \cup X_{i+3} \cup X_{i+4} \cup X_{i+5}$ .

*Proof.* Consider a vertex  $x_1 \in X_i$ . Suppose there is a vertex  $x_2 \in X_{i+2}$  such that  $x_1x_2 \in E$ . This creates a  $C_6(x_1, x_2, i+4, i+5, i+6, i)$ , which is forbidden. Therefore, we have  $X_i \textcircled{0} X_{i+2}$ , and by symmetry  $X_i \textcircled{0} X_{i+5}$ . Now suppose that there is a vertex  $x_3 \in X_{i+3}$  such that  $x_1x_3 \in E$ . This creates a  $C_4(x_3, x_1, i+2, i+3)$ , which is forbidden. Therefore, we have  $X_i \textcircled{0} X_{i+3}$ , and by symmetry,  $X_i \textcircled{0} X_{i+4}$ .  $\square$

**Observation 3.8**  $X_i \textcircled{1} Y_i \cup Y_{i+1} \cup Y_{i+4}$ .

*Proof.* Consider a vertex  $x \in X_i$ . Suppose there is a vertex  $y \in Y_i$  such that  $xy \notin E$ . This creates a  $4K_1(x, y, i+3, i+5)$ , which is forbidden. Therefore,  $X_i \textcircled{1} Y_i$ . Now suppose there is a vertex  $y \in Y_{i+1}$  with  $xy \notin E$ . This creates a  $4K_1(x, y, i+3, i+6)$ , which is forbidden. Therefore  $X_i \textcircled{1} Y_{i+1}$ . Finally, suppose there is a vertex  $y \in Y_{i+4}$  with  $xy \notin E$ . This creates a  $4K_1(x, y, i+3, i+6)$ , which is forbidden. Therefore  $X_i \textcircled{1} Y_{i+4}$ . Consequently,  $X_i \textcircled{1} Y_i \cup Y_{i+1} \cup Y_{i+4}$ .  $\square$

**Observation 3.9**  $X_i \textcircled{0} Y_{i+2} \cup Y_{i+3} \cup Y_{i+5} \cup Y_{i+6}$

*Proof.* Consider a vertex  $x \in X_i$ . Suppose there is a vertex  $y \in Y_{i+2}$  with  $xy \in E$ . This creates a  $C_4(x, y, i+6, i)$ , which is forbidden. Therefore, we have  $X_i \textcircled{0} Y_{i+2}$ , and by symmetry,  $X_i \textcircled{0} Y_{i+6}$ . Now suppose that there is a vertex  $y \in Y_{i+3}$  with  $xy \in E$ . This creates a  $C_4(x, y, i+3, i+2)$ , which is forbidden. Therefore,  $X_i \textcircled{0} Y_{i+3}$ , and by symmetry,  $X_i \textcircled{0} Y_{i+5}$ .  $\square$

**Observation 3.10**  $X_i \textcircled{1} Z_i \cup Z_{i+1} \cup Z_{i+4} \cup Z_{i+5} \cup Z_{i+6}$

*Proof.* Consider a vertex  $x \in X_i$ . Suppose there is a vertex  $z \in Z_i$  such that  $xz \notin E$ . This creates a  $C_4(x, i, z, i+2)$ , which is forbidden. Therefore, we have  $X_i \textcircled{1} Z_i$ , and by symmetry,  $X_i \textcircled{1} Z_{i+5}$ . Now suppose there is a vertex  $z \in Z_{i+1}$  such that  $xz \notin E$ . This creates a  $C_6(i, i+6, i+5, z, i+2, x)$ , which is forbidden. Therefore, we have  $X_i \textcircled{1} Z_{i+1}$ , and by symmetry,  $X_i \textcircled{1} Z_{i+4}$ . Finally, suppose that there is a vertex  $z \in Z_{i+6}$  such that  $xz \notin E$ . This creates a  $C_4(x, i, z, i+2)$ , which is forbidden. Therefore, we have  $X_i \textcircled{1} Z_{i+6}$ , and we are done.  $\square$

**Observation 3.11**  $X_i \textcircled{0} Z_{i+2} \cup Z_{i+3}$

*Proof.* Consider a vertex  $x \in X_i$ . Suppose there is a vertex  $z \in Z_{i+2}$  such that  $xz \in E$ . This creates a  $C_4(x, z, i+6, i)$ , which is forbidden. Therefore,  $X_i \textcircled{0} Z_{i+2}$ . Next, suppose that there is a vertex  $z \in Z_{i+3}$  such that  $xz \in E$ . This also creates a  $C_4(x, z, i+3, i+2)$ , which is forbidden. Therefore,  $X_i \textcircled{0} Z_{i+3}$ .  $\square$

**Observation 3.12**  $X_i \textcircled{1} W$

*Proof.* Suppose that there are vertices  $x \in X_i$  and  $w \in W$  such that  $xw \notin E$ . This creates a  $C_4(i, x, i+2, w)$ , which is forbidden. Therefore we have  $X_i \textcircled{1} W$ .  $\square$

Observations 3.6–3.12 together imply the following lemma.

**Lemma 3.13** *For every  $i$ , the set  $X_i$  is uniform in the partition  $\mathcal{P}$ .*  $\square$

Next, we examine the sets  $Y_i$ .

**Observation 3.14** *At most two of the sets  $Y_1, Y_2, \dots, Y_7$  can be non-empty. In particular, if  $Y_i \neq \emptyset$ , then  $Y_{i+1} = Y_{i+2} = Y_{i+5} = Y_{i+6} = \emptyset$ , and we have  $Y_{i+3} \neq \emptyset$  or  $Y_{i+4} \neq \emptyset$ , but not both.*

*Proof.* Suppose  $Y_i \neq \emptyset$ . Let  $y$  be a vertex in  $Y_i$ . Suppose the set  $Y_{i+1}$  contains a vertex  $y_1$ . If  $yy_1 \notin E$ , then there exists a  $4K_1(y, y_1, i+3, i+6)$ , which is forbidden. If  $yy_1 \in E$ , then there is a  $C_4(y, y_1, i+5, i+4)$ , which is forbidden. So we have  $Y_{i+1} = \emptyset$ , and by symmetry,  $Y_{i+6} = \emptyset$ . Now, suppose  $Y_{i+2}$  is non-empty and contains a vertex  $y_2$ . If  $yy_2 \in E$ , then there is a  $C_4(y, i+4, i+3, y_2)$ , a contradiction. But if  $yy_2 \notin E$ , then there exists a  $C_6(y, i+4, i+3, y_2, i+6, i)$ , a contradiction. So, we have  $Y_{i+2} = \emptyset$ , and by symmetry,  $Y_{i+5} = \emptyset$ .

The first part of this proof shows that if  $Y_{i+3} \neq \emptyset$ , then  $Y_{i+4} = \emptyset$ . So only one of the two sets  $Y_{i+3}, Y_{i+4}$  can be non-empty.  $\square$

**Observation 3.15**  $Y_i \textcircled{1} Y_{i+3} \cup Y_{i+4}$

*Proof.* Consider a vertex  $y \in Y_i$ . Suppose there is a vertex  $y_3 \in Y_{i+3}$  such that  $yy_3 \notin E$ . There is a  $C_4(y, i, y_3, i+4)$ , a contradiction. So, we have  $Y_i \textcircled{1} Y_{i+3}$ , and by symmetry,  $Y_i \textcircled{1} Y_{i+4}$ .  $\square$

**Observation 3.16** *If  $Y_i \neq \emptyset$ , then  $Z_{i+5} = Z_{i+6} = \emptyset$*

*Proof.* Assume that  $Y_i \neq \emptyset$  and  $Z_{i+5} \neq \emptyset$ . Consider vertices  $y \in Y_i$  and  $z \in Z_{i+5}$ . If  $yz \in E$ , then there exists a  $C_4(y, z, i+5, i+4)$ , which is forbidden. However, if  $yz \notin E$ , then there exists a  $C_6(y, i, z, i+2, i+3, i+4)$ , which is also forbidden. Therefore if  $Y_i \neq \emptyset$ , then  $Z_{i+5} = \emptyset$ , and by symmetry,  $Z_{i+6} = \emptyset$ .  $\square$

**Observation 3.17**  $Y_i \textcircled{1} W \cup Z_i \cup Z_{i+1} \cup Z_{i+3} \cup Z_{i+4}$

*Proof.* Consider a vertex  $y \in Y_i$ . Let  $w \in W$ . If  $yw \notin E$ , then there is a  $C_4$   $(i, w, i + 4, y)$ , which is forbidden. Therefore,  $Y_i \textcircled{1} W$ .

Consider a vertex  $z \in Z_i \cup Z_{i+1} \cup Z_{i+3} \cup Z_{i+4}$ . Then  $z(i+4) \in E$ , and  $z$  is adjacent to a vertex  $i' \in \{i, i + 1\}$ . If  $zy \notin E$ , there there is a  $C_4$   $(i', y, i + 4, z)$ , which is forbidden. Therefore,  $Y_i \textcircled{1} Z_i \cup Z_{i+1} \cup Z_{i+3} \cup Z_{i+4}$ .  $\square$

**Observation 3.18**  $Y_i \textcircled{0} Z_{i+2}$

*Proof.* Consider vertices  $y \in Y_i$  and  $z \in Z_{i+2}$  such that  $yz \in E$ . This creates a  $C_4$   $(y, z, i + 6, i)$ , which is forbidden. Therefore  $Y_i \textcircled{0} Z_{i+2}$ .  $\square$

Observations 3.14–3.18 together imply the following lemma.

**Lemma 3.19** *For every  $i$ , the set  $Y_i$  is uniform in the partition  $\mathcal{P}$ .*  $\square$

Next, we examine the sets  $Z_i$ .

**Observation 3.20** *If  $Z_i \neq \emptyset$ , then  $Z_{i+2} = \emptyset$  and  $Z_{i+5} = \emptyset$ .*

*Proof.* Suppose that  $Z_i \neq \emptyset$ . Also, suppose  $Z_{i+2} \neq \emptyset$ . Consider vertices  $z_i \in Z_i$  and  $z_{i+2} \in Z_{i+2}$ . If  $z_i z_{i+2} \in E$ , then there is a  $C_4$   $(z_i, z_{i+2}, i + 6, i)$ , a contradiction. If  $z_i z_{i+2} \notin E$ , then there is a  $C_4$   $(z_i, i + 4, z_{i+2}, i + 2)$ , a contradiction. So  $Z_{i+2}$  is empty, and by symmetry,  $Z_{i+5}$  is empty.  $\square$

**Observation 3.21** *There can exist at most 3 distinct sets of 5-vertices for  $C_7$ .*

*Proof.* Follows from Observation 3.20.

**Observation 3.22**  $Z_i \textcircled{1} W \cup Z_{i+1} \cup Z_{i+3} \cup Z_{i+4} \cup Z_{i+6}$

*Proof.* Consider a vertex  $z_i \in Z_i$  and a vertex  $z \in V(G) - C_7$  with  $z \neq z_i$ . If  $z$  is adjacent to two non-adjacent vertices, say  $a$  and  $b$ , of the set  $\{i, i + 1, i + 2, i + 3, i + 4\}$ , then  $zz_i \in E$ , for otherwise there is a  $C_4$   $(z, a, z_i, b)$ . Observe that any vertex in  $W \cup Z_{i+1} \cup Z_{i+3} \cup Z_{i+4} \cup Z_{i+6}$  is adjacent to two non-adjacent vertices of  $\{i, i + 1, i + 2, i + 3, i + 4\}$ . The Observation follows.  $\square$

Observations 3.20–3.22 together imply the following lemma.

**Lemma 3.23** *For every  $i$ , the set  $Z_i$  is uniform in the partition  $\mathcal{P}$ .*  $\square$

We can now prove our main results.

*Proof of Theorem 2.7.* Let  $G$  be a  $(4K_1, C_4, C_6)$ -free graph with a  $C_7$ . Define the sets  $X_i, Y_i, Z_i, W$  as above. Let  $G' = G - C_7$ . Observation 3.4 implies that the sets  $X_i, Y_i, Z_i, W$  form a partition of  $G'$ . Lemmas 3.13, 3.19, and 3.23 show that each of the sets  $X_i, Y_i, Z_i$  is uniform. Now the set  $W$  is also uniform because every other set is uniform to  $W$ . Thus, the partition is uniform.  $\square$

*Proof of Theorem 2.3.* Let  $G$  be a  $(4K_1, C_4, C_6)$ -free graph with a  $C_7$ . The graph  $G' = G - C_7$  has bounded clique width by Theorem 2.7 and Theorem 2.6,  $G'$  has bounded clique width. Thus,  $G$  has bounded clique width by Observation 2.2.  $\square$

*Proof of Theorem 2.4.* Let  $G$  be a  $(4K_1, \text{even hole})$ -free graph with a  $C_7$ . By Theorem 2.3,  $G$  has bounded clique width. By Theorem 2.1,  $G$  can be optimally colored in polynomial time.  $\square$

So, if our graphs contain a  $C_7$ , we know how to color them. If they do not contain a  $C_7$ , then we know they must contain a  $C_5$ , for otherwise they are perfect and we would know how to color them. In the next section, we discuss the case of the  $C_5$ .

## 4 When the graphs contains a $C_5$

In this section, we assume  $G$  is  $(4K_1, C_4, C_6)$ -free. For all the claims below, we will also assume that  $G$  contains an induced  $C_5$  with vertices  $(i, i+1, i+2, i+3, i+4)$ . Let  $R$  denote the set of 0-vertices for this  $C_5$ , let  $F_i$  be the set of 1-vertices adjacent to  $i$ , let  $T_i$  be the set of 2-vertices with neighbors  $(i, i+1)$ , let  $X_i$  be the set of 3-vertices with neighbors  $(i, i+1, i+2)$  and let  $W$  denote the set of 5-vertices.

The following observation is immediate.

**Observation 4.1** *The sets  $F_i, T_i, X_i, R, W$  form a partition of the vertex set of  $G - C_5$*   
 $\square$

**Observation 4.2** *Each of  $F_i, T_i, X_i, R, W$  form a clique.*

*Proof.* Consider two non-adjacent vertices  $x, y$  of  $G$ . If both  $x, y$  belong to  $F_i$ , then  $x$  and  $y$  and some two non-adjacent vertices of the  $C_5$  form a  $4K_1$ , a contradiction. Similarly, we can see that  $x, y$  cannot both belong to  $T_i$ , or to  $R$ . If  $x, y$  both belong to  $X_i$  or to  $W$ , then  $x, y$  and some two non-adjacent vertices of  $C_5$  form a  $C_4$ .  $\square$

**Observation 4.3**  $R \overset{(1)}{\circlearrowleft} F_i \cup T_i$ .

*Proof.* Consider a vertex  $r \in R$  and a vertex  $s \in F_i \cup T_i$ . If  $rs \notin E$ , then  $r, s$  and some two non-adjacent vertices in the  $C_5$  form a  $4K_1$ .  $\square$

**Observation 4.4** If  $F_i \neq \emptyset$  then  $F_j = \emptyset$  for all  $j \neq i$ .

*Proof.* Consider vertices  $f_i \in F_i$ ,  $f_j \in F_j$ , with  $i \neq j$ . We must have  $f_i f_j \in E$ , for otherwise  $f_i, f_j$  and some two vertices in the  $C_5$  form a  $4K_1$ . If  $j = i + 1$ , then there is a  $C_4 (f_i, f_j, j, i)$ . So we have  $j \neq i + 1$ , and by symmetry,  $j \neq i - 1$ . If  $j = i + 2$ , then there is a  $C_6 (f_i, f_j, j, j + 1, j + 2, i)$ . So we have  $j \neq i + 2$ , and by symmetry,  $j \neq i - 2$ .  $\square$

**Observation 4.5**  $F_i \textcircled{1} T_i \cup T_{i+2} \cup T_{i+4}$ .

*Proof.* Let  $f \in F_i$ , and let  $t \in T_i \cup T_{i+2}$ . If  $ft_i \notin E$ , then  $f, t$  and some two non-adjacent vertices of the  $C_5$  form a  $4K_1$ . So  $F_i \textcircled{1} T_i \cup T_{i+2}$ . By symmetry (with the case  $T_i$ ), we have  $F_i \textcircled{1} T_{i+4}$ .  $\square$

**Observation 4.6**  $F_i \textcircled{0} T_{i+1} \cup T_{i+3}$ .

*Proof.* Let  $f \in F_i$ , and let  $t \in T_{i+1}$ . If  $ft \in E$ , then there is a  $C_4 (f, t, i + 1, i)$ . So we have  $F_i \textcircled{0} T_{i+1}$ , and by symmetry,  $F_i \textcircled{0} T_{i+3}$ .  $\square$

**Observation 4.7**  $F_i \textcircled{0} X_{i+1}$ .

*Proof.* Let  $f \in F_i$ , and let  $x \in X_{i+1}$ . If  $fx \in E$ , then there is a  $C_4 (f, x, i + 1, i)$ . So we have  $F_i \textcircled{0} X_{i+1}$ .  $\square$

We note that vertices of  $F_i$  may have neighbors and non-neighbors in  $X_i \cup X_{i+3} \cup X_{i+4}$ .

**Observation 4.8**  $T_i \textcircled{0} T_j$  for all  $j \neq i$ .

*Proof.* Consider vertices  $t_i \in T_i, t_{i+1} \in T_{i+1}$ . If  $t_i t_{i+1} \in E$ , then there is a  $C_6 (t_i, t_{i+1}, i + 2, i + 3, i + 4, i)$ . So we have  $T_i \textcircled{0} T_{i+1}$ , and by symmetry,  $T_i \textcircled{0} T_{i+4}$ . Now consider a vertex  $t_{i+2} \in T_{i+2}$ . If  $t_i t_{i+2} \in E$ , then there is a  $C_4 (t_i, i + 1, i + 2, t_{i+2})$ . So we have  $T_i \textcircled{0} T_{i+2}$ , and by symmetry,  $T_i \textcircled{0} T_{i+3}$ .  $\square$

**Observation 4.9**  $T_i \textcircled{0} X_{i+2}$ .

*Proof.* Consider vertices  $t_i \in T_i, x_{i+2} \in X_{i+2}$ . If  $t_i x_{i+2} \in E$ , then there is a  $C_4 (t_i, x_{i+2}, i + 2, i + 1)$ .  $\square$

**Observation 4.10**  $X_i \textcircled{0} X_{i+2}$ .

*Proof.* Consider vertices  $x_i \in X_i, x_{i+2} \in X_{i+2}$ . If  $x_i x_{i+2} \in E$ , then there is a  $C_4 (x_i, x_{i+2}, i + 4, i)$ .  $\square$

In the next section, we will use the results of this section to prove Theorem 2.5.

## 5 Clique cutset decomposition

In this section, we present a proof of Theorem 2.5. We will need to introduce definitions and background for the problem.

Consider a graph  $G$ . A *clique cutset* of  $G$  is a set of vertices  $S$  such that  $S$  is a clique and  $G - S$  is disconnected. Consider the following procedure to decompose  $G$ . If  $G$  has a clique cutset  $C$ , then  $G$  is decomposed into subgraphs  $G_1 = G[V_1]$  and  $G_2 = G[V_2]$  where  $V = V_1 \cup V_2$  and  $C = V_1 \cap V_2$  ( $G[X]$  denotes the subgraph of  $G$  induced by  $X$  for a subset  $X$  of vertices of  $V(G)$ ). Given optimal colourings of  $G_1, G_2$ , we can obtain an optimal colouring of  $G$  by identifying the colouring of  $C$  in  $G_1$  with that of  $C$  in  $G_2$ . In particular, we have  $\chi(G) = \max(\chi(G_1), \chi(G_2))$ . If  $G_i$  ( $i \in \{1, 2\}$ ) has a clique cutset, then we can recursively decompose  $G_i$  in the same way. This decomposition can be represented by a binary tree  $T(G)$  whose root is  $G$  and the two children of  $G$  are  $G_1$  and  $G_2$ , which are in turn the roots of subtrees representing the decompositions of  $G_1$  and  $G_2$ . Each leaf of  $T(G)$  corresponds to an induced subgraph of  $G$  that contains no clique cutset; we will call such graph an *atom*. Algorithmic aspects of the clique cutset decomposition are studied in Tarjan [18] and Whiteside [19]. In particular, the decomposition tree  $T(G)$  can be constructed in  $O(nm)$  time such that the total number atoms is at most  $(n - 1)$  [18] (Here, as usual,  $n$ , resp.,  $m$ , denotes the number of vertices, resp., edges, of the graph  $G$ ). This discussion can be summarized by the theorem below.

**Theorem 5.1** ([18], [19]) *Let  $G$  be a graph. If every atom of  $G$  can be colored in polynomial time, then  $G$  can be colored in polynomial time.*  $\square$

We will need the following theorem that illustrates the structure of  $(4K_1, C_4, C_6, C_5$ -twin)-free graphs.

**Theorem 5.2** *Let  $G$  be a  $(4K_1, C_4, C_6, C_5$ -twin)-free graph. If  $G$  contains a  $C_5$ , then one of the following holds:*

- (i)  $G$  contains a clique cutset.
- (ii)  $G$  contains a  $C_7$ .
- (iii)  $G$  has bounded clique width.
- (iv)  $G$  is the join of a (possibly empty) clique and a  $C_5$ . In this case, (iii) is also satisfied.

*Proof of Theorem 5.2.* Let  $G$  be a  $(4K_1, C_4, C_6, C_5$ -twin)-free graph and suppose  $G$  contains a  $C_5$ . Assume that  $G$  contains no clique cutset and no  $C_7$ , for otherwise we are done. Define the sets  $F_i, T_i, X_i, W, R$  as above. Note that  $X_i = \emptyset$  for all  $i$  because

$G$  contains no  $C_5$ -twin. By Observation 4.4, at most one set  $F_i$  can be non-empty. We will assume that this one set, if it exists, is  $F_1$ . Define  $T = T_1 \cup \dots \cup T_5$ .

We will show that

$$R = \emptyset. \tag{1}$$

Suppose  $R \neq \emptyset$ . Note that  $W \cup T \cup F_1$  is a cutset, separating  $R$  from the  $C_5$ . Let  $C$  be a minimal  $(R, C_5)$ -separator of  $G$  that is contained in  $W \cup T \cup F_1$ . By assumption,  $C$  is not a clique. Consider two non-adjacent vertices  $a, b$  in  $C$ . By the minimality of  $C$ , there is a chordless path  $P$  with endpoints being  $a, b$ , and interior vertices belonging to  $R$ . Since  $R$  is a clique,  $P$  has at most three edges.

Suppose first that  $P$  has three edges. Enumerate the vertices of  $P$  as  $x, r_1, r_2, y$  with  $r_i \in R$ . Since  $r_2$  is not adjacent to  $a$ , by Observation 4.3, we have  $a \in W$ . Similarly, we have  $b \in W$ . But by Observation 4.2,  $ab$  is an edge, a contradiction.

So  $P$  has two edges. Enumerate the vertices of  $P$  as  $x, r, y$  with  $r \in R$ . Note that both  $a, b$  have neighbors in the  $C_5$ . No vertex  $c \in C_5$  can be adjacent to both  $a, b$ , for otherwise, there is a  $C_4$   $(c, a, r, b)$ . So we have  $a, b \in F_1 \cup T$ , in particular,  $a, b \notin W$ . Since  $ab$  is not an edge, by Observation 4.2, either  $a$  or  $b$ , or both, belongs to some  $T_s$ . We may assume  $b \in T_s$ , that is,  $b$  is a 2-vertex.

Let  $i$  be a vertex in the  $C_5$  that is adjacent to  $a$ . Let  $j$  be a vertex in  $C_5$  that is adjacent to  $b$  and is closest to  $i$  in the  $C_5$ . Then  $P' = (a, i, i+1, \dots, j)$  is an induced path. If  $P'$  has length at least four, the  $P \cup P'$  induces a chordless cycle of length at least six, a contradiction. So we know  $j = i+1$ . Since  $b \in T_s$  and  $bi$  is an edge, we know  $b(i+3), b(i-1) \notin E$ . It follows that  $b(i+2) \in E$ . The vertex  $a$  may or may not be adjacent to  $i-1$ . If  $a(i-1) \in E$ , let  $P''$  be the chordless path  $b, i+2, i+3, i-1, a$ ; otherwise, let  $P''$  be the chordless path  $b, i+2, i+3, i-1, i, a$ . Then  $P''$  and  $r$  together induces a  $C_6$  or  $C_7$ , a contradiction. We have established (1).

Next, we claim that

$$F \neq \emptyset. \tag{2}$$

Suppose  $F = \emptyset$ . If  $T = \emptyset$ , then  $G$  is the join of  $W$  and the  $C_5$ , and we are done. (Note that in this case  $G$  has clique width three). We may assume some  $T_i$  is not empty. By Observation 4.8, the vertices in  $T_i$  have no neighbors in  $T_j$  with  $i \neq j$ . So  $W \cup \{i, i+1\}$  is a clique cutset separating  $T_i$  from  $\{i+2, i+3, i+4\}$ , a contradiction. We have established (2).

Now, we may assume  $F = F_1$  is not empty. Suppose  $T_2 \neq \emptyset$ . By Observations 4.8 and 4.6,  $T_2$ 's neighbors belong to  $W \cup \{2, 3\}$ . But then  $W \cup \{2, 3\}$  is a clique cutset of  $G$ , a contradiction. So we have  $T_2 = \emptyset$ , and by symmetry,  $T_4 = \emptyset$ .

Suppose that  $T_3 \neq \emptyset$ . If  $|T_3| \geq 2$ , then there is a  $C_5$ -twin  $(f, 1, 2, 3, t, t')$  for any  $f \in F_1$  and  $t, t' \in T_3$  (by Observation 4.5,  $f$  is adjacent to  $t, t'$ ). So we have  $|T_3| = 1$ .

If  $T_1 \neq \emptyset$ , then there is a  $C_5$ -twin  $(f, t_3, 3, 2, t_1, 1)$  for any  $f \in F_1$ ,  $t_1 \in T_1$ , and  $t_3 \in T_3$ . So we have  $T_1 = \emptyset$ , and by symmetry,  $T_5 = \emptyset$ . Consider the graph  $G'$  obtained from  $G$  by removing the six vertices of  $C_5 \cup T_3$ . The partition  $W, F_1$  is a near-uniform partition of  $G'$ . By Theorem 2.6,  $G'$  has bounded clique width. By Observation 2.2,  $G$  has bounded clique width.

So, we may assume that  $T_3 = \emptyset$ . If  $T_1 = \emptyset$ , then  $\{1, 5\} \cup W$  is a clique cutset separation  $F_1 \cup T_5$  from  $\{2, 3, 4\}$ , a contradiction. So we have  $T_1 \neq \emptyset$ , and by symmetry,  $T_5 \neq \emptyset$ . Consider vertices  $t_5 \in T_5, f \in F_1, t_1 \in T_1$ . By Observations 4.8 and 4.5, we have  $ft_5, ft_1 \in E$ , and  $t_1t_5 \notin E$ . Now, there is a  $C_7$   $(t_5, f, t_1, 2, 3, 4, 5)$  and so (ii) holds.  $\square$

We are now in position to prove Theorem 2.5.

*Proof of Theorem 2.5.* Let  $G$  be a  $(4K_1, C_4, C_6, C_5\text{-twin})$ -free graph. We may assume that  $G$  is not perfect, for otherwise, we may use the algorithm of [12] to color  $G$ . Since  $G$  is  $C_4$ -free,  $G$  contains no anti-hole of length at least six. So  $G$  must contain a  $C_5$  or  $C_7$ . If  $G$  contains a  $C_7$ , then we are done by Theorem 2.4. So, we may assume that  $G$  contains a  $C_5$ , but no  $C_7$ . By Theorem 5.1, we only need to show that every atom of  $G$  can be colored in polynomial time. Let  $A$  be an atom of  $G$  (an induced subgraph with no clique cutset). By Theorem 5.2,  $A$  has bounded clique width, thus it can be colored in polynomial time by Theorem 2.1.  $\square$

## 6 Conclusions

In this paper, we studied the complexity of VERTEX COLORING for  $(4K_1, C_4, C_6)$ -free graphs. We showed the problem admits a polynomial time algorithm when the graph in our class has a  $C_7$ . We have not solved the problem when the graph contains a  $C_5$ . We leave this as an open problem. In addition, we designed a polynomial time algorithm for VERTEX COLORING for  $(4K_1, C_4, C_6, C_5\text{-twin})$ -free graphs. We note that the more general problem to color a  $(4K_1, C_4)$ -free graph in polynomial time is still open.

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