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On the core of ordered submodular cost games

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Abstract. A general ordertheoretic linear programming model for the study of matroid-type greedy algorithms is introduced. The primal restrictions are given by so-called weakly increasing submodular functions on antichains. The LP-dual is solved by a Monge-type greedy algorithm. The model offers a direct combinatorial explanation for many integrality results in discrete optimization. In particular, the submodular intersection theorem of Edmonds and Giles is seen to extend to the case with a rooted forest as underlying structure. The core of associated polyhedra is introduced and applications to the existence of the core in cooperative game theory are discussed.

Key words. core – *N*-person game – greedy algorithm – Monge property – order – polymatroid – poset – submodular

1. Introduction

The present investigation is motivated by two fundamental questions. The first arises from cooperative game theory, where so-called *convex* games (cf. Shapley [1971]) have the attractive property to possess not only a non-empty *core* but allow efficient optimization of linear functions over the core. Can this class of games be extended to a larger class with the same features?

Cores of convex games are also known as *base polytopes* of submodular structures (cf. Fujishige [1991]), for which the greedy algorithm is known to be a fundamental algorithmic optimization technique. Extending the work of Queyranne *et al*. [1993], it was shown in Faigle and Kern [1996] that the greedy algorithm for polymatroids and the Monge algorithm for transportation problems with a suitable cost structure are just algorithmic manifestations of the same primal-dual pair of linear programs involving submodular constraints and submodular costs respectively that can, more generally, be defined relative to an underlying order structure given by a rooted forest. Hence the question arises how this model generalizes to arbitrary (partial) orders.

It turns out that a generalization to arbitrary orders is not possible unless we impose some restrictions on the class of submodular functions under considerations. We show in Sect. 2 that a full analog of the fundamental algorithmic properties of the previous models can be obtained when we restrict ourselves to submodular functions that are *weakly*

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increasing relative to the underlying order structure. In Sect. 3, we derive integrality properties for the pairs of submodular linear programs, which offer the (primal and dual) greedy algorithm as an explanation for many min-max properties of discrete structures. In particular, we extend the Intersection Theorem of Edmonds and Giles [1977] from unordered ground sets to rooted forests.

The core of a submodular structure is introduced in Sect. 4. In contrast to the situation with (unordered) polymatroids, "maximal" feasible vectors may have different component sums. By definition, the core consists of the feasible vectors of maximal component sum. It can be shown that the greedy algorithm can be modified to optimize arbitrary linear functions over the core relative to a weakly increasing submodular function.

We discuss the relationship with the core of cooperative games in Sect. 5. Taking a different look than suggested by Bondareva's [1963] and Shapley's [1967] *balancedness conditions* we are able to tie the existence of the core of an (arbitrary) cooperative game to the integrality of an LP-relaxation of a natural partitioning problem for the groundset of "players". The special case of an order structure with a submodular function on the collection of antichains then yields a far-reaching extension of the classical convex games.

2. A Greedy Algorithm for a class of submodular programs

In this section, we extend the model of Faigle and Kern [1996] to a wider class of structures and show that the same greedy algorithm works optimally.

Let E be a (finite) set and consider the (partial) order $P = (E, \leq)$. With any $S \subseteq E$ we associate the *ideal* generated by S via

$$id(S) := \{x \in E \mid x \leq s \text{ for some } s \in S\}.$$

Denoting by S^+ the collection of maximal elements of the order P restricted to S, we note that S^+ is an *antichain, i.e.*, a subset of pairwise incomparable elements, and that every antichain A arises as $A = (id(A))^+$. So we can define two binary operations on the set $\mathcal{A}$ of antichains by setting for $A, B \in \mathcal{A}$,

$$\begin{aligned} A \vee B &:= (id(A) \cup id(B))^+ \\ A \wedge B &:= (id(A) \cap id(B))^+ . \end{aligned}$$

We remark that $(\mathcal{A}, \vee, \wedge)$ is a *distributive lattice* (see, *e.g.*, Birkhoff [1967]).

Let $f : \mathcal{A} \to \mathbb{R}$ be given. Throughout our investigations we will assume that f is *normalized*, *i.e.*, $f(\emptyset) = 0$. Let furthermore $c : E \to \mathbb{R}$ be a weighting of E and consider both the linear program *LP*:

$$\begin{aligned} &\max c^T x \\ &s.t. \;\; x(A) \;\leq\; f(A) \;\; \text{for all } A \in \mathcal{A}, \end{aligned} \tag{1}$$

where we use the shorthand notation $x(A) = \sum_{e\in A} x_e$ for vectors $x \in \mathbb{R}^E$, and its dual *DLP*:

$$\begin{aligned} \min \sum_{A\in\mathcal{A}} & f(A)y_A \\ s.t. \quad \sum_{A\ni e} y_A &= c_e \text{ for all } e \in E \\ y_A &\geq 0 \text{ for all } A \in \mathcal{A}. \end{aligned} \tag{2}$$

It is straightforward to see that the following algorithm yields a feasible solution for *DLP* (cf. Faigle and Kern [1996]).

(Dual) Greedy Algorithm:

Initialize: $y_A \leftarrow 0$ for all $A \in \mathcal{A}$;
$X \leftarrow E$;
$w \leftarrow c$;
$\pi \leftarrow \emptyset$;
Iterate: WHILE $X \neq \emptyset$ DO:
determine some $e \in X^+$ with w_e minimal ;
$y_{X^+} \leftarrow w_e$;
$\pi \leftarrow e\pi$;
$w_a \leftarrow [w_a - w_e]$ for all $a \in X^+$;
$X \leftarrow [X \setminus e]$;

A run of the Greedy Algorithm will produce a linear extension

$$\pi = e_1e_2 \dots e_n$$

of P, namely the reverse order in which the algorithm discards the elements of E. (Recall that a *linear extension* of P is a permutation $\pi = e_1e_2 \dots e_n$ of the groundset E such that $e_i \leq e_j$ in P implies $i \leq j$).

With the linear extension π we associate the *primally greedy* vector x^π as the (unique) vector $x \in \mathbb{R}^E$ satisfying for $i = 1, \dots, n$,

$$x(E_i^+) = f(E_i^+) ,$$

where $E_i = \{e_1, e_2, \dots , e_i\}$.

Denoting by y the greedy solution vector for *DLP* and letting the vector x^π be defined as above, it follows that

$$c^T x^\pi = f^T y .$$

Hence both x^π and y are optimal solutions for the linear programs *LP* and *DLP* whenever x^π is a feasible solution for *LP*. We will now introduce a class of structural constraints relative to which feasibility of x^π can be proved.

The function $f : \mathcal{A} \to \mathbb{R}$ is said to be *submodular* or *concave* (relative to P) if for all $A, B \in \mathcal{A}$,

$$f(A \vee B) + f(A \wedge B) \leq f(A) + f(B).$$

In order to illustrate this concept of concavity, consider the complete bipartite graph $K_{n,n}$ with nonnegative costs $c(i, j)$ on the edges (i, j). Recall that the costs are said to have the *Monge property* (cf. Burkard *et al.* [1996]) if for all i_1, i_2 and j_1, j_2,

$$c(i_1 \vee i_2, j_1 \vee j_2) + c(i_1 \wedge i_2, j_1 \wedge j_2) \leq c(i_1, j_1) + c(i_2, j_2) \; ,$$

where we set for any two integers $s, t \in \mathbb{N}$,

$$\begin{aligned} s \vee t &:= \max\{s, t\} \\ s \wedge t &:= \min\{s, t\} \; . \end{aligned}$$

It is straightforward to check that in this model of edge-weighted bipartite graphs the Monge property amounts exactly to the concavity of the cost function on the 2-element antichains. Furthermore, it is easy to extend the cost function with Monge property to a concave function defined for *all* antichains. For example, we may choose a constant M larger than any edge cost and assign to the singleton with index i the cost $c(i) = i \cdot M$.

Remark. In the special case where the order P is a union of pairwise disjoint linear orders, our model is essentially the submodular model of Queyranne *et al.* [1993].

□

Remark. A function f is *supermodular* (a.k.a. "*convex*") if $(-f)$ is submodular. If f is defined for all subsets of E and $f(\emptyset) = 0$ holds, then $f^*(S) = f(E) - f(E \setminus S)$ gives rise to a function f^* such that $(f^*)^* = f$. Moreover, f is convex if and only if f^* is concave.

Reversing the inequalities appropriately, it is straightforward to see that one may obtain a theory for cores associated with supermodular functions that is completely analogous to our submodular model here. Cooperative game theory traditionally prefers the model of *convex* games (where a "profit" is to be allocated) to the concave "cost" model (see Shapley [1971]). It is not difficult to verify that the "concave core" of f equals the "convex core" relative to f^*.

□

Unfortunately, submodularity of f is not necessarily sufficient to guarantee feasibility of x^π (cf. Example 4.1 in Faigle and Kern [1996]). We require f to satisfy an additional condition.

Recall that $b \in E$ is an *upper neighbor* of $a \in E$ relative to P if $a < b$ holds in P and there is no $c \in E$ with $a < c < b$. We say that the function $f : \mathcal{A} \to \mathbb{R}$ is *weakly increasing* if for every $e \in E$ with at least 2 upper neighbors relative to P the following property holds:

$$A \cup e \in \mathcal{A} \;\text{ implies }\; f(A \cup e) \geq f(A) \; .$$

For example, f is trivially weakly increasing if every element of E has at most one upper neighbor relative to P (which is the defining property of the *rooted forests* investigated in Faigle and Kern [1996]).

For our feasiblity proof, we need a technical lemma. So, for some minimal element $e \in E$, consider the induced order $P' = P \setminus \{e\}$ on the ground set $E' = E \setminus \{e\}$ and denote by $\mathcal{A}'$ the collection of antichains of P'. We define the function $f' : \mathcal{A}' \to \mathbb{R}$ via

$$f'(A) := \begin{cases} f(A \cup e) - f(e) & \text{if } A \cup e \in \mathcal{A} \\ f(A) & \text{otherwise.} \end{cases}$$

Lemma 1. *Assume that* $f : \mathcal{A} \to \mathbb{R}$ *is submodular and weakly increasing, and let* $e \in E$ *be a minimal element with respect to* P*. Then also* $f' : \mathcal{A}' \to \mathbb{R}$ *is weakly increasing and submodular.*

Proof. The minimality of e and the submodularity of f together immediately imply $f(A) \geq f'(A)$ for all $A \in \mathcal{A}'$. Hence it is straightforward to see that f' is weakly increasing. We want to show that f' is submodular.

Let $A, B \in \mathcal{A}'$ be arbitrary antichains. If $A \cup e$ was an antichain relative to P, then also $(A \wedge B) \cup e$ was in $\mathcal{A}$. Moreover, $(A \vee B) \cup e$ was an antichain in P if and only if $B \cup e$ was an antichain. So in either case, f' satisfies the submodular inequality for A and B because f was submodular.

We may therefore assume that neither $A \cup e$ nor $B \cup e$ (and hence nor $(A \vee B) \cup e$) are antichains in P. If also $(A \wedge B) \cup e$ was no antichain, f' coincides with f relative to (A, B) and submodularity follows.

Consider finally the case where $(A \wedge B) \cup e$ was indeed an antichain in P. This can only mean that $(A \wedge B) \cup e$ is precisely the infimum of A and B relative to the lattice $\mathcal{A}$ of antichains. So the submodularity of f yields under the present conditions

$$f'(A \vee B) + f((A \wedge B) \cup e) \;\leq\; f'(A) + f'(B)\ .$$

It suffices now to show that $f'(A \wedge B) \leq f((A \wedge B) \cup e)$, *i.e.*, $f(e) \geq 0$, holds. To this end, we note that under the present conditions the element e necessarily must have at least 2 upper neighbors. (If e had only one upper neighbor e', say, e' would be dominated by members of both A and B and $A \wedge B \cup e$ would be no antichain). Because f is weakly increasing, we thus conclude that $f(e) \geq f(\emptyset) = 0$, as required.

$\square$

Theorem 1. *Assume that* $f : \mathcal{A} \to \mathbb{R}$ *is submodular and weakly increasing, and let* π *be a linear extension of* E *relative to the order* $P = (E, \leq)$*. Then the vector* x^π *satisfies for all* $A \in \mathcal{A}$*,*

$$x(A) \;\leq\; f(A)\ .$$

Proof. We proceed by induction on the size $|E|$ of the underlying ground set E. Note that $\pi' = e_2, e_3, \ldots, e_n$ is a linear extension of P' whenever $\pi = e_1, e_2, \ldots, e_n$ is a linear extension of P.

Fix the minimal element e_1 and define the function f' as in Lemma 1 relative to $e = e_1$. By induction, we may assume for all $A \in \mathcal{A}'$,

$$x^{\pi'}(A) \;\leq\; f'(A)\ .$$

By construction, we have $x^\pi = (f(e_1), x^{\pi'})$. Hence

$$x^\pi(A) \;\leq\; f(A)$$

must hold for all $A \in \mathcal{A}$.

$\square$

Corollary 1. *If* f *is submodular and weakly increasing, then the greedy algorithm solves the linear programs LP and DLP optimally.*

$\square$

Remark. The construction of the vector x^π reduces to the greedy algorithm of Edmonds [1970] in the case of a trivial order P (see also Ichishi [1981]). In the general case, however, it is *not* "greedy" in the sense that it would build up a linear extension x^π by successively adjoining elements with largest possible weights. In fact, such a naive "greedy algorithm" does not work (cf. the next example). Our greedy algorithm is motivated rather by the well-known *NW-corner rule* or *Monge greedy algorithm* for the bipartite assignment problem (cf. Burkard *et al.* [1994]).

□

Example. Consider the set $E = \{a, b, c, d\}$ and order P with the only non-trivial order relations $a < d$ and $b < c$. Let $\mathcal{A}$ consist of all antichains of P and define $f : \mathcal{A} \to \mathbb{R}$ by $f(\emptyset) = 0$ and $f(A) = 1$ otherwise.

Relative to the weighting $w_a = 5, w_b = 4, w_c = 3, w_d = 1$, the "naive" greedy algorithm would construct the linear extension $\pi = abcd$ with associated vector $x^\pi = (1, 0, 0, 1)$. The linear extension $\psi = bacd$, however, yields a better vector $x^\psi = (0, 1, 1, 0)$.

□

It is not difficult to extend our model to more general families $\mathcal{A}$ of antichains that are closed under the operations $\vee$ and $\wedge$ as follows.

Let $\mathcal{D}$ denote the family of *all* ideals of P and let $\mathcal{L} \subseteq \mathcal{D}$ be a subfamily that is closed under union and intersection. Set

$$\mathcal{A}(\mathcal{L}) \ := \ \{L^+ \mid L \in \mathcal{L}\} \ .$$

Note that $\mathcal{A}(\mathcal{L})$ is closed under $\vee$ and $\wedge$. If the corresponding linear program $LP(\mathcal{L})$ has an optimal solution at all, we may assume w.l.o.g. that each element of E occurs in some antichain in $\mathcal{A}(\mathcal{L})$ (thus, in particular, $E \in \mathcal{L}$).

Let us say that the elements $e, f \in E$ are *equivalent* ($e \sim f$) (relative to $\mathcal{A}(\mathcal{L})$) if for every $A \in \mathcal{A}(\mathcal{L})$, $e \in A$ holds exactly when $f \in A$ is true. Set

$$[e] \ := \ \{f \in E \mid f \sim e\}.$$

Lemma 2. *For all $e \in E^+$, $E \setminus [e] \in \mathcal{L}$.*

Proof. Observe that $[e] \subseteq E^+$ holds. Thus $E \setminus [e]$ is an ideal in P (and hence a member of $\mathcal{D}$).

Suppose there exists some $L \in \mathcal{L}$ that properly contains $E \setminus [e]$. Then there must exist some $f \in [e]$ with $f \in L$. Because $f \in L^+$, we conclude $[e] \subseteq L$, *i.e.*, $L = E$. So $E \setminus [e]$ is the intersection of all members of $\mathcal{L}$ containing it and, therefore, also itself a member of $\mathcal{L}$.

□

Lemma 3. *If* $\max\{\sum c_e x_e \mid x(A) \leq f(A)$ *for all* $A \in \mathcal{A}(\mathcal{L})\}$ *is bounded, then $c_e = c_f$ whenever $e \sim f$.*

□

Arguing with the equivalence classes $[e]$ instead of the elements e, Lemmas 2 and 3 now allow us to derive the analog of Lemma 1 and Theorem 1 in the same way.

3. Intersection of submodular structures

In this section, we will investigate integrality properties of linear programs that are defined by submodular functions on antichains or functions that are expressible as minima of pairs of submodular functions. We also allow for lower and upper bounds $l, u : E \to \mathbb{R} \cup \{-\infty, \infty\}$ on primally feasible vectors.

Let $P = (E, \leq)$ be an ordered set and denote by $\mathcal{A}$ the collection of all antichains. (We remark here that it suffices to require $\mathcal{A}$ just to be a collection of antichains that is closed under the operations $\vee$ and $\wedge$).

Consider an arbitrary function $h : \mathbb{R} \to \mathcal{A}$, with $h(\emptyset) = 0$, together with a weighting $c \in \mathbb{R}^E$. As in the previous section, we are interested in the linear program

$$\begin{array}{ll} \max & c^T x \\ s.t. & \sum_{e \in A} x_e \leq h(A) \text{ for all } A \in \mathcal{A} \\ & \quad x_e \leq u_e \quad \text{for all } e \in E \\ & \quad x_e \geq l_e \quad \text{for all } e \in E \end{array} \tag{3}$$

and its dual

$$\begin{array}{lllllll} \min & h^T y + u^T s - l^T t & & & & & \\ s.t. & \sum_{\substack{A \in \mathcal{A} \\ e \in A}} y_A + & s_e & - & t_e & = c_e & \text{for all } e \in E \\ & & & & y, s, t \geq 0\,. & & \end{array} \tag{4}$$

Say that the linear inequalities that occur as constraints in (3) and the linear inequalities occurring as constraints in (4) form a *totally dual integral* pair of linear inequalities provided the following is true: the maximum in (3) is achieved by an integral vector x if l, u and h are integral and, furthermore, the minimum in (4) is attained by an integral vector (y, s, t) if c is integral (provided both linear programs are feasible).

Theorem 2. *Assume that h is submodular and weakly increasing. Then (3) and (4) form a totally dual integral pair.*

Proof. Let (y^*, s^*, t^*) be an optimal solution of (4) and consider the linear program

$$\begin{array}{ll} \min & h^T u \\ s.t. & \sum_{\substack{A \in \mathcal{A} \\ e \in A}} u_A = c_e^* \text{ for all } e \in E \\ & \quad u \quad \geq 0, \end{array} \tag{5}$$

where $c^* = c - s^* + t^*$.

By the results of the previous section, (5) can be solved by the (dual) greedy algorithm. So we can assume that y^*, in fact, is this solution. Set

$$\mathcal{L} := \{A \in \mathcal{A} \mid y_A^* > 0\}\,.$$

Problem (5) now is equivalent to

$$\begin{array}{lll} \min \sum\limits_{A\in\mathcal{L}} h(A)y_A & & \\ s.t. \quad \sum\limits_{\substack{A\in\mathcal{L}\\ e\in A}} y_A & + s_e - t_e = c_e \text{ for all } e\in E \\ \quad\quad y,s,t & \geq 0. \end{array} \tag{6}$$

If $\mathcal{L} = \{A_1, \ldots, A_k\}$, we may assume $id(A_i) \subset id(A_j)$ for $i < j$. From this, it is easy to see that the matrix M with rows indexed by E and columns indexed by $\mathcal{L}$ such that

$$M_{e,A} = \begin{cases} 1 & \text{if } e\in A \\ 0 & \text{otherwise} \end{cases}$$

has the *consecutive* 1*'s property*, *i.e.*, in each row of M the 1's occur consecutively. Such a matrix is well-known to be totally unimodular (see, *e.g.*, Schrijver [1986, p.279]). (Recall that a matrix is said to be *totally unimodular* if the determinant of every square submatrix takes on a value in $\{0, -1, 1\}$.)

Because M is totally unimodular and the columns associated with the variables s and t correspond to identity matrices, it is clear that the constraints of (6) are totally unimodular. Hence (6) and, therefore, (5) has an integral optimal solution if c is an integral vector.

Finally, if l, u and h are integral, our argument shows that the optimal objective function value of (5) and, by linear programming duality, of (3) is an integer whenever c is integral. By Theorem (2.9) of Hoffman [1982], the latter implies that the vertices of the feasibility region of the linear program (3) are integral, which yields the theorem.

□

Theorem 2 allows us, for example, to derive the Theorem of Greene [1976] in the same spirit as in Hoffman [1982]:

Example. For every fixed $k \in \mathbb{N}$, the function $h(A) = k$ for $A \neq \emptyset$ is submodular and (trivially) weakly increasing on the collection of antichains. Consider the case where $c_e = 1, l_e = 0$, and $u_e = 1$ for all $e \in E$.

By Theorem 2, (4) has an optimal integral solution. It is clear that this solution must have $(0, 1)$-components and thus corresponds to a partition of E into antichains that is minimal relative to the weight function $d_k(A) = \min\{k, |A|\}$.

An integral solution x for (3) necessarily has $(0, 1)$-components and corresponds to a subset $X \subseteq E$ that contains no antichain of size larger than k. By the Theorem of Dilworth [1950], such a subset X can be covered by k (or less) chains relative to the order P on E.

The equality of the optimal objective function values in (3) and (4) now yields Greene's Theorem.

□

We would like to extend Theorem 2 to the case where h can be expressed as the minimum of two submodular functions. The difficulty thereby is that the matrix M

occurring in the proof will in general not be totally unimodular. Therefore, we restrict the class of orders P under consideration to the class of rooted forests.

We say that a collection $\mathcal{C} \subseteq \mathcal{A}$ of antichains is a *chain* in $\mathcal{A}$ if for all $A, B \in \mathcal{C}$ either $id(A) \subseteq id(B)$ or $id(B) \subseteq id(A)$ holds. For our next lemma, we consider two chains $\mathcal{C}_1$ and $\mathcal{C}_2$ of antichains and the incidence matrix M with rows indexed by E and columns indexed by $\mathcal{L} = \mathcal{C}_1 \cup \mathcal{C}_2$, where

$$M_{e,A} \;=\; \begin{cases} 1 & \text{if } e \in A \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4. *Assume that the order P is a rooted forest and let M be the incidence matrix of two chains $\mathcal{C}_1$ and $\mathcal{C}_2$ of antichains relative to P. Then M is totally unimodular.*

Proof. Assume first that P is the trivial order with no proper comparabilities. Indexing the columns of M in increasing order relative to the cardinality of the members of $\mathcal{C}_1$ and in decreasing order relative to $\mathcal{C}_2$, it is clear that M has the consecutive 1's property and, therefore, is totally unimodular.

If P has non-trivial comparabilities, the idea is now to add some row to other rows of M so that the resulting matrix M' is the incidence matrix relative to an order P' with strictly fewer comparabilities than P. By induction on the number of comparabilites, we can then assume that M' is totally unimodular. Because M can be recovered from M' by elementary row operations, also M must be totally unimodular.

Choose some $e \in E^+$ with at least one lower neighbor and let $\{e_1, \dots, e_k\}$ be the set of all lower neighbors of e. Then $M_{e,A} = 1$ in M implies $M_{e_i,A} = 0$ for $i = 1, \dots, k$. Hence we can add the row e of M to each of the rows $e_1, \dots, e_k$ and obtain a (0,1)-matrix M'.

Let P' be the order that coincides with P on the set $E \setminus e$ but has e incomparable with every element in $E \setminus e$. Because P is a rooted forest, also P' is a rooted forest and the elements $e_1, \dots, e_k$ are maximal relative to P'. Hence, if A is an antichain in P with $e \in A$, then $A \cup \{e_1, \dots, e_k\}$ is an antichain in P'.

Replace now each antichain A with $e \in A$ in $\mathcal{C}_i$ $(i = 1, 2)$ by $A \cup \{e_1, \dots, e_k\}$. This yields two chains $\mathcal{C}'_1$ and $\mathcal{C}'_2$ of antichains relative to P' with incidence matrix M'.

□

Note that Lemma 4 may fail to hold if P is not a rooted forest as the following example demonstrates.

Example. Let P on $E = \{a, b, c, d, e\}$ be given by the non-trivial order relations $a < b$, and $a < c$. Consider the two chains $\mathcal{C}_1 = \{ad, bd, bcd\}$ and $\mathcal{C}_2 = \{ae, cde\}$ of antichains and let M be the corresponding incidence matrix. If γ is the sum of the columns of M, each component of γ is an even integer. So $\gamma/2$ is integral and

$$Mx \;=\; \gamma/2 \;\;,\;\; x \;\geq\; 0$$

is feasible but has no integral solution as is straightforward to check.

□

Our next result generalizes the intersection theorem of Edmonds and Giles [1977] from trivially ordered sets to rooted forests.

Theorem 3. *Let P be a rooted forest with collection $\mathcal{A}$ of antichains and let $f, g : \mathcal{A} \to \mathbb{R}$ be submodular. If $h : \mathcal{A} \to \mathbb{R}$ satisfies $h(A) = \min\{f(A), g(A)\}$ for all $A \in \mathcal{A}$, then (3) and (4) form a totally dual integral pair of linear inequalities.*

Proof. We re-write the linear programs as

$$\begin{array}{lrcll} \max & c^T x & & & \\ s.t. & \sum_{e\in A} x_e & \leq & f(A) & \text{for all } A \in \mathcal{A} \\ & \sum_{e\in A} x_e & \leq & g(A) & \text{for all } A \in \mathcal{A} \\ & x_e & \leq & u_e & \text{for all } e \in E \\ & x_e & \geq & l_e & \text{for all } e \in E \end{array} \tag{7}$$

and

$$\begin{array}{ll} \min & f^T y + g^T z + u^T s - l^T t \\ s.t. & \sum_{\substack{A \in \mathcal{A} \\ e \in A}} y_A + z_A + s_e - t_e = c_e \text{ for all } e \in E \\ & y, z, s, t \geq 0 \,. \end{array} \tag{8}$$

Let (y^*, z^*, s^*, t^*) be an optimal solution for (8). Considering the modified vectors $c_y^* = c - z^* - s^* + t^*$ and $c_z^* = c - y^* - s^* + t^*$, we conclude as in the proof of Theorem 2 that the supports

$$\mathcal{L}_y = \{A \in \mathcal{A} \mid y_A^* > 0\}$$

$$\mathcal{L}_z = \{A \in \mathcal{A} \mid z_A^* > 0\}$$

can be assumed to be chains of antichains relative to P.

Let M be the incidence matrix of E *vs.* $\mathcal{L}_y \cup \mathcal{L}_z$. By Lemma 4, M is totally unimodular. Hence the theorem follows with exactly the same argument as in the proof of Theorem 2.

□

4. The core of submodular polyhedra

In this section, we let P be an arbitrary order on the set E with family $\mathcal{A}$ of antichains and assume that the function $f : \mathcal{A} \to \mathbb{R}$ is normalized, *i.e.*, $f(\emptyset) = 0$, submodular and weakly increasing. So we know that every primally greedy vector is feasible for the linear program LP .

In contrast to the situation in classical submodular structures, *i.e.*, the case where P is the trivial order on E, different Greedy vectors may have different component sums.

Example. Let $E = \{a, b, c\}$ and P have the only non-trivial order relation $b < c$. Define $f(A) = 1$ for every non-empty antichain A. With respect to the linear extensions $\pi = abc$ and $\psi = bac$ of P, we obtain the greedy vectors $x^\pi = (0, 1, 0)$ and $x^\psi = (1, 0, 1)$ and observe $x^\pi(E) = 1 < 2 = x^\psi(E)$.

□

Generalizing the notion of a *base polyhedron* of a polymatroid (see Fujishige [1991]), we define the *core* of f to be set $core(f)$ of all optimal solutions to the following linear program

$$\begin{array}{ll} \max & x(E) \\ s.t. & x(A) \leq f(A) \text{ for all } A \in \mathcal{A} \end{array} \tag{9}$$

Note that $core(f)$ is a bounded polyhedron. We denote by $F(E)$ the component sum $x(E)$ for any $x \in core(f)$.

Our aim is to show that the Greedy Algorithm of Sect. 2 can be modified to optimize any linear objective function over $core(f)$. In the context of the k-chain covering problem discussed in Sect. 2, our problem here is to determine among all k-chain covers *of maximal cardinality* one of maximal weight.

The problem of maximizing the linear function $w^T x$ over $core(f)$ is the linear programming problem *CLP*:

$$\begin{array}{llcll} \max & w^T x & & & \\ s.t. & x(A) & \leq & f(A) & \text{for all } A \in \mathcal{A} \\ & x(E) & = & F(E)\ . & \end{array} \tag{10}$$

For any nonnegative parameter value λ, we consider the Lagrangian parametrization $L(\lambda)$ of our problem:

$$\begin{array}{llcll} \max & w^T x + \lambda(x(E) - F(E)) & & & \\ s.t. & x(A) & \leq & f(A) & \text{for all } A \in \mathcal{A}. \end{array} \tag{11}$$

Instead of $L(\lambda)$, we may, equivalently, solve the linear program $LP(\lambda)$:

$$\begin{array}{ll} \max & w(\lambda)^T x \\ s.t. & x(A) \leq f(A) \text{ for all } A \in \mathcal{A}, \end{array} \tag{12}$$

where $w(\lambda)_e = w_e + \lambda$ for all $e \in E$. Clearly, every optimal solution x^* for $LP(\lambda)$ with $x^*(E) = F(E)$ will be an optimal solution for our original problem.

Problem $LP(\lambda)$ can be solved with the Greedy Algorithm from Sect. 2. The difficulty is, however, that the solution x thus obtained does not necessarily satisfy the condition $x(E) = F(E)$. We have to introduce more terminology.

The order P decomposes into a (unique) *standard partition* of antichains

$$P = \bigcup_{j \geq 0} P_j$$

with $P_0 := E^+$ and $P_j := (E \setminus (P_0 \cup \ldots \cup P_{j-1}))^+$ for $j \geq 1$.

A linear extension $\pi = e_1, e_2, \ldots, e_n$ of the elements of E is *standard* if every element of P_j occurs after any element of P_{j+1} for all $j \geq 0$, *i.e.*, if π respects the standard partition (the maximal elements come last).

Proposition 1. *Assume that $f : \mathcal{A} \to \mathbb{R}$ is given as above and that the linear extension $\pi = e_1 e_2 \ldots e_n$ of E is standard. Then x^π lies in $core(f)$.*

Proof. Relative to the unit weighting $w_e = 1$ and $\lambda = 0$, every standard linear extension π is in accordance with the Greedy Algorithm.

$\square$

Returning to the arbitrary weighting w, it is straightforward to see that the Greedy Algorithm can be forced to generate a standard linear extension if λ is large enough.

Proposition 2. *Given w, there exists a $\lambda^* \geq 0$ such that for all $\lambda \geq \lambda^*$, the Greedy Algorithm for $DLP(\lambda)$ generates a standard linear extension of E.*

Proof. Choose $\lambda^* \geq 3\,W$, where $W = \sum_{e \in E} |w_e|$.

In the first step, the algorithm will select an element e_n in the topmost block P_0 of the standard partition with smallest weight w_{e_n} and reduce the weights of the elements in P_0 by $\lambda^* + w_{e_n}$. By the choice of λ^*, the algorithm will proceed to remove all other elements in P_0 .

Then an element e_i in the next block P_1 is selected with smallest current reduced weight w'_{e_i} and the weight of the elements in P_1 is reduced by $\lambda^* + w'_{e_i}$. Because none of the remaining elements in P_1 will ever have a reduced weight larger than W, the algorithm will remove all of P_1 before proceeding to P_2 *etc*.

$\square$

In view of the preceding discussion, we may solve the core optimization problem *CLP* above as follows.

We choose λ^* as in Proposition 2 and apply the Greedy Algorithm to $DLP(\lambda^*)$. The latter yields a standard linear extension π. The associated vector x^π will be an optimal solution for the problem $LP(\lambda^*)$. Because $x^\pi(E) = F(E)$ holds, x^π is also an optimal solution for *CLP*.

For an actual implementation of the Greedy Algorithm, the value λ^* does not have to be computed explicitly. Consider the following algorithm.

Modified Greedy Algorithm:

Initialize: $X \leftarrow E$;
$Z \leftarrow E^+$;
$\pi \leftarrow \emptyset$;
$w \leftarrow w(\lambda)$;
Iterate: WHILE $X \neq \emptyset$ DO:
determine some $e \in Z$ with w_e minimal ;
$\pi \leftarrow e\pi$;
$w_a \leftarrow [w_a - w_e]$ for all $a \in X^+$;
$X \leftarrow [X \setminus e]$;
$Z \leftarrow [Z \setminus e]$;
IF $Z = \emptyset$ THEN $Z \leftarrow X^+$;

Note that, in contrast to the Greedy Algorithm of Sect. 2, the Modified Greedy Algorithm does not explicitly construct a dual solution but a linear extension π. The variable Z in the algorithm assures that π follows the standard partition of P and hence will be standard.

For the choice $\lambda = \lambda^*$ the Greedy Algorithm and the Modified Greedy Algorithm generate the same (standard) π. Moreover, the Modified Greedy Algorithm yields the same π for any choice of λ. Indeed, the size of λ^* only ensures that the Greedy Algorithm follows the standard partition. Within a block of the standard partition, the selection of an element e is carried out according to the reduced size w'_e relative to the original weighting w and is independent of the size of λ^*. So it suffices to run the Modified Greedy Algorithm with $\lambda = 0$ in order to generate the optimal standard linear arrangement π.

We summarize in the following theorem.

Theorem 4. *Given w, the Modified Greedy Algorithm generates a standard linear extension π of E. Moreover, if $f : \mathcal{A} \to \mathbb{R}$ is submodular weakly increasing, the associated vector x^π is an optimal solution for the core optimization problem CLP.*

□

5. Applications to cooperative game theory

The basic model of cooperative game theory comprises a set N of *players* the subsets $S \subseteq N$ of which are *coalitions*. There is a *characteristic function* $v : 2^N \to \mathbb{R}$ that assigns to each coalition S its *value* $v(S)$. We assume v to be normalized, *i.e.*, $v(\emptyset) = 0$. In our presentation here, we will furthermore think of $v(S)$ as the *cost* generated by S. A *solution concept* is a method to divide the value $v(N)$ of the grand coalition N, *i.e.*, the total cost, among the individual players in a “fair” way.

The concept of the *core* of a game goes back to von Neumann and Morgenstern [1944] and suggests to allocate a vector $x \in \mathbb{R}^N$ such that no coalition S is allocated more than its true cost, *i.e.*, such that $x(S) > v(S)$ does not occur. The core $Core(v)$ is thus defined to be the polyhedron consisting of all vectors $x \in \mathbb{R}^N$ that satisfy the following system of inequalities:

$$\begin{aligned} x(S) &\le v(S) \quad \text{for all } S \subseteq N \\ x(N) &\ge v(N) \ . \end{aligned} \tag{13}$$

In a slightly more general model, we assume v to be given for a subfamily $\mathcal{E}$ of *essential* coalitions. Then $v : \mathcal{E} \to \mathbb{R}$ induces the characteristic function $\bar{v} : 2^N \to \mathbb{R}$ via

$$\bar{v}(S) := \min \left\{ \sum_j v(E_j) \mid E_j \in \mathcal{E}, E'_j\text{s partition } S \right\}$$

with the understanding that $\bar{v}(\emptyset) = 0$ and $\bar{v}(S) = \infty$ if S cannot be partitioned into members of $\mathcal{E}$. We call the cooperative game $(N, \bar{v})$ arising from $(N, \mathcal{E}, v)$ a *partition game*.

Note that $\bar{v}$ is *subadditive*, *i.e.*, $S \cap T = \emptyset$ implies

$$\bar{v}(S \cup T) \le \bar{v}(S) + \bar{v}(T) \ .$$

Moreover, $\bar{v} = v$ holds if and only if v is subadditive on $\mathcal{E} = 2^N$. In general, we observe

Proposition 3. *Assume $\mathcal{E} = 2^N$. Then*

$$Core(v) = \begin{cases} \emptyset & \text{if } v(N) > \bar{v}(N) \\ Core(\bar{v}) & \text{if } v(N) = \bar{v}(N) \ . \end{cases}$$

□

It is generally a non-trivial problem to decide whether $v(N) = \bar{v}(N)$ holds.

Example. (Deng and Papadimitriou [1994]). Let G be an edge-weighted complete graph on the node set N. For each $S \subseteq N$, take $w(S)$ to be the sum of the edge-weights in the subclique induced by S and let the value of S be given by $v(S) = w(N) - w(N \setminus S)$. Then $v(N) = \bar{v}(N)$ holds if and only if G contains no negative cut, which is *NP*-hard to decide. Moreover, $Core(v) \neq \emptyset$ holds if and only if $v(N) = \bar{v}(N)$.

□

Recall that the classical *balancedness conditions* of Bondareva [1963] exhibit the $Core(v)$ of a cooperative game (N, v) to be non-empty if and only if for every integer $m \in \mathbb{N}$ and subsets $S_1, \ldots, S_m$ of N,

$$\frac{1}{m} \sum_i 1_{S_i} \ = \ 1_N \ \text{ implies } \ \frac{1}{m} \sum_i v(S_i) \ \geq \ v(N) \ ,$$

where 1_S denotes the characteristic function of $S \subseteq N$.

For our purposes, it is important that the balancedness conditions can be replaced by the simple to state existence condition of an optimal solution with *integral* components for a related linear program. (This observation implies, for example, also the main result in Sharkey [1990]). The condition says that the value $\bar{v}(N)$ can be computed via the natural linear programming relaxation.

Theorem 5. *Let $(N, \mathcal{E}, v)$ be a partition game. Then $Core(\bar{v}) \neq \emptyset$ if and only if the following linear program has an integral optimal solution:*

$$\begin{aligned} \min & \sum_{E \in \mathcal{E}} v(E) y_E \\ s.t. & \qquad \sum_{E \ni e} y_E = 1 \quad \text{for all } e \in N \\ & \qquad\quad y_E \geq 0 \quad \text{for all } E \in \mathcal{E} \ . \end{aligned} \tag{14}$$

Proof. Consider the associated linear programming dual (D):

$$\begin{aligned} & \max \ x(N) \\ & s.t. \ \ x(E) \leq v(E) \quad \text{for all } E \in \mathcal{E} \ . \end{aligned} \tag{15}$$

Since each partition of N into members of $\mathcal{E}$ yields an (integral) feasible solution for the linear program in the statement of the theorem, linear programming duality implies $x(N) \leq \bar{v}(N)$ for every feasible solution x for (D).

Hence a feasible solution x for (D) with $x(N) = \bar{v}(N)$ exists if and only if a minimal cost partition of N corresponds to an optimal linear programming solution.

□

Corollary 2. *If Core($\bar{v}$) $\neq \emptyset$, then Core($\bar{v}$) is exactly the set of optimal solutions for the linear program*

$$\begin{aligned} &\max\ x(N) \\ &s.t.\ \ x(E) \leq v(E) \quad \textit{for all } E \in \mathcal{E}\ . \end{aligned} \tag{16}$$

□

Corollary 3. *Assume that the value function v of a game is submodular and weakly increasing relative to some order P. Then Core($\bar{v}$) $\neq \emptyset$.*

Proof. The integrality condition of Theorem 5 is implied by total dual integrality and the choice $c^T = (1, \ldots, 1)$.

□

For the same reason, games whose value function satisfies the conditions of Theorem 3 are seen to have a non-empty core.

From Corollary 2 and Corollary 3 we conclude that, in the case of submodularity, the gametheoretic notion of *core* coincides with the notion of the *core* introduced in Sect. 4. Hence we may employ the Modified Greedy Algorithm of the previous section in order to compute the optimal core vector relative to a linear "utility function" $c^T x$ on the set N of players.

We illustrate an application of the integrality results of Sect. 3 to a generalization of the classical assignment games to 3 dimensions. Generalizations to higher dimensions are then straightforward to obtain.

Example (3d-Assignment Game). Let N be the set of $3n$ nodes of the complete 3-partite graph $K_{n,n,n}$ and $\mathcal{E}$ the collection of all triples (i, j, k), $1 \leq i, j, k \leq n$. A (partial) 3*d-assignment* is a collection $\mathcal{M} \subseteq \mathcal{E}$ of pairwise non-incident triples (with respect to each of the three components).

Let a cost function $h : \mathcal{E} \to \mathbb{R}$ be given and define the *cost* $\bar{h}(S)$ of a non-empty subset $S \subseteq N$ as the cost of a minimal assignment covering S if such an assignment exists and "∞" otherwise. Then $\bar{h}$ defines a partition game on N.

$(N, \bar{h})$ need not have a non-empty core. (Otherwise, Theorem 5 would allow us to compute cost-minimal $3d$-assignments via a polynomial linear programming relaxation. Because the $3d$-assignment problem is *NP*-complete, $P = NP$ would then follow).

Considering three unrelated copies of the chain $\{1 < 2 \ldots < n\}$, one for each component, we obtain a rooted forest P on N, where each triple (i, j, k) corresponds to a 3-element antichain. Moreover, $\mathcal{E}$ is closed under the operations $\vee$ and $\wedge$ relative to P.

Theorem 3 now implies a non-empty core of $(N, \bar{h})$, whenever h can be expressed as the minimum of two submodular functions on $\mathcal{E}$.

□

The above example can be cast into the following gametheoretic setting. Let A,B, and C be three pairwise disjoint sets of "players" with $|A| = |B| = |C| = n$. The players are to form teams (a, b, c) with one member from each set. The "profit" gained from forming such a team is $p(a, b, c) \geq 0$. How should the teams be formed and the total profit be distributed among the 3n players in the best possible way?

Consider the associated "cost" function $h(a,b,c) = M - p(a,b,c)$, where M is an arbitrary constant. An equivalent problem formulation now asks for forming n teams such that the total cost relative to h is minimized and the total cost is distributed among the $3n$ players in an acceptable way.

If we define "acceptable" via the notion of the core, it is not clear whether an acceptable cost distribution exists at all. We describe a very special case, where Theorem 3 guarantees the non-emptyness of the core.

Assume that A, B, and C can be geometrically represented as points on three parallel lines in $\mathbb{R}^2$ and the points are ordered in the same direction along the lines. Each team (a,b,c) corresponds to a triangle $\Delta(a,b,c)$ with vertices a, b, and c on the three lines. If one associates with each triangle $\Delta(a,b,c)$ the sum $\gamma(a,b,c)$ of the pairwise Euclidean distances of its vertices, *i.e.*, its circumference, one obtains a submodular function on the collection of feasible teams.

If the constant C offers an alternative cost per team (due to another cost scheme for which a team may opt), then the induced cost function

$$h(a,b,c) = \min\{C, \gamma(a,b,c,)\}$$

satisfies the conditions of Theorem 3 because C is submodular on the collection of triangles.

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