# Perturbation Analysis of Singular Semidefinite Programs and Its Applications to Control Problems 

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#### Abstract

We consider sensitivity of a semidefinite program under perturbations in the case that the primal problem is strictly feasible and the dual problem is weakly feasible. When the coefficient matrices are perturbed, the optimal values can change discontinuously as explained in concrete examples. We show that the optimal value of such a semidefinite program changes continuously under conditions involving the behavior of the minimal faces of the perturbed dual problems. In addition, we determine what kinds of perturbations keep the minimal faces invariant, by using the reducing certificates, which are produced in facial reduction. Our results allow us to classify the behavior of the minimal face of a semidefinite program obtained from a control problem.


Keywords: Semidefinite programming, sensitivity, facial reduction, minimal face, H-infinity feedback control problem

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## 1 Introduction

A semidefinite program is the problem of maximizing a linear function subject to the constraint that an affine combination of matrices is positive semidefinite, where the constraint is called a linear matrix inequality. Semidefinite programs have various applications, such as discrete optimization, polynomial optimization and control problems (e.g. [1, 2] ). If the feasible sets of a semidefinite program and the dual problem satisfy the constraint qualifications, both of which are called strict feasibility, then interior point methods compute an approximation to an exact solution efficiently; see, e.g., [3, 4].

[^0]With a lack of strict feasibility, interior point methods are numerically unstable and often give wrong optimal values [5-7]. To avoid such numerical instability, we can use the technique called facial reduction, which finds the minimal face among the faces of the positive semidefinite cone containing a feasible set. Such a face is called the minimal face of a semidefinite program [8-11]. Then, we obtain a semidefinite program that satisfies strict feasibility and has the same optimal value as the original problem. For applications of facial reduction, see the monograph [12] and the references therein.

The first contribution of this paper is to provide sufficient conditions for continuity of the optimal value under perturbations, in the case that the primal problem is strictly feasible and the dual problem is feasible but not strictly feasible (Theorem 3.1). In that case, if we perturb the constant matrix in the constraint of the primal problem, then it can be shown from the general theory of convex analysis that the optimal value changes continuously [13, Corollary 7.5.1]. For more detailed analysis, see [14]. However if we also perturb the coefficient matrices of the variables, then the optimal value may change discontinuously (Example 3.3, 3.4). Here one of the keys to the phenomenon is the behavior of the minimal face of the dual problem under the perturbation. By using concrete examples, we argue that our sufficient conditions are hard to remove.

In the case that both of the primal and dual problems are strictly feasible, continuity of the optimal value can be shown by Gol'šhteĕn [15, Theorem 17]. Moreover if perturbations are restricted on the constant matrices in the constraint of a semidefinite program, several authors have studied stability of optimal solutions; see, e.g., [16-18]. Perturbation analysis of general nonlinear programming has been studied thoroughly by Bonnans and Shapiro [19].

The second contribution is to obtain sufficient conditions for the perturbations to keep the minimal face invariant (Proposition 4.3, 4.4). If the minimal face does not change under a perturbation, then one of the conditions in Theorem 3.1 is satisfied. These results give a new insight to perturbation analysis of semidefinite programs. Here we use reducing certificates, which are generated by facial reduction to find the minimal face [10]. We remark that reducing certificates are often obtained without solving semidefinite programs if the problems are generated from combinatorial optimization problems, matrix completion problems, sums of squares problems [12] or $H_{\infty}$ control problems [20]. Using these conditions, we investigate a semidefinite program generated from an $H_{\infty}$ state feedback control problem.

The organization of this paper is as follows: preliminaries on semidefinite programs and facial reduction are given in Section 2, In Section 3, we show the main result on continuity of the optimal value of a semidefinite program. In Section [ we give sufficient conditions on the perturbations under which the minimal face does not change. We devote Section 5 to applications to a control problem and numerical experiements. The conclusions are given in Section 6 .

## 2 Preliminaries on Semidefinite Program and Facial Reduction

### 2.1 Semidefinite Program

Let $\mathbb{S}^{n}, \mathbb{S}_{+}^{n}$ and $\mathbb{S}_{++}^{n}$ be the sets of $n \times n$ symmetric matrices, positive semidefinite matrices and positive definite matrices, respectively. In this paper, the primal semidefinite program (SDP)
$(D)$ and its dual (D) are formulated as follows:

$$
\begin{align*}
& \sup _{y, Z}\left\{b^{T} y: A_{0}-\sum_{k=1}^{m} y_{k} A_{k}=Z, y \in \mathbb{R}^{m}, Z \in \mathbb{S}_{+}^{n}\right\},  \tag{P}\\
& \inf _{X}\left\{A_{0} \bullet X: A_{k} \bullet X=b_{k}(k \in[m]), X \in \mathbb{S}_{+}^{n}\right\}, \tag{D}
\end{align*}
$$

where $A_{0}, A_{1}, \ldots, A_{m} \in \mathbb{S}^{n}, b \in \mathbb{R}^{m},[m]:=\{1, \ldots, m\}$, and the inner product $A \bullet B$ is defined by $\sum_{i, j=1}^{n} A_{i j} B_{i j}$ for $A, B \in \mathbb{S}^{n}$.

Problem ( $(P)$ is said to be strictly feasible if there exists a feasible solution $(y, Z)$ in $(\underline{P})$ such that $Z \in \mathbb{S}_{++}^{n}$. Problem (D) is said to be strictly feasible if there exists a feasible solution $X$ in (D) such that $X \in \mathbb{S}_{++}^{n}$. We say that (D) (resp. (D)) is weakly feasible, if (D) (resp. (D)) is feasible but not strictly feasible.

Throughout this paper, we deal with only the case where both (P) and its dual (D) are feasible. We say that $(P)$ is nonsingular if both $(P)$ and $(D)$ are strictly feasible and the coefficient matrices $A_{1}, \ldots, A_{m}$ are linearly independent. We say that $(P)$ is singular if the coefficient matrices are linear dependent or at least one of $(\underline{P})$ and $(D)$ is weakly feasible.

### 2.2 Facial Reduction for SDP

The definition of a face of a general convex set is provided in [13. The following lemma provides results on a facial structure of $\mathbb{S}_{+}^{n}$, e.g. [21, 10].

Lemma 2.1. 1. Any face of $\mathbb{S}_{+}^{n}$ is either the empty set, $\left\{O_{n \times n}\right\}, \mathbb{S}_{+}^{n}$, or

$$
\left\{Q\left(\begin{array}{cc}
O_{(n-r) \times(n-r)} & O_{(n-r) \times r} \\
O_{r \times(n-r)} & M
\end{array}\right) Q^{T}: M \in \mathbb{S}_{+}^{r}\right\},
$$

where $Q$ is an $n \times n$ nonsingular matrix.
2. The set $\mathbb{S}_{+}^{n}+F^{\perp}$ is closed for all faces $F$ of $\mathbb{S}_{+}^{n}$, where $F^{\perp}$ stands for the set $\left\{Z \in \mathbb{S}^{n}\right.$ : $Z \bullet X=0(\forall X \in F)\}$.

We call $Q$ in Part 1 of Lemma 2.1 the nonsingular matrix associated to the face. It follows from this property that for any $U \in \mathbb{S}_{+}^{n}$, the set $\mathbb{S}_{+}^{n} \cap\{U\}^{\perp}$ is a face of $\mathbb{S}_{+}^{n}$, where $\{U\}^{\perp}=$ $\left\{X \in \mathbb{S}^{n}: X \bullet U=0\right\}$. The property given in Part 2 of Lemma 2.1, which is called niceness, implies that $F^{*}=\mathbb{S}_{+}^{n}+F^{\perp}$ for all faces $F$ of $\mathbb{S}_{+}^{n}$. Here $F^{*}$ is the dual cone of $F$, i.e. $F^{*}=$ $\left\{Z \in \mathbb{S}^{n}: Z \bullet X \geq 0(\forall X \in F)\right\}$.

We define the minimal face of (D) and introduce facial reduction for (D). The minimal face of $(\bar{D})$ is defined as the intersection of all faces of $\mathbb{S}_{+}^{n}$ that contain the feasible region of (D). We denote the minimal face by $F_{\min }$. The following result on the minimal face is obtained by [10] and Part 2 in Lemma 2.1.

Lemma 2.2. [10, SDP version of Section 28.2.6 and Lemma 28.4] Assume that (D) and (D) are feasible. Let $F$ be a face of $\mathbb{S}_{+}^{n}$ that contains $F_{\min }$ and rint $F$ be its relative interior. Then the following are equivalent;

1. $F \neq F_{\min }$;
2. There exists $(y, U, V) \in \mathbb{R}^{m} \times \mathbb{S}_{+}^{n} \times F^{\perp}$ such that

$$
\begin{equation*}
b^{T} y=0, \quad-\sum_{k \in[m]} y_{k} A_{k}=U+V \text { and } U+V \notin F^{\perp} \tag{1}
\end{equation*}
$$

3. $\left\{X \in \operatorname{rint} F: A_{k} \bullet X=b_{k}(k \in[m])\right\}=\emptyset$.

If $U$ satisfies the system in (2, then we have $F_{\min } \subseteq F \cap\{U\}^{\perp} \subsetneq F$.
We call the above system (11) the discriminant system of the facial reduction for (D), and a solution $(y, U, V)$ a reducing certificate.

The facial reduction for SDP in e.g. [10, 11] is a procedure based on Lemma 2.2, It generates a sequence $\left\{F_{i}\right\}_{i=0}^{s}$ of faces of $\mathbb{S}_{+}^{n}$ such that

$$
F_{0}=\mathbb{S}_{+}^{n}, F_{i}=F_{i-1} \cap\left\{U^{i}\right\}^{\perp}(i=1, \ldots, s) \text { and } F_{s}=F_{\min }
$$

Therefore, the iterative process can be expressed as

$$
\mathbb{S}_{+}^{n}=F_{0} \xrightarrow{\left(y^{1}, U^{1}, V^{1}\right)} F_{1} \xrightarrow{\left(y^{2}, U^{2}, V^{2}\right)} F_{2} \xrightarrow{\left(y^{3}, U^{3}, V^{3}\right)} \ldots \xrightarrow{\left(y^{s}, U^{s}, V^{s}\right)} F_{s}=F_{\min },
$$

where we call $\left\{\left(y^{i}, U^{i}, V^{i}\right)\right\}_{i=1}^{s}$ a facial reduction sequence for (D). Here we note that $U^{i}, V^{i}$ need to satisfy $U^{i}+V^{i} \notin F_{i-1}^{\perp}$. Examples of facial reduction for SDP can be seen in e.g. 10, Example 28.3] and [11, Example 3.1].

If the discriminant system (11) has multiple solutions, then we have flexibility in choosing a facial reduction sequence for (D). Cheung and Wolkowicz [14, Proposition B.1] prove that any two facial reduction sequences must be of the same length when a reducing certificate $(y, U, V)$ is selected at each iteration so that $U$ has the maximal rank. The length is called the degree of singularity for (D). The degree of singularity is used in [14] for the sensitivity analysis of SDPs and in 22 for the error bounds.

Although we deal with only the feasible SDPs in the present paper, we introduce a study on the infeasibility briefly. Infeasibility of SDP has two types as well as feasibility, i.e. strong infeasibility and weak infeasibility. The authors in [23, 24] discuss a characterization of infeasibility by facial reduction.

## 3 Main Result

### 3.1 Stability of Singular Semidefinite Programs

We define the perturbed problems for ( $(P)$ by

$$
\begin{align*}
& \sup _{y, Z}\left\{b(t)^{T} y: \sum_{k \in[m]} y_{k} A_{k}(t)+Z=A_{0}(t), y \in \mathbb{R}^{m}, Z \in \mathbb{S}_{+}^{n}\right\}  \tag{t}\\
& \inf _{X}\left\{A_{0}(t) \bullet X: A_{k}(t) \bullet X=b_{k}(t)(k \in[m]), X \in \mathbb{S}_{+}^{n}\right\} \tag{t}
\end{align*}
$$

where $t \geq 0, A_{k}(t) \in \mathbb{S}^{n}, b(t) \in \mathbb{R}^{m}$ are continuous at $t=0$, and $A_{k}(0)=A_{k}, b(0)=b$.
In this subsection, the following conditions are imposed on the initial SDP:

## Condition 1.

(C1) (D) is feasible, and (D) is strictly feasible;
(C2) $A_{1}, \ldots, A_{m}$ are linearly independent.
Then, by applying the facial reduction to (D), there exist a nonsingular matrix $Q$ and $r \in \mathbb{N}$ such that

$$
\inf _{X_{3}}\left\{Q^{T} A_{0} Q \bullet\left(\begin{array}{cc}
O & O \\
O & X_{3}
\end{array}\right): Q^{T} A_{k} Q \bullet\left(\begin{array}{cc}
O & O \\
O & X_{3}
\end{array}\right)=b_{k}(k \in[m]), X_{3} \in \mathbb{S}_{+}^{r}\right\} \quad\left(F(D)_{0}\right)
$$

has the same optimal value as (DI), and $F(D)_{0}$ is strictly feasible due to Lemma 2.2. Here, for $n \times n$ matrix $M$, we denote by $M_{3}$ the right bottom block of the partitioning

$$
M=\left(\begin{array}{cc}
M_{1} & M_{2}^{T}  \tag{2}\\
M_{2} & M_{3}
\end{array}\right),
$$

where the partitioning is uniquely determined by Lemma 2.1 for the minimal face of (D) with $M_{1} \in \mathbb{S}^{n-r}, M_{2} \in \mathbb{R}^{r \times(n-r)}, M_{3} \in \mathbb{S}^{r}$. We call $M_{3}$ the third block of $M$ associated to the minimal face of (D). Then we can rewrite $F(D)_{0}$ as follows:

$$
\begin{equation*}
\inf _{X}\left\{\left(Q^{T} A_{0} Q\right)_{3} \bullet X:\left(Q^{T} A_{k} Q\right)_{3} \bullet X=b_{k}(k \in[m]), X \in \mathbb{S}_{+}^{r}\right\} . \tag{D}
\end{equation*}
$$

For $A=\left(a_{i j}\right)_{1 \leq i, j \leq n} \in \mathbb{S}^{n}$, we define $\operatorname{vec}(A)$ as the vectorization of $A$, i.e.,

$$
\operatorname{vec}(A)=\left(a_{11}, a_{12}, \ldots, a_{1 n}, a_{21}, a_{22}, \ldots, a_{n 1}, \ldots, a_{n n}\right)^{T} .
$$

Let $r\left(A_{1}, \ldots, A_{m}\right)$ be the rank of the matrix $\left(\operatorname{vec}\left(A_{1}\right), \ldots, \operatorname{vec}\left(A_{m}\right)\right)$.
The following theorem is the main result of this paper.
Theorem 3.1. Under Condition $\mathbb{1}$, suppose that the minimal face $F_{\min }$ of (D) can be written as

$$
F_{\min }=\left\{Q\left(\begin{array}{cc}
O_{(n-r) \times(n-r)} & O_{(n-r) \times r} \\
O_{r \times(n-r)} & X
\end{array}\right) Q^{T}: X \in \mathbb{S}_{+}^{r}\right\}
$$

for some nonsingular matrix $Q \in \mathbb{R}^{n \times n}$ and $r \in \mathbb{N}$. In addition, we suppose that the set $\left\{\left(A_{0}(t), \ldots, A_{m}(t), b(t)\right): 0 \leq t \leq \delta\right\}$ satisfies the following assumptions for some $\delta>0$ :

1. ( $\overline{\left.D_{t}\right)}$ is feasible for each $t \in[0, \delta]$;
2. For each $t \in[0, \delta]$, there exists a nonsingular matrix $Q(t)$ such that $\lim _{t \rightarrow 0} Q(t)=Q$, and the minimal face of $\left(\overline{D_{t}}\right)$ can be written as

$$
\left\{Q(t)\left(\begin{array}{cc}
O_{(n-r) \times(n-r)} & O_{(n-r) \times r} \\
O_{r \times(n-r)} & X
\end{array}\right) Q(t)^{T}: X \in \mathbb{S}_{+}^{r}\right\} ;
$$

3. For each $t \in[0, \delta]$, we have

$$
r\left(\left(Q(t)^{T} A_{1}(t) Q(t)\right)_{3}, \ldots,\left(Q(t)^{T} A_{m}(t) Q(t)\right)_{3}\right)=r\left(\left(Q^{T} A_{1} Q\right)_{3}, \ldots,\left(Q^{T} A_{m} Q\right)_{3}\right)
$$

where $M_{3}$ is the third block of $M \in \mathbb{S}^{n}$ associated with the minimal face of (D).
Then the optimal value of $\left(\overline{D_{t}}\right)$ varies continuously at $t=0$.

The following is an immediate corollary.
Corollary 3.2. Under Condition (1, suppose that there exists $\delta>0$ such that ( $D_{t}$ ) has a nonempty feasible set and the same minimal face as (D), and

$$
r\left(\left(Q^{T} A_{1}(t) Q\right)_{3}, \ldots,\left(Q^{T} A_{m}(t) Q\right)_{3}\right)=r\left(\left(Q^{T} A_{1} Q\right)_{3}, \ldots,\left(Q^{T} A_{m} Q\right)_{3}\right)
$$

for $t \in[0, \delta]$. Then the optimal value of ( $\left.\overline{D_{t}}\right)$ varies continuously at $t=0$.
Before proceeding to the proof, we investigate examples and show that the rank condition or the condition on the face can not be removed from Theorem 3.1 and Corollary 3.2,

Example 3.3. The following example satisfies the condition on the face but does not satisfy the rank condition. We set $b=(0,2,2)^{T}$ and

$$
A_{0}=\left(\begin{array}{ccccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array}\right), A_{1}=\left(\begin{array}{ccccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), A_{2}=\left(\begin{array}{cccc}
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right), A_{3}=\left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

in (D) and (D). Then $A_{1}, A_{2}, A_{3}$ are linearly independent, $(P)$ is strictly feasible, and ( $D$ ) is weakly feasible. The optimal value is 0 and an optimal pair is $X=\left(\begin{array}{cccc}0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{array}\right), y=$ $(0,0,0), Z=\left(\begin{array}{cccc}0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1\end{array}\right)$. The minimal face of $(D)$ is

$$
F_{\min }=\left\{\left(\begin{array}{cc}
O_{2 \times 2} & O_{2 \times 2} \\
O_{2 \times 2} & X_{3}
\end{array}\right) \in \mathbb{S}_{+}^{4}: X_{3} \in \mathbb{S}_{+}^{2}\right\}
$$

If we perturb the matrices as

$$
A_{i}(t)=A_{i}(i=0,1,2), A_{3}(t)=\left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1+t & 0 \\
0 & 0 & 0 & 1-t
\end{array}\right),
$$

then ( $\overline{D_{t}}$ ) remains feasible for each $t>0$. In fact the feasible points of $\left(\overline{D_{t}}\right)$ can be written as $X=\left(\begin{array}{cccc}0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \alpha \\ 0 & 0 & 1\end{array}\right)(-1 \leq \alpha \leq 1)$. Thus the minimal face of $\left(\overline{D_{t}}\right)$ is equal to $F_{\min }$ for each $t>0$. Now the dimension of the span of the third blocks of the matrices $A_{1}, A_{2}, A_{3}$ is 1 , while that of $A_{1}(t), A_{2}(t), A_{3}(t)$ is 2 for each $t>0$. The optimal value of $\left(D_{t}\right)$ is 1 and the optimal pairs are $X=\left(\begin{array}{cccc}0 & 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 & \alpha \\ 0 & 0 & 1\end{array}\right)(-1 \leq \alpha \leq 1), y=\left(\beta, \frac{1+t}{2 t},-\frac{1}{2 t}\right), Z=\left(\begin{array}{cccc}-\beta & \frac{1}{2 t} & 0 & 0 \\ \frac{1}{2 t} & -\beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right) \quad\left(\beta \leq-\frac{1}{2 t}\right)$ for each $t>0$. Thus the optimal value changes discontinuously at $t=0$.
Example 3.4. The following example satisfies the rank condition but does not satisfy the condition on the face. We set $b=(2,2,2,0)^{T}$ and

$$
A_{0}=\left(\begin{array}{lll}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right), A_{1}=\left(\begin{array}{lll}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{array}\right), A_{2}=\left(\begin{array}{lll}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{array}\right), A_{3}=\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{array}\right), A_{4}=\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{array}\right)
$$

in (D) and (D). Then $A_{1}, \ldots, A_{4}$ are linearly independent, (D) is strictly feasible, and (D) is weakly feasible. The optimal value is $\frac{1}{2}$, and the optimal pairs are $X=\left(\begin{array}{lll}0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & \frac{1}{2}\end{array}\right), y=$ $\left(-\frac{1}{4}, \frac{1}{2}-y_{3}, y_{3}, 0\right), Z=\left(\begin{array}{ccc}-y_{3} & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1\end{array}\right), y_{3} \leq 0$. The minimal face of (D) is

$$
F_{\min }=\left\{\left(\begin{array}{cc}
0 & O_{1 \times 2} \\
O_{2 \times 1} & X_{3}
\end{array}\right) \in \mathbb{S}_{+}^{3}: X_{3} \in \mathbb{S}_{+}^{2}\right\}
$$

If we perturb the matrices as

$$
A_{i}(t)=A_{i}(i=0,1,2), A_{3}(t)=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 1-t^{2} \\
0 & 1-t^{2} & 0
\end{array}\right), A_{4}(t)=\left(\begin{array}{ccc}
0 & 0 & 1 \\
0 & -2 t & 0 \\
1 & 0 & 0
\end{array}\right),
$$

then $\left(\overline{D_{t}}\right)$ is strictly feasible for each $t>0$. In fact, $X=\left(\begin{array}{ccc}2 t^{2} & 2 & 2 \\ 0 & 2 & 1 \\ 2 t & 1 & 1\end{array}\right)$ are strict feasible points of $\left(D_{t}\right)$. Thus the minimal face of $\left(\overline{D_{t}}\right)$ is $\mathbb{S}_{+}^{3}$ for each $t>0$. Since the span of the third blocks of the matrices $A_{1}(t), \ldots, A_{4}(t)$ has the same basis as that of $A_{1}, \ldots, A_{4}$ for each $t>0$, the rank condition is satisfied. However the optimal value of ( $\overline{D_{t}}$ ) is 2 with $X=\left(\begin{array}{ccc}2 t^{2} & t & 2 t \\ t & 2 & 1 \\ 2 t & 1 & 2\end{array}\right), y=$ $\left(2, \frac{1}{t^{2}}-1,-\frac{1}{t^{2}}, \frac{1}{t}\right), Z=\left(\begin{array}{ccc}\frac{1}{t^{2}} & 0 & -\frac{1}{t} \\ 0 & 0 & 0 \\ -\frac{1}{t} & 0 & 1\end{array}\right)$ being the unique optimal pair for each $t>0$. Thus the optimal value changes discontinuously at $t=0$.

Example 3.5. Consider the same SDP as in Example 3.4. If we perturb the matrices as

$$
A_{i}(t)=A_{i}(i=0,1,4), A_{2}(t)=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & t
\end{array}\right), A_{3}(t)=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & t
\end{array}\right),
$$

the minimal face of each perturbed problem is equal to $F_{\min }$ in Example 3.4. Here the condition on the face and the rank condition are satisfied for sufficiently small $t>0$. Thus Theorem 3.1 guarantees the continuity of the optimal value. In fact, the optimal value of $\left(\overline{D_{t}}\right)$ is $\frac{2 t+4-4 \sqrt{t+1}}{t^{2}}$ and converges to $\frac{1}{2}$ as $t \rightarrow 0$. The optimal pairs are

$$
\begin{aligned}
& X=\left(\begin{array}{lll}
0 & 0 & \frac{0}{0} \\
0 & \frac{2 \sqrt{t+1-2}}{} \\
0 & \frac{2 \sqrt{t+1-2}}{t} & \frac{2 t+4-4 \sqrt{t+1}}{t^{2}}
\end{array}\right), y=\left(-\frac{\sqrt{t+1}(t+2)-2 t-2}{t^{3}+t^{2}}, \beta,-\frac{\sqrt{t+1}+\beta t^{2}+(\beta-1) t-1}{t^{2}+t}, 0\right), \\
& Z=\left(\begin{array}{ccc}
\frac{\sqrt{t+1}+\beta t^{2}+(\beta-1) t-1}{t^{2}+t} & 0 & 0 \\
0 & \frac{\sqrt{t+1}(t+2)-2 t-2}{t^{3}+2}+\frac{\sqrt{t+1}-t-1}{t^{2}+t} \\
0 & \frac{\sqrt{t+1} t-t-1}{t^{2}+t} & \frac{1}{\sqrt{t+1}}
\end{array}\right)
\end{aligned}
$$

for all $\beta$ such that $(1,1)$ st element of $Z$ is nonnegative.
Proof of Theorem 3.1. By the assumptions 1 and 2 in Theorem 3.1, the optimal value of $\left(\overline{D_{t}}\right)$ is equal to

$$
\inf _{X}\left\{\left(Q(t)^{T} A_{0}(t) Q(t)\right)_{3} \bullet X:\left(Q(t)^{T} A_{k}(t) Q(t)\right)_{3} \bullet X=b_{k}(t)(k \in[m]), X \in \mathbb{S}_{+}^{r}\right\}, \quad\left(F\left(D_{t}\right)\right)
$$

and $F\left(D_{t}\right)$ has a nonempty feasible set for each $t \in[0, \delta]$. Thus if continuity of the optimal value of $\left[F\left(D_{t}\right)\right.$ at $t=0$ is shown, then that of the optimal value of $\left(\overline{D_{t}}\right)$ is also shown. For each $t \in[0, \delta]$, we have that the dual of $F\left(D_{t}\right)$ is

$$
\sup _{y, Z}\left\{b(t)^{T} y: \sum_{k \in[m]} y_{k}\left(Q(t)^{T} A_{k}(t) Q(t)\right)_{3}+Z=\left(Q(t)^{T} A_{0} Q(t)\right)_{3}, Z \in \mathbb{S}_{+}^{r}\right\} . \quad\left(F\left(D_{t}\right)^{\prime}\right)
$$

Then $F\left(D_{t}\right)$ has the same optimal value as $F\left(D_{t}\right)^{\prime}$ because $F\left(D_{t}\right)$ and $F\left(D_{t}\right)^{\prime}$ are strictly feasible. In fact, strict feasibility of $F\left(D_{t}\right)$ follows from the properties of facial reduction. Since, for a strictly feasible point $(\tilde{y}, Z)$ of $\left(\overrightarrow{P_{t}}\right),\left(\tilde{y},\left(Q(t)^{T} \tilde{Z} Q(t)\right)_{3}\right)$ is also a strictly feasible point of $F\left(D_{t}\right)^{\prime}$, and hence $F\left(D_{t}\right)^{\prime}$ is strictly feasible. Therefore, the proof is done by showing Theorem 3.6,

Theorem 3.6. If both (DI) and (D) are strictly feasible, (D) is feasible, and $r\left(A_{1}(t), \ldots, A_{m}(t)\right)=$ $r\left(A_{1}, \ldots, A_{m}\right)$ for each sufficiently small $t>0$, then the optimal value of ( $\left(A_{t}\right)$ varies continuously at $t=0$.

We will prove Theorem 3.6 in Subsection 3.2,
Remark 3.7. The coefficient matrices $A_{1}, \ldots, A_{m}$ in (P) are usually assumed to be linearly independent in the literature. However the coefficient matrices in $F(D)$ can be linearly dependent even if the initial SDP has linearly independent constraints. In fact, the coefficient matrices of the reduced SDPs are linearly dependent in Examples 3.3, 3.4 and 3.5. Thus we need to consider SDPs with linearly dependent coefficient matrices in Theorem 3.6.

As in Example 3.3 and 3.4, if $r\left(A_{1}(t), \ldots, A_{m}(t)\right)=r\left(A_{1}, \ldots, A_{m}\right)$ and (D) is weakly feasible, then the optimal value of $\left(\overline{\left.D_{t}\right)}\right.$ can vary discontinuously. We present an additional example and show that the feasibility condition on $\left(\overline{D_{t}}\right)$ or the rank condition can not be removed from Theorem 3.6,

Example 3.8. In (D) and (D), we set $b=(2,2)^{T}$,

$$
A_{0}=\left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right), A_{1}=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right), A_{2}=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) .
$$

Then (D) and (D) are strictly feasible. The optimal value is 0 , and the optimal pairs are $X=\left(\begin{array}{c}2 \\ 2 \\ 0 \\ 0\end{array}\right), y=(\alpha,-\alpha), Z=\left(\begin{array}{ll}0 & 0 \\ 0 & 1\end{array}\right)$ for any $\alpha \in \mathbb{R}$. However, if we take $A_{2}(t)=\left(\begin{array}{cc}1+t & 0 \\ 0 & 1\end{array}\right)$, then $r\left(A_{1}(t), A_{2}(t)\right)=r\left(A_{1}, A_{2}\right)=1$ but $\left(\overline{D_{t}}\right)$ is infeasible. Therefore feasibility of $\left(\overline{\left.D_{t}\right)}\right.$ can not be derived from the rank condition and needs to be assumed.

On the other hand, if we take $A_{2}(t)=\left(\begin{array}{ccc}1+t & 0 \\ 0 & 1-t\end{array}\right)$, then ( $\left.\overline{D_{t}}\right)$ is feasible and $r\left(A_{1}(t), A_{2}(t)\right)=2$ for all $t>0$. The optimal value is 1 , and the optimal pair is $X=\left(\begin{array}{c}1 \\ \beta \\ 1\end{array}\right) \quad(-1 \leq \beta \leq 1), y=$ $\left(\frac{1+t}{2 t},-\frac{1}{2 t}\right), Z=\left(\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right)$. Thus the optimal value varies discontinuously at $t=0$ without the rank condition.

### 3.2 Proof of Theorem 3.6

First, we recall an existence theorem for optimal solutions to an SDP with a focus on the linear independence of the coefficient matrices.

Theorem 3.9. [25, Theorem 4.1 and Corollary 4.1] Suppose (P) is strictly feasible and (D) is feasible. Then (D) has a nonempty compact optimal set and the same optimal value as (P). Also, suppose that (D) is feasible and (D) is strictly feasible. If the coefficient matrices $A_{1}, \ldots, A_{m}$ are linearly independent, then (D) has a nonempty compact optimal set and the same optimal value as (D).

Remark 3.10. 1. Suppose that (D) is feasible and (D) is strictly feasible. However, we do not assume that the coefficient matrices $A_{1}, \ldots, A_{m}$ are linearly independent. Then easy arguments show that (P) has a nonempty optimal set and the same optimal value as (D). Here we lost the compactness of the optimal set of (D).
2. The set of the optimal solutions $(y, Z)$ of ( $(P)$ is unbounded when the matrices $A_{1}, \ldots, A_{m}$ are linearly dependent. However Lemma 3.13 bellow tells that the image of the optimal solutions under the projection $(y, Z) \mapsto Z$ is bounded if (D) and (D) are strictly feasible.

We note that we do not assume the linear independence of the coefficient matrices $A_{1}, \ldots, A_{m}$ in the following arguments. We will use the symbol $S(t)=\left(\operatorname{vec}\left(A_{1}(t)\right), \ldots, \operatorname{vec}\left(A_{m}(t)\right)\right) \in$ $\mathbb{R}^{n^{2} \times m}$ and the symbol $\left(S(t)^{T}\right)^{\dagger}$ for the Moor-Penrose generalized inverse of $S(t)^{T}$ [26].
Lemma 3.11. Suppose $X_{0}$ is a strictly feasible point of (DI). If ( $D_{t}$ ) is feasible and $r\left(A_{1}(t), \ldots, A_{m}(t)\right)=r\left(A_{1}, \ldots, A_{m}\right)$ for each $t \in[0, \delta]$, then there exist strictly feasible points $X_{t}$ of ( $\left(\overline{\left.D_{t}\right)}\right.$ for all sufficiently small $t>0$ such that $X_{t} \rightarrow X_{0}$ as $t \rightarrow 0$.
Proof. We can write the equality constraints of (D) and (D) by $S(0)^{T} \operatorname{vec}(X)=b$ and $S(t)^{T} \operatorname{vec}(X)=b(t)$, respectively. Note that $A_{k}(0)=A_{k}(k \in[m]), b=b(0)$. We set

$$
\begin{aligned}
& \operatorname{vec}\left(X_{0}\right)=\left(I-\left(S(0)^{T}\right)^{\dagger} S(0)^{T}\right) \operatorname{vec}\left(X_{0}\right)+\left(S(0)^{T}\right)^{\dagger} b(0) \text { and } \\
& \operatorname{vec}\left(X_{t}\right)=\left(I-\left(S(t)^{T}\right)^{\dagger} S(t)^{T}\right) \operatorname{vec}\left(X_{0}\right)+\left(S(t)^{T}\right)^{\dagger} b(t) .
\end{aligned}
$$

Then we can check that $S(0)^{T} \operatorname{vec}\left(X_{0}\right)=b$ and $S(t)^{T} \operatorname{vec}\left(X_{t}\right)=b(t)$, by using the fact that $S(t)^{T}\left(S(t)^{T}\right)^{\dagger} v=v$ if and only if $v \in \operatorname{Im} S(t)^{T}$. Since we have $\operatorname{rank}(S(t))=r\left(A_{1}(t), \ldots, A_{m}(t)\right)$ for all $t \geq 0$, it follows from the assumption on the rank and [27. Theorem 5.2] that $\left(S(t)^{T}\right)^{\dagger} \rightarrow$ $\left(S(0)^{T}\right)^{\dagger}$ as $t \rightarrow 0$. Therefore $X_{t} \rightarrow X_{0}$ as $t \rightarrow 0$.

Remark 3.12. Unlike $\left(\overline{D_{t}}\right)$, we can easily prove that $\left(\overline{P_{t}}\right)$ have strictly feasible points ( $y_{t}, Z_{t}$ ) for all sufficiently small $t \geq 0$ without assuming the rank condition. If (P) is strictly feasible, there exists $y_{0} \in \mathbb{R}^{m}$ such that $A_{0}-\sum_{k} y_{0, k} A_{k} \in \mathbb{S}_{++}^{n}$. Then we have that $Z_{t}:=A_{0}(t)$ $\sum_{k} y_{0, k} A_{k}(t) \in \mathbb{S}_{++}^{n}$ for all sufficiently small $t \geq 0$. For each $t>0$, $\left(y_{0}, Z_{t}\right)$ is a strictly feasible point of $\left(\overline{\left.P_{t}\right)}\right.$ and converges to a strict feasible point of (Pl).
Let $\mathcal{U}(t)$ be the set of optimal solutions of $\left(\overline{D_{t}}\right)$, and

$$
\mathcal{V}(t)=\left\{Z \in \mathbb{S}^{n}:(y, Z) \text { is optimal to }\left(\overline{P_{t}}\right) \text { for some } y \in \mathbb{R}^{m}\right\} .
$$

Lemma 3.13. Suppose that (P) is strictly feasible. If there exist strictly feasible points $X_{t}$ of (D) for all sufficiently small $t \geq 0$ such that $X_{t} \rightarrow X_{0}$ as $t \rightarrow 0$, then both sets $\mathcal{U}(t)$ and $\mathcal{V}(t)$ are nonempty and uniformly bounded; i.e., there exist $\delta>0$ and compact sets $C_{1}, C_{2}$ such that

$$
\mathcal{U}(t) \subset C_{1}, \mathcal{V}(t) \subset C_{2} \quad(0 \leq t \leq \delta)
$$

Proof. Since $\left(\overline{D_{t}}\right)$ and $\left(\overline{P_{t}}\right)$ have strictly feasible points, Remark 3.10 ensures that they have the same optimal value and that $\mathcal{U}(t)$ and $\mathcal{V}(t)$ are nonempty for all sufficiently small $t \geq 0$. For a strictly feasible point $\left(y_{0}, Z_{0}\right)$ of (P), we set $y_{t}=y_{0}$ and $Z_{t}=A_{0}(t)-\sum_{k} y_{0, k} A_{k}(t)$. Then $\left(y_{t}, Z_{t}\right)$ is a strictly feasible point of $\left(\overline{P_{t}}\right)$ for each small $t \geq 0$ as explained in Remark 3.12. Let $X$ and $(y, Z)$ be arbitrary optimal solutions to $\left(\overline{D_{t}}\right)$ and $\left(\overline{P_{t}}\right)$ respectively. Since $X_{t}$ and $\left(y_{t}, Z_{t}\right)$ are feasible points, we have

$$
A_{k}(t) \bullet\left(X-X_{t}\right)=0, \sum_{k \in[m]}\left(y_{k}-y_{t, k}\right) A_{k}(t)+Z-Z_{t}=0 .
$$

Then it follows that $\left(X-X_{t}\right) \bullet\left(Z-Z_{t}\right)=0$ and hence that $X \bullet Z_{t}+X_{t} \bullet Z=X_{t} \bullet Z_{t}$. Moreover, positive semidefiniteness of $X_{t}$ and $Z$ guarantees that $X \bullet Z_{t} \leq X_{t} \bullet Z_{t}$. Thus, by positive definiteness of $Z_{t}$, there exists $\epsilon>0$ such that for all sufficiently small $t>0$, we have

$$
\|X\| \leq \frac{X_{t} \bullet Z_{t}}{\lambda_{\min }\left(Z_{t}\right)}<\frac{X_{0} \bullet Z_{0}+\epsilon}{\lambda_{\min }\left(Z_{0}\right)-\epsilon},
$$

where $\lambda_{\min }(M)$ is the smallest eigenvalue of a matrix $M$. Therefore, $\mathcal{U}(t)$ is uniformly bounded for all sufficiently small $t>0$. Similar arguments are applied to $\mathcal{V}(t)$.

The following lemma is well-known, and the proof is omitted.
Lemma 3.14. Suppose that (D) has the same optimal value as $(\mathbb{D})$ and that both of $(\bar{D})$ and (P) have optimal solutions. We define the function $L: \mathbb{S}^{n} \times \mathbb{R}^{m} \rightarrow \mathbb{R}$ as follows:

$$
L(X, y)=A_{0} \bullet X+\sum_{k \in[m]} y_{k}\left(b_{k}-A_{k} \bullet X\right) .
$$

Then $\tilde{X}$ and $\left(\tilde{y}, A_{0}-\sum_{k} \tilde{y}_{k} A_{k}\right)$ are optimal solutions of (D) and (D) respectively if and only if $(\tilde{X}, \tilde{y}) \in \mathbb{S}_{+}^{n} \times \mathbb{R}^{m}$ satisfies

$$
L(\widetilde{X}, y) \leq L(\tilde{X}, \tilde{y}) \leq L(X, \tilde{y}), \forall(X, y) \in \mathbb{S}_{+}^{n} \times \mathbb{R}^{m}
$$

Lemma 3.15. Let $S$ be a matrix $\left(\operatorname{vec}\left(A_{1}\right) \ldots \operatorname{vec}\left(A_{m}\right)\right) \in \mathbb{R}^{n^{2} \times m}$. If $(\tilde{y}, \widetilde{Z})$ is an optimal solution to ( $(\mathbb{P})$, then $\left(y_{*}, \widetilde{Z}\right)$ is also an optimal solution to ( $\left.\mathbb{P}\right)$, where $y_{*}=S^{\dagger}\left(\operatorname{vec}\left(A_{0}\right)\right.$ $\operatorname{vec}(\widetilde{Z}))$.
Proof. By feasibility of $(\tilde{y}, \widetilde{Z})$, we have $S \tilde{y}=\operatorname{vec}\left(A_{0}\right)-\operatorname{vec}(\widetilde{Z})$. Since $S S^{\dagger} v=v$ if and only if $v \in \operatorname{Im} S$, we see that $S y_{*}=\operatorname{vec}\left(A_{0}\right)-\operatorname{vec}(\widetilde{Z})$. Then we obtain $y_{*} \in \tilde{y}+\operatorname{ker} S$. Here we have $\operatorname{ker} S \subset(\operatorname{Span}\{b\})^{\perp}$ since otherwise the optimal value of $(\underline{D})$ is infinity and hence this contradicts finiteness of the optimal value. Thus $b^{T} y_{*}=b^{T} \tilde{y}$, and therefore, $\left(y_{*}, \widetilde{Z}\right)$ is optimal.

Lemma 3.16 plays an essential role in the proof of Theorem 3.6. Lemma 3.13 and 3.16 ensure outer semicontinuity of the set-valued map $t \mapsto \mathcal{U}(t) \times \mathcal{V}(t)$; see [28, Section 5.B]. In the following, $\mathbb{B}$ denotes the closed unit ball in $\mathbb{S}^{n}$. We define, for $X \in \mathbb{S}^{n}$ and $C \subset \mathbb{S}^{n}$,

$$
d(X, C)=\inf \{\|X-Y\|: Y \in C\} .
$$

Lemma 3.16. Suppose that (P) is strictly feasible. If there exist strictly feasible points $X_{t}$ of (Dt) for all sufficiently small $t \geq 0$ such that $X_{t} \rightarrow X_{0}$ as $t \rightarrow 0$, then for any $\epsilon>0$, there exists $\eta>0$ such that

$$
\mathcal{U}(t) \subset \mathcal{U}(0)+\epsilon \mathbb{B}, \mathcal{V}(t) \subset \mathcal{V}(0)+\epsilon \mathbb{B} \quad(0 \leq t \leq \eta) .
$$

Proof. By Remark [3.10, $\left(\overline{D_{t}}\right)$ and $\left(\overline{P_{t}}\right)$ have optimal solutions and the same optimal value. Suppose that the conclusion is false. Then there exist $\epsilon>0,\left\{t_{j}\right\}, X\left(t_{j}\right) \in \mathcal{U}\left(t_{j}\right)$ and $Z\left(t_{j}\right) \in$ $\mathcal{V}\left(t_{j}\right)$ such that $t_{j} \rightarrow 0$ and

$$
\begin{equation*}
d\left(X\left(t_{j}\right), \mathcal{U}(0)\right) \geq \epsilon, d\left(Z\left(t_{j}\right), \mathcal{V}(0)\right) \geq \epsilon, \tag{3}
\end{equation*}
$$

for all $j$. Recall that $S(t)$ denotes the matrix $\left(\operatorname{vec}\left(A_{1}(t)\right) \cdots \operatorname{vec}\left(A_{m}(t)\right)\right)$. Let $y\left(t_{j}\right)=S\left(t_{j}\right)^{\dagger}\left(\operatorname{vec}\left(A_{0}\right)-\right.$ $\left.\operatorname{vec}\left(Z\left(t_{j}\right)\right)\right)$. Then, Lemma 3.15 implies that the feasible solution $\left(y\left(t_{j}\right), Z\left(t_{j}\right)\right)$ is optimal for $\left(P_{t_{j}}\right)$ for each $j$. We define

$$
L(X, y, t)=A_{0}(t) \bullet X+\sum_{k \in[m]} y_{k}\left(b_{k}(t)-A_{k}(t) \bullet X\right) .
$$

Then, we note that $L(X, y, 0)$ is equal to $L(X, y)$ defined in Lemma 3.14. By Lemma 3.14, we have

$$
L\left(X\left(t_{j}\right), y, t_{j}\right) \leq L\left(X\left(t_{j}\right), y\left(t_{j}\right), t_{j}\right) \leq L\left(X, y\left(t_{j}\right), t_{j}\right), \forall(X, y) \in \mathbb{S}_{+}^{n} \times \mathbb{R}^{m}
$$

Since Lemma 3.13 ensures that $\left\{\left(X\left(t_{j}\right), Z\left(t_{j}\right)\right)\right\}$ is uniformly bounded, we may assume that

$$
\left(X\left(t_{j}\right), y\left(t_{j}\right), Z\left(t_{j}\right)\right) \rightarrow(\tilde{X}, \tilde{y}, \widetilde{Z})
$$

as $j \rightarrow \infty$ for some ( $\tilde{X}, \tilde{y}, \widetilde{Z}$ ). Thus we have

$$
L(\widetilde{X}, y, 0) \leq L(\widetilde{X}, \tilde{y}, 0) \leq L(X, \tilde{y}, 0), \forall(X, y) \in \mathbb{S}_{+}^{n} \times \mathbb{R}^{m}
$$

By applying Lemma 3.14 again, $\widetilde{X}$ and $(\tilde{y}, \widetilde{Z})$ are optimal for $(\mathbb{P})$ and (D) respectively. This contradicts the inequalities (3).

Proof of Theorem 3.6. By Lemma 3.11 and 3.16, we have that for any $\epsilon>0$ and $X(t) \in \mathcal{U}(t)$, there exist $\eta>0$ and $\widetilde{X}^{t} \in \mathcal{U}(0)$ such that for $t \in[0, \eta]$,

$$
\left|A_{0}(t) \bullet X(t)-A_{0} \bullet \widetilde{X}^{t}\right| \leq k_{1}\left\|X(t)-\widetilde{X}^{t}\right\|+k_{2}\left\|A_{0}(t)-A_{0}(0)\right\|<\epsilon
$$

for some $k_{1}, k_{2}>0$. This completes the proof of Theorem 3.6.
Corollary 3.17. If both (D) and (D) are strictly feasible and $A_{1}, \ldots, A_{m}$ are linearly independent, then the optimal value of $\left(\overline{D_{t}}\right)$ varies continuously at $t=0$.

Proof. By strict feasibility and the linear independence condition, for all sufficiently small $t>0,\left(\overline{P_{t}}\right)$ and $\left(\overline{D_{t}}\right)$ are feasible, and the rank condition is satisfied.

## 4 Behavior of a Minimal Face under Perturbations

In this section, the behavior of a minimal face under perturbations is investigated. In particular, we give criteria for perturbations to keep the minimal face invariant. We slightly simplify the situations and consider the following perturbed problem:

$$
\begin{equation*}
\inf _{X}\left\{A_{0} \bullet X:\left(A_{k}+E_{k}(t)\right) \bullet X=b_{k}(k \in[m]), X \in \mathbb{S}_{+}^{n}\right\}, \tag{t}
\end{equation*}
$$

where $E_{k}(t)=A_{k}(t)-A_{k}$ for all $k \in[m]$. We note $E_{k}(t) \rightarrow 0$ as $t \rightarrow 0$ since we assume that $A_{k}(t)$ are continuous at $t=0$ and $A_{k}(0)=A_{k}$. Throughout this section, we assume the following conditions:

## Condition 2.

1. (D) is feasible, and (D) is strictly feasible;
2. $A_{1}, \ldots, A_{m}$ are linearly independent;
3. ( $D_{t}$ ) is feasible for each sufficiently small $t>0$.

We say that $\left.\left\{\overline{D_{t}}\right\rangle\right\}_{t \geq 0}$ satisfies the rank condition if there exist an associated nonsingular matrix $Q$ to the minimal face of (D) and $\delta>0$ such that for all $t \in[0, \delta]$,

$$
r\left(\left(Q^{T}\left(A_{1}+E_{1}(t)\right) Q\right)_{3}, \ldots,\left(Q^{T}\left(A_{m}+E_{m}(t)\right) Q\right)_{3}\right)=r\left(\left(Q^{T} A_{1} Q\right)_{3}, \ldots,\left(Q^{T} A_{m} Q\right)_{3}\right)
$$

where the submatrix $M_{3}$ for $M \in \mathbb{S}^{n}$ is determined by the minimal face of (D) as in (2). Here we note that $Q$ in the left hand side does not depend on $t$. We start with the following lemma.

Lemma 4.1. Let $F_{\min }$ and $F_{\min }^{t}$ be the minimal faces of (D) and $\left(\overline{D_{t}}\right)$ respectively. Suppose $\left\{\left(\overline{D_{t}}\right)\right\}_{t \geq 0}$ satisfies the rank condition. If there exists $\delta>0$ such that $F_{\min }^{t} \subset F_{\min }$ for all $t \in[0, \delta]$, we have $F_{\min }^{t}=F_{\min }$ for all sufficiently small $t>0$.

Proof. By Lemma [2.2, the reduced problem $F(D)$ of (D) has a strictly feasible point which solves $\left(Q^{T} A_{k} Q\right)_{3} \bullet X=b_{k}(k \in[m]), X \in \mathbb{S}_{++}^{r}$ for some $r>0$, where $Q$ is an associated nonsingular matrix to the minimal face $F_{\text {min }}$ of (D). For each $t \in[0, \delta]$, feasibility of ( $\left(D_{t}\right)$ and $F_{\text {min }}^{t} \subset F_{\text {min }}$ imply that there exists $\tilde{X} \in F_{\min }$ such that $\left(A_{k}+E_{k}(t)\right) \bullet \tilde{X}=b_{k}(k \in[m])$. It follows from the representation of $F_{\min }$ with $Q$ that

$$
\left(Q^{T}\left(A_{k}+E_{k}(t)\right) Q\right)_{3} \bullet X=b_{k}(k \in[m]), X \in \mathbb{S}_{+}^{r}
$$

is feasible. Consider the following problem obtained by perturbing $F(D)$;

$$
\begin{equation*}
\inf _{X}\left\{\left(Q^{T} A_{0} Q\right)_{3} \bullet X:\left(Q^{T}\left(A_{k}+E_{k}(t)\right) Q\right)_{3} \bullet X=b_{k}(k \in[m]), X \in \mathbb{S}_{+}^{r}\right\} . \tag{4}
\end{equation*}
$$

Here, $F(D)$ has a strictly feasible point, (4) is feasible, and the rank condition is satisfied. Thus Lemma 3.11 implies that for each sufficiently small $t>0$, (4) has a strictly feasible point. It means that $\left\{X \in \operatorname{rint} F_{\min }:\left(A_{k}+E_{k}(t)\right) \bullet X=b_{k}(k \in[m])\right\} \neq \emptyset$ for each sufficiently small $t>0$. Since $F_{\min }$ is a face of $\mathbb{S}_{+}^{n}$ containing $F_{\min }^{t}$, we have $F_{\min }=F_{\min }^{t}$ by Lemma 2.2.
Example 4.2. Lemma 4.1 does not hold without the assumption $F_{\min }^{t} \subset F_{\min }$. In Example 3.4, the perturbation is of the same type as this section is considering. Condition 圆 and the rank condition are satisfied, but the minimal faces $F_{\min }^{t}$ of $\left(\overline{D_{t}}\right)$ are not equal to $F_{\min }$. Here $F_{\min }^{t}$ are not included in $F_{\text {min }}$.

We first give simple sufficient conditions that can be shown easily.
Proposition 4.3. For a facial reduction sequence $\left\{\left(\hat{y}^{i}, \widehat{U}^{i}, \widehat{V}^{i}\right)\right\}_{i=1}^{s}$ of (D), let the minimal face of (D) be $F_{\min }$ and $\hat{K}=\left\{k: \hat{y}_{k}^{i}=0(\forall i=1, \ldots, s)\right\}$. Suppose that $\left\{\left(D_{t}\right)\right\}_{t \geq 0}$ satisfies the rank condition and $E_{k}(t)=O_{n \times n}(k \notin \hat{K})$. Then the minimal faces of $\left(D_{t}\right)$ are equal to $F_{\min }$ for all sufficiently small $t>0$.

Proof. Let $\left\{F_{i}\right\}_{i=1}^{s}$ be the sequence of faces generated by the facial reduction sequence $\left\{\left(\hat{y}^{i}, \widehat{U}^{i}, \widehat{V}^{i}\right)\right\}_{i=1}^{s}$ of (D). Since $E_{k}(t)=O_{n \times n}$ for all $k \notin \hat{K}$, we have

$$
-\sum_{k \in[m]} \hat{y}_{k}^{i}\left(A_{k}+E_{k}(t)\right)=-\sum_{k \in[m]} \hat{y}_{k}^{i} A_{k}=\widehat{U}^{i}+\widehat{V}^{i}
$$

for $i=1, \ldots, s$. Thus $\left\{\left(\hat{y}^{i}, \widehat{U}^{i}, \widehat{V}^{i}\right)\right\}_{i=1}^{s}$ is a facial reduction sequence of $\left(\overline{D_{t}}\right)$ up to the $s$-th iteration. It is summarized as

$$
\left(\underline{D_{t}}\right) \quad \mathbb{S}_{+}^{n} \xrightarrow{\left(\hat{y}^{1}, \widehat{U}^{1}, \widehat{V}^{1}\right)} F_{1} \xrightarrow{\left(\hat{y}^{2}, \widehat{U}^{2}, \widehat{V}^{2}\right)} F_{2} \xrightarrow{\left(\hat{y}^{3}, \widehat{U}^{3}, \widehat{V}^{3}\right)} \cdots \xrightarrow{\left(\hat{y}^{s}, \widehat{U}^{s}, \widehat{V}^{s}\right)} F_{s}=F_{\text {min }} .
$$

Thus the minimal faces of $\left(\overline{D_{t}}\right)$ are contained in $F_{\text {min }}$. In addition, since $\left\{\left(\overline{D_{t}}\right\rangle\right\}_{t \geq 0}$ satisfies the rank condition, it follows from Lemma 4.1 that the minimal faces of $\left(D_{t}\right)$ are equal to $F_{\min }$ for sufficiently small $t>0$.

Next, we will use the positive eigenvectors of reducing certificates to give sufficient conditions for the minimal face to be invariant under a peturbation.

Proposition 4.4. Let $\left\{\left(\hat{y}^{i}, \widehat{U}^{i}, \widehat{V}^{i}\right)\right\}_{i=1}^{s}$ be a facial reduction sequence of (D), $F_{0}=\mathbb{S}_{+}^{n}$ and $F_{1}, \ldots, F_{s}$ be the generated faces. In addition, let

$$
L_{i}=\operatorname{Span}\left\{q q^{T}: q \text { is an eigenvector of } \widehat{U}^{i} \text { associated with a positive eigenvalue }\right\}
$$

Suppose that $\left\{\left(\overline{D_{t}}\right)\right\}_{t \geq 0}$ satisfies the rank condition, and for each $i=1, \ldots, s$,

$$
\sum_{k \in[m]} \hat{y}_{k}^{i} E_{k}(t)+v^{i}(t) \in L_{i}
$$

for some $v^{i}(t) \in F_{i-1}^{\perp}$ with $v^{i}(t) \rightarrow O_{n \times n}$ as $t \rightarrow 0$. Then ( $D_{t}$ have the same minimal face as (D) for all sufficiently small $t>0$.

Before proceeding to the proof, we present an example and a remark.
Example 4.5. The SDP in Example 3.4 has a facial reduction sequence consisting of only one certificate $(\hat{y}, \widehat{U}, \widehat{V})=\left((0,1,-1,0)^{T},\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right), O_{3 \times 3}\right)$, and hence $L_{1}=\operatorname{Span}\{\widehat{U}\}$. If we perturb the matrices as

$$
A_{1}(t)=\left(\begin{array}{ccc}
3 t & 4 t & 5 t \\
4 t & t & 0 \\
5 t & 0 & t
\end{array}\right), A_{2}(t)=\left(\begin{array}{ccc}
0 & 3 t & 2 t \\
3 t & 0 & 1 \\
2 t & 1 & t
\end{array}\right), A_{3}(t)=\left(\begin{array}{ccc}
1+4 t & 3 t & 2 t \\
3 t & 0 & 1 \\
2 t & 1 & t
\end{array}\right), A_{4}(t)=\left(\begin{array}{ccc}
2 t & 5 t & 1+3 t \\
5 t & t & -t \\
1+3 t & -t & 0
\end{array}\right),
$$

then the corresponding $E_{i}(t)$ satisfy $\sum_{k=1}^{4} \hat{y}_{k} E_{k}(t)=-4 t \widehat{U} \in L_{1}$. Thus the conditions of Proposition 4.4 are satisfied, and the minimal face is invariant under the perturbation for sufficiently small $t>0$. In fact, since the minimal face of ( (D) is contained in $\mathbb{S}_{+}^{n} \cap\{\widehat{U}\}^{\perp}$ and $\left(\begin{array}{ccc}0 & 0 & 0 \\ 0 & 2-t & 1-t / 2 \\ 0 & 1-t / 2 & 1\end{array}\right)$ is a feasible point, the minimal face of ( $D_{t}$ ) is equal to $F_{\min }$ in Example 3.3. More generally, to apply Proposition 4.4, it suffices that we choose $E_{i}(t)$ such that ( $D_{t}$ ) are feasible, the rank condition holds, and $E_{2}(t)-E_{3}(t)=\alpha_{t} \widehat{U}$ for some $\alpha_{t} \in \mathbb{R}$.

Remark 4.6. In particular, the inclusion in Proposition 4.4 holds if we have

$$
-\sum_{k \in[m]} \hat{y}_{k}^{i} E_{k}(t) \in \alpha^{i}(t) \widehat{U}^{i}+F_{i-1}^{\perp}
$$

with $\alpha^{i}(t) \rightarrow 0$ as $t \rightarrow 0$ for each $i=1, \ldots, s$.
Proof of Proposition 4.4. Since $\left\{\left(\hat{y}^{i}, \widehat{U}^{i}, \widehat{V}^{i}\right)\right\}_{i=1}^{s}$ is a facial reduction sequence of (D), we have that $\widehat{U}^{i} \in \mathbb{S}_{+}^{n}, \widehat{V}^{i} \in F_{i-1}^{\perp}$ and that $-\sum_{k} \hat{y}_{k}^{i} A_{k}=\widehat{U}^{i}+\widehat{V}^{i} \notin F_{i-1}^{\perp}$ for each $i=1, \ldots, s$.

Let us fix $i$. Let $\left\{q_{l}\right\}$ be the set of the eigenvectors of $\widehat{U}^{i}$ that are associated with positive eigenvalues, orthogonal to each other, and $\left\|q_{l}\right\|=1$. Then every matrix in $L_{i}$ can be written as a linear combination of $q_{l} q_{l}^{T}$. By the assumption, there exist $\alpha_{l}(t) \in \mathbb{R}$ and $v(t) \in F_{i-1}^{\perp}$ such that $-\sum_{k} \hat{y}_{k}^{i} E_{k}(t)=\sum_{l} \alpha_{l}(t) q_{l} q_{l}^{T}+v(t)$ and $v(t) \rightarrow O_{n \times n}$. Since $\sum_{k} \hat{y}_{k}^{i} E_{k}(t) \rightarrow O_{n \times n}$ as $t \rightarrow 0$ and $\left\{q_{l} q_{l}^{T}\right\}$ is linearly independent, we have $\alpha_{l}(t) \rightarrow 0$ for each $l$. We set

$$
U^{i}=\widehat{U}^{i}+\sum_{l} \alpha_{l}(t) q_{l} q_{l}^{T}, \quad V^{i}=\widehat{V}^{i}+v(t)
$$

Then $V^{i} \in F_{i-1}^{\perp}$. Since $\widehat{U}^{i}$ can be written as $\sum_{l} \lambda_{l} q_{l} q_{l}{ }^{T}$, where $\lambda_{l}$ is the positive eigenvalue of $\widehat{U}^{i}$ corresponding to $q_{l}$, we see that $U^{i} \in \mathbb{S}_{+}^{n}$ for all sufficiently small $t>0$. Thus we have

$$
-\sum_{k} \hat{y}_{k}^{i}\left(A_{k}+E_{k}(t)\right)=\widehat{U}^{i}+\widehat{V}^{i}+\sum_{\ell} \alpha_{\ell}(t) q_{\ell} q_{\ell}^{T}+v(t)=U^{i}+V^{i}
$$

Since $\widehat{U}^{i}+\widehat{V}^{i} \notin F_{i-1}^{\perp}$ by the definition of the facial reduction sequence and $F_{i-1}^{\perp}$ is closed, we also have $U^{i}+V^{i} \notin F_{i-1}^{\perp}$ for all sufficiently small $t>0$. In addition, we obtain that

$$
\begin{aligned}
F_{i-1} \cap\left\{U^{i}\right\}^{\perp} & =F_{i-1} \cap\left\{\widehat{U}^{i}+\sum_{l} \alpha_{l}(t) q_{l} q_{l}^{T}\right\}^{\perp} \\
& =F_{i-1} \cap\left\{\sum_{l}\left(\lambda_{l}+\alpha_{l}(t)\right) q_{l} q_{l}^{T}\right\}^{\perp}=F_{i-1} \cap\left\{\widehat{U}^{i}\right\}^{\perp}=F_{i} .
\end{aligned}
$$

Therefore, we have shown that $\left\{U^{i}\right\}_{i=1}^{s}$ also generates the faces $F_{1}, \ldots, F_{s}$ and that $\left(\hat{y}^{i}, U^{i}, V^{i}\right)$ is a reducing certificate of $\left(\overline{D_{t}}\right)$ at the $i$-th iteration for each $i=1, \ldots, s$. Thus $F_{s}$ contains the minimal face of $\left(\overline{D_{t}}\right)$ for each sufficiently small $t>0$. In addition, since $\left.\left\{\overline{D_{t}}\right\rangle\right\}_{t \geq 0}$ satisfies the rank condition, Lemma 4.1 implies that the minimal face of $\left(\overline{D_{t}}\right)$ is equal to $F_{s}$ for each sufficiently small $t>0$.

## 5 Application to a Control Problem

### 5.1 A Singular SDP in $H_{\infty}$ State Feedback Control Problem

We present a singular SDP arising from $H_{\infty}$ state feedback control problem. The $H_{\infty}$ control problem is one of the most successful applications of SDP and is the problem for designing a controller that achieves stabilization with some guaranteed performance based on the $H_{\infty}$ norm. In particular, the $H_{\infty}$ state feedback control problem is a special case of the $H_{\infty}$ control problem. See, e.g., [29, 2] for the detail on the SDP formulation.
In this section, we deal with the following SDP problem:

$$
\sup \left\{\begin{array}{ccl}
-\gamma:\left(\begin{array}{ccc}
-\operatorname{He}\left(A Y_{1}+B_{2} Y_{2}\right) & & \\
-C_{1} Y_{1}-D_{12} Y_{2} & \gamma I_{2} & \\
-B_{1}^{T} & -D_{11}^{T} & \gamma I_{2}
\end{array}\right) \in \mathbb{S}_{+}^{6}, & Y_{2} \in \mathbb{R}^{2 \times 2},  \tag{P0}\\
& \gamma \in \mathbb{R}
\end{array}\right\}
$$

where $\operatorname{He}(X)=X+X^{T}$ for $X \in \mathbb{R}^{n \times n}$ and the blanks in the matrices stand for the transpose of the lower triangular block part. Also, the matrices $A, B_{1}, B_{2}, C_{1}, D_{11}$ and $D_{12}$ are defined as follows:

$$
\left(\begin{array}{c|c|c}
A & B_{1} & B_{2}  \tag{5}\\
\hline C_{1} & D_{11} & D_{12}
\end{array}\right)=\left(\begin{array}{cc|cc|c}
-1 & -1 & -1 & -1 & 0 \\
1 & 0 & -1 & 0 & 1 \\
\hline 2 & -1 & -1 & 0 & 2 \\
-1 & 2 & -1 & 0 & -1
\end{array}\right)
$$

Its dual can be formulated as follows:

$$
\inf \left\{-\left(\begin{array}{ccc}
O & &  \tag{6}\\
O & O & \\
B_{1}^{T} & D_{11}^{T} & O
\end{array}\right) \bullet Z: \begin{array}{l}
\operatorname{He}\left(A^{T} Z_{11}+C_{1}^{T} Z_{21}\right) \in \mathbb{S}_{+}^{2} \\
I_{p} \bullet Z_{22}+I_{m_{1}} \bullet Z_{33}=1, \\
B_{2}^{T} Z_{11}+D_{12}^{T} Z_{21}=O, \\
Z=\left(Z_{i j}\right)_{1 \leq i, j \leq 3} \in \mathbb{S}_{+}^{6}
\end{array}\right\}
$$

To adjust our SDP problem of interest to the form of ( $\mathbb{P}$ ), we define the coefficient matrices $A_{k} \quad(k \in[6] \cup\{0\})$ and vector $b$ by $A_{k}=\left(\begin{array}{cc}A_{k, 1} & O \\ O & A_{k, 2}\end{array}\right)$ and $b=\left(\begin{array}{llllll}0 & 0 & 0 & 0 & 0 & -1\end{array}\right)^{T}$, where $A_{k, 1} \in \mathbb{S}^{6}$ and $A_{k, 2} \in \mathbb{S}^{2}$ for all $k$. Furthermore, we rewrite variables $Y_{1}, Y_{2}, \gamma$ as follows:

$$
Y_{1}=\left(\begin{array}{ll}
y_{1} & y_{2} \\
y_{2} & y_{3}
\end{array}\right), Y_{2}=\left(\begin{array}{ll}
y_{4} & y_{5}
\end{array}\right), y_{6}=\gamma
$$

It follows from [20, Theorems 3.3 and 3.5$]$ that ( $(\mathrm{P} 0)$ is strictly feasible but its dual problem is weakly feasible. Thus we can say that ( $(\mathrm{P} 0)$ is singular.

We compare computational results on ( (P0) with the following three perturbed SDPs for (P0): For $\epsilon=1.0 \mathrm{e}-16$,
(P1) SDP obtained by perturbing the $(2,2)$ nd element of $A_{5,1}$ into $-2(1+\epsilon)$,
(P2) SDP obtained by perturbing the $(2,3)$ rd and $(3,2)$ nd elements in $A_{5,1}$ into $-2(1+\epsilon)$, and
(P3) SDP obtained by perturbing the $(2,4)$ th and $(4,2)$ nd elements of $A_{5,1}$ into $1+\epsilon$.
We apply SDPA-GMP [30] to solve (P0) to (P3) with stopping tolerances $\delta(\delta=1.0 \mathrm{e}-10,1.0 \mathrm{e}-30$ and $1.0 \mathrm{e}-50$ ) and set the floating point computation to approximately 300 significant digits. We set maxIteration $=10000$ and betaStar $=$ betaBar $=$ gammaStar $=0.5$ for parameters of SDPA-GMP. See [30] for more details on parameters. Table 1 shows the numerical results. We observe the following:

Table 1: Computed values for $(\overline{\mathrm{P} 0})$, its perturbed problems (P1), (P2) and (P3)

|  | $\delta=1.0 \mathrm{e}-10$ | $\delta=1.0 \mathrm{e}-30$ | $\delta=1.0 \mathrm{e}-50$ |
| :---: | :---: | :---: | :---: |
| (1P0) | -2.2360679775444764 | -2.2360679774997897 | -2.2360679774997897 |
| (P1) | -2.2360072694172072 | -2.1078335768712432 | -1.4142135623730950 |
| (P2) | -2.2360072694172055 | -2.0000000000000000 | -2.0000000000000000 |
| (P3) | -2.2360072665294605 | -1.4142135623730950 | -1.4142135623730950 |

- The computed values of ( $\overline{\mathrm{P} 0})$ are almost same for all $\delta$, whereas the values for perturbed problems (P1), (P2) and (P3) are different. These significant differences imply that one needs to choose suitable tolerances $\delta$ in order to use the floating point computation with longer significant digits for singular SDPs.
- We can verify that the optimal value of ( (P0) is $-\sqrt{5}$, while the optimal value of the perturbed problem (P1) is $-\sqrt{2}$. These differences show that a small perturbation of coefficient matrices $A_{k}$ in ( $\overline{\mathrm{P} 0}$ ) may yield a significant change of the optimal value of ( P 0 )


### 5.2 Behavior of Minimal Faces under Perturbations for Our Example

We show that matrix-wise perturbations make the minimal face of the dual problem of ( P 0 ) invariant or full-dimensional, i.e., $\mathbb{S}_{+}^{8}$.

Let $A=\left(\begin{array}{ll}a_{11} & a_{12} \\ a_{21} & a_{22}\end{array}\right), B_{2}=\binom{b_{1}}{b_{2}}, C_{1}=\left(\begin{array}{ll}c_{11} & c_{12} \\ c_{21} & c_{22}\end{array}\right), D_{12}=\binom{d_{11}}{d_{21}}$, and let $B_{1}$ and $D_{11}$ be the same matrices as in (51). Then the first constraint in ( (P0) means that

$$
\left.\left(\begin{array}{ccccc}
-2 a_{11} y_{1}-2 a_{12} y_{2}-2 b_{1} y_{4} & & & \\
-a_{21} y_{1}-\left(a_{11}+a_{22}\right) y_{2}-a_{12} y_{3}-b_{2} y_{4}-b_{1} y_{5} & -2 a_{21} y_{2}-2 a_{22} y_{3}-2 b_{2} y_{5} & & & \\
-c_{11} y_{1}-c_{12} y_{2}-d_{1} y_{4} & -c_{11} y_{2}-c_{12} y_{3}-d_{1} y_{5} & y_{6} & & \\
-c_{21} y_{1}-c_{22} y_{2}-d_{2} y_{4} & -c_{21} y_{2}-c_{22} y_{3}-d_{2} y_{5} & 0 & y_{6} & \\
1 & 1 & 1 & 1 & y_{6} \\
1 & 0 & 0 & 0 & 0
\end{array}\right) y_{6}\right)
$$

is contained in $\mathbb{S}_{+}^{6}$. The related part with $a_{11}$ in the above matrix can be extracted as

$$
\begin{aligned}
a_{11}\left(\begin{array}{cccccc}
-2 y_{1} & -y_{2} & 0 & 0 & 0 & 0 \\
-y_{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) & =a_{11} y_{1}\left(\begin{array}{cccccc}
-2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)+a_{11} y_{2}\left(\begin{array}{cccccc}
0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \\
& =: a_{11}\left(y_{1} E_{1,1}+y_{2} E_{2,1}\right) .
\end{aligned}
$$

Since a perturbation on $a_{11}$ affects the coefficient matrices of $y_{1}$ and $y_{2}$, the corresponding perturbing matrices are $E_{1}(t)=\left(\begin{array}{cc}t E_{1,1} & O \\ O & O_{2 \times 2}\end{array}\right), E_{2}(t)=\left(\begin{array}{cc}t E_{2,1} & O \\ O & O_{2 \times 2}\end{array}\right)$ and $E_{k}(t)=O_{8 \times 8}(k=$ $3, \ldots, 6)$. We remark that we need to consider block matrices with two blocks for the perturbation because the coefficient matrices for $y_{1}$ also appear in the constraint $Y_{1} \in \mathbb{S}_{+}^{2}$ of (P0).

Consider the problem (Dt) perturbed with $\left\{E_{k}(t)\right\}$. Then one can verify that the length of the facial reduction sequence for $\left(D_{t}\right)$ is one and that it is $\{(y, U, V)\}$, where

$$
\left\{\begin{array}{l}
y=(1,0,0,-1,0,0)^{T}, U=\left(\begin{array}{cc}
U_{1} & O \\
O & U_{2}
\end{array}\right), V=\left(\begin{array}{cc}
V_{1} & O \\
O & V_{2}
\end{array}\right)  \tag{7}\\
V_{1}=O_{6 \times 6}, V_{2}=O_{2 \times 2}, U_{1}=\left(\begin{array}{cc}
1 & 0^{T} \\
0 & O_{5 \times 5}
\end{array}\right), U_{2}=\left(\begin{array}{cc}
1 & 0 \\
0 & 0
\end{array}\right)
\end{array}\right.
$$

Let $e_{1} \in \mathbb{R}^{6}$ and $f_{1} \in \mathbb{R}^{2}$ be the unit vectors whose first entry is 1 and others are zero. Then the positive eigenvalues of $U$ are 2,1 , and the associated eigenvectors are $\left(e_{1}, 0_{2}^{T}\right)^{T},\left(0_{6}^{T}, f_{1}^{T}\right)^{T}$ respectively. Here $0_{p}$ is the $p$-dimensional zero vector for a given positive integer $p$. Since we have that

$$
-\left(1 \cdot E_{1}(t)+0 \cdot E_{2}(t)\right) \in \operatorname{Span}\left\{\left(\begin{array}{cc}
e_{1} e_{1}^{T} & O \\
O & O_{2 \times 2}
\end{array}\right),\left(\begin{array}{cc}
O_{6 \times 6} & O \\
O & f_{1} f_{1} T
\end{array}\right)\right\}
$$

and that $\left\{\left(\overline{D_{t}}\right)\right\}_{t \geq 0}$ satisfies the rank condition, Proposition 4.4 implies that this perturbation does not change the minimal face of the dual problem.

We can apply similar arguments to see behavior of the minimal face of the dual problem for the other perturbations and observe the followings:

- The minimal face is invariant under the matrix-wise perturbation with respect to $a_{11}$, $a_{12}, a_{22}, c_{12}, c_{22}$ and $b_{1}$. The optimal value of $\left(D_{t}\right)$ changes continuously at $t=0$ due to Theorem 3.1.
- The other perturbations, i.e. $a_{21}, c_{11}, c_{21}, b_{2} d_{1}$ and $d_{2}$, make the minimal face of the dual problem to be $\mathbb{S}_{+}^{8}$, which implies that the perturbed problem is strictly feasible. However, we have numerically confirmed that the optimal value of $\left(\overline{D_{t}}\right)$ also varies continuously in this case. It is a future study to find other conditions that ensure the continuity of the optimal value under any matrix-wise perturbations.
- Hence if we perturb matrices $A, B_{2}, C_{1}$ and $D_{12}$ in the structured form, the minimal face may be different, but can not be smaller.


## 6 Conclusions

We consider perturbations of the coefficient matrices of a semidefinite program, in the case that the primal problem is strictly feasible and the dual problem is weakly feasible. We give sufficient
conditions for continuity of the optimal value. These conditions involve the behavior of the minimal faces of the perturbed dual problems and the submatrices of the coefficient matrices associated with the minimal faces. By using examples, it is argued that these conditions are hard to remove. We further obtain sufficient conditions for the perturbations to keep the minimal face invariant. A facial reduction sequence, which is obtained in the process of facial reduction, plays the central role. Then our results are applied to a semidefinite program obtained from an $H_{\infty}$ control problem. By presenting numerical experiments with interior point methods, we also discuss the importance of computations with arbitrary precision arithmetic, together with an appropriate parameter for the stopping criteria, in order to obtain an approximation to the optimal value of a singular semidefinite program.

In the future work, it is worth considering to use a facial reduction sequence to analyze other properties of a semidefinite program. In addition, it may be interesting to find combinatorial structures in the elements of perturbing matrices that preserve the minimal face of a semidefinite program.

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