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Abstract

In the case of the proportional output reduction strategy with a single output, the Variable-Returns-to-Scale (VRS) Zero-Sum Gains Data Envelopment Analysis (ZSG-DEA) efficiency scores can be obtained from the VRS conventional DEA efficiency scores by means of the Target's Assessment Theorem (TAT). Using TAT as a departure point, two relations for computing the ZSG-DEA efficiency scores appear in the literature. Our objective in this note is to compare, contrast and challenge them on both theoretical and empirical grounds. For the latter, three different data sets are used.

Keywords

Output Interdependency; ZSG-DEA Efficiency; Conventional DEA Efficiency

1 Introduction

Conventional Data Envelopment Analysis (DEA) assumes that the output of each Decision Making Unit (DMU) is independent of that of any other DMU. This implies that each DMU may expand its output as far as it is needed to improve its efficiency independently of the other DMUs. However, this does not hold in the presence of output interdependency that occurs, for example, if (*i*) outputs are ranks in a contest where the higher is a participant's ranking, the lower is the ranking of another participant; (*ii*) the aggregate output of DMUs is *a priori* fixed as when market share or the total number of wins during a league season are considered as outputs; (*iii*) DMUs use inputs to both expand their output and shrink that of their rivals as in head-to-head competition; or (*iv*) the aggregate desirable (undesirable) output is regulated by quotas (permits).

To deal with such cases, Lins et al. (2003) proposed the Zero-Sum Gains DEA (ZSG-DEA) model that provides efficiencies adjusted for output interdependency.[1] This model is operationalized by means of the equal, the proportional or the minimal output reduction strategy; see Lins et al. (2003), Collier et al. (2011), and Yang et al. (2011). In the ZSG-DEA model, the extra output that each DMU under evaluation may require to become efficient is taken from all other DMUs in such a way that the sum of output gain and output losses across DMUs equals zero and the aggregate resultant output equals the aggregate actual output.

One difficulty with this model is that it is non-linear and only under certain circumstances, it can be simplified. For example, Lins et al. (2003) have shown that, in the case of the proportional output reduction strategy with a single output, the Variable-Returns-to-Scale (VRS) ZSG-DEA efficiency scores are related to the VRS conventional DEA efficiency scores by means of the Target's Assessment Theorem (TAT), which states that the potential output of each inefficient DMU in the ZSG-DEA model is a fraction of its potential output in the conventional DEA model. This implies that the distance of each inefficient DMU from the ZSG-DEA efficient frontier is *always* shorter than its distance from the conventional DEA efficient frontier or, in other

[1] Lins et al. (2003) estimated the ZSG-DEA efficiency scores of countries in the Olympic Games by using as a single output the *a priori* fixed number of their total (gold, silver and bronze) medals won.

words, that the ZSG-DEA efficiency score of each inefficient DMU is *always* greater than its conventional DEA efficiency score. In addition, Lins et al. (2003) have shown that the ZSG-DEA and the conventional DEA models result in the same set of intensity variables; this is known as the Benchmarks' Contribution Equality Theorem (BCET)[2] and it implies that the efficient frontiers in both models are formed by the same DMUs or, in other words, that the ZSG-DEA (in)efficient DMUs are also DEA (in)efficient and *vice versa*.

The relation for computing the ZSG-DEA efficiency scores by means of the TAT under the above circumstances is not however directly operational. In an attempt to simplify things, two alternative relations appear in the literature for computing the ZSG-DEA efficiency scores; see Hu and Fang (2010) and Bi et al. (2014). The main objective of this note is to compare, contrast and challenge them on both theoretical and empirical grounds. In theoretical terms, we examine whether they are consistent with the postulates of the ZSG-DEA model, namely the TAT and the BCET. Specifically, we expect the resulting ZSG-DEA efficiency scores to be (*i*) between zero and one, (*ii*) greater-than-or-equal-to their conventional DEA efficiency scores, and (*iii*) less than (equal to) one if their conventional DEA efficiency scores are less than (equal to) one. In empirical terms, we use three different data sets to examine their behavior and relationship.

2 Theoretical Framework

The envelopment form of the output-oriented VRS ZSG-DEA model is given as (Lins et al., 2003):

$$
\begin{array}{lll}
\underset{\hat{h}^k, \lambda_k^i, \tilde{y}_r^i}{Max} & \hat{h}^k & \\
s.t. & \sum_{i=1}^{I} \lambda_k^i x_m^i \leq x_m^k, & m = 1, \dots, M; \\
& \sum_{i=1}^{I} \lambda_k^i \tilde{y}_r^i \geq \hat{h}^k y_r^k, & r = 1, \dots, R; \\
& \sum_{i=1}^{I} \lambda_k^i = 1, & \\
& \lambda_k^i \geq 0, & i = 1, \dots, k, \dots, I;
\end{array} \qquad (1)
$$

[2] Gomes and Lins (2008) coined the names TAT and BCET since these are respectively referred to as Theorem and Corollary in Lins et al. (2003).

where $\hat{h}^k$ refers to the expansion factor ($1 \leq \hat{h}^k < \infty$), x to input quantities, y to output quantities, $\tilde{y}$ to output quantities after accounting for gain and losses among DMUs, λ_k^i to the intensity variables estimated in the k^{th} "run" of (1), m is used to index inputs, r to index outputs, and i to index DMUs. The ZSG-DEA efficiency scores are given as $\hat{\theta}^k = \frac{1}{\hat{h}^k}$. The only difference between the above and the corresponding conventional DEA model is in the left-hand side of the second constraint in (1) where we have $\tilde{y}$ instead of y. In (1), $\tilde{y}$ is a choice variable defined as $\tilde{y}^k = y^k + z^k = \hat{h}^k y^k$ for the k^{th} DMU and as $\tilde{y}^i = y^i - l_k^i$ for all other DMUs, where z^k and l_k^i refer respectively to output gain and output losses. Then, (1) may be rewritten as:

$$\begin{array}{lll} \underset{\hat{h}^k, \lambda_k^i, l_{kr}^i, z_r^k}{Max} & \hat{h}^k & \\ s.t. & \sum_{i \neq k} \lambda_k^i x_m^i + \lambda_k^k x_m^k \leq x_m^k\,, & m = 1, \dots, M; \\ & \sum_{i \neq k} \lambda_k^i \left(y_r^i - l_{kr}^i\right) + \lambda_k^k (y_r^k + z_r^k) \geq \hat{h}^k y_r^k\,, & r = 1, \dots, R; \\ & \sum_{i \neq k} \lambda_k^i + \lambda_k^k = 1, & \\ & \lambda_k^i \geq 0, & i = 1, \dots, k, \dots, I; \end{array} \qquad (2)$$

If $\hat{h}^k = 1$, then (2) is identical to the VRS conventional DEA model since in this case the k^{th} DMU requires no output gain to become ZSG-DEA efficient (i.e., $z_r^k = 0 \; \forall \, r = 1, \dots, R$) and thus, no other DMU is forced to lose output from it (i.e., $l_{kr}^i = 0 \; \forall \, i \neq k, r = 1, \dots, R$). This implies that the same DMUs are on both the conventional DEA and the ZSG-DEA efficient frontiers, which in turn implies that the ZSG-DEA (in)efficient DMUs are also DEA (in)efficient and *vice versa*. On the other hand, if $\hat{h}^k > 1$, then (2) seems to resemble the super-efficiency DEA model (Andersen and Petersen, 1993) because in this case $\lambda_k^k = 0$ and thus, there is no second term in the left-hand side of the first, second and third constraint in (2).[3] Although, a difference may be that (2) also estimates both the output gain of the k^{th} DMU and the output loss of each DMU $i \neq k$.

Noticeably, (2) is a non-linear model that can be simplified only under certain circumstances. One such a case considered by Lins et al. (2003) is for the proportional

[3] For this reason, Bi et al. (2014) wrote the ZSG-DEA model by excluding the DMU under evaluation from the reference set.

output reduction strategy with a single output (i.e., $r = 1$), where $z^k = y^k(\hat{h}^k - 1) \geq 0$ and $l_k^i = \frac{y^i z^k}{Y - y^k} \geq 0 \ \forall\, i \neq k$ with $Y = \sum_{i=1}^{I} y^i$. In this case, (2) is written as:

$$
\begin{aligned}
& \underset{\hat{h}^k,\, \lambda_k^i}{Max} \quad \hat{h}^k \\
& s.t. \quad \sum_{i \neq k} \lambda_k^i x_m^i + \lambda_k^k x_m^k \leq x_m^k\,, \qquad m = 1, \dots, M; \\
& \qquad\ \ \sum_{i \neq k} \lambda_k^i y^i RC^k + \lambda_k^k \hat{h}^k y^k \geq \hat{h}^k y^k\,, \\
& \qquad\ \ \sum_{i \neq k} \lambda_k^i + \lambda_k^k = 1, \\
& \qquad\ \ \lambda_k^i \geq 0, \qquad i = 1, \dots, k, \dots, I;
\end{aligned}
\tag{3}
$$

where $0 < RC^k = 1 - \frac{y^k(\hat{h}^k - 1)}{Y - y^k} \leq 1$ is the reduction coefficient estimated in the k^{th} "run" of (3).[4] Lins et al. (2003) have shown that $\hat{\theta}^k$ can be obtained, without using an optimizer, from the conventional DEA efficiency scores (θ^k) by means of the TAT, which states that the potential output of the k^{th} DMU evaluated by means of (3) is equal to its potential output evaluated by using the conventional DEA model multiplied by its reduction coefficient, namely:

$$
\frac{y^k}{\hat{\theta}^k} = \left(\frac{y^k}{\theta^k}\right) RC^k \tag{4}
$$

The above relation is not however directly operational since RC^k contains z^k, which in turn contains $\hat{\theta}^k$, the variable that we want to estimate.

Using (4) as a point of departure, two relations appear in the literature for computing the VRS ZSG-DEA efficiency scores under the proportional output reduction strategy with a single output. One, due to Hu and Fang (2010), is given as:

$$
\breve{\theta}^k = \frac{\theta^k y^k (Y - y^k) + {y^k}^2}{y^k (Y - y^k + 1)} = \frac{\theta^k (Y - y^k) + y^k}{Y - y^k + 1} \tag{5}
$$

and the other, due to Bi et al. (2014), is given in terms of the expansion factor, namely $\hat{h}^k = \frac{h^k Y}{Y - y^k + h^k y^k}$, which can be converted into ZSG-DEA efficiency score terms as:

[4] If $\hat{h}^k > 1$, then $\lambda_k^k = 0$ and thus (3) is the same as the model in Bi et al.'s (2014) equation (3).

$$\hat{\theta}^k = \frac{Y - y^k + \frac{y^k}{\theta^k}}{\frac{Y}{\theta^k}} = \frac{\theta^k(Y - y^k) + y^k}{Y} \tag{6}$$

in order to be directly comparable with (5). One can verify that (6) is implied by (4) but we were unable to show that the same is true for (5). To prove the former, substitute z^k into RC^k and then rewrite (4) as follows:

$$\frac{1}{\hat{\theta}^k} = \frac{1 - \frac{y^k\left(\frac{1}{\hat{\theta}^k} - 1\right)}{Y - y^k}}{\theta^k} = \frac{\hat{\theta}^k(Y - y^k) - y^k\left(1 - \hat{\theta}^k\right)}{\theta^k\hat{\theta}^k(Y - y^k)} \tag{7}$$

which implies that:

$$\hat{\theta}^k Y - y^k = \theta^k(Y - y^k) \tag{8}$$

Then, by solving (8) for $\hat{\theta}^k$, we can obtain (6). Notice that (6) may also be written as:

$$\hat{\theta}^k = \theta^k + s^k - \theta^k s^k \tag{9}$$

which states that the VRS ZSG-DEA efficiency score of the k^{th} DMU is equal to its VRS conventional DEA efficiency score plus its output share ($s^k = \frac{y^k}{Y}$) minus their product.

If one takes (5) at face value, it may at first glance be seen that the only difference with (6) is in their denominators, unless $y^k = 1$. In particular, if $y^k < (>) 1$ then $\breve{\theta}^k < (>) \hat{\theta}^k$. Apart from this, (5) may imply (depending on the values of y^k) ZSG-DEA efficiency scores that are: (*i*) less than their conventional DEA efficiency scores; (*ii*) greater-than-one; and (*iii*) less than (equal to) one despite that their conventional DEA efficiency scores are equal to (less than) one. As these results are inconsistent with the postulates of the ZSG-DEA model, namely the TAT and the BCET, doubts are raised about the use of (5).

To see that, consider *first* the case that (5) may imply ZSG-DEA efficiency scores that are less than their conventional DEA efficiency scores, i.e., $\breve{\theta}^k < \theta^k$ for $\theta^k < 1$. This is inconsistent with the TAT, which implies exactly the opposite since in this case the reduction coefficient, which measures the vertical distance between the conventional DEA and the ZSG-DEA efficient frontier at the evaluated DMU's input level, is positive but less than one (see Figure 1).[5] To prove that the difference between the ZSG-DEA efficiency scores implied by (5) and their conventional DEA efficiency scores can be negative, substitute (5) into $\breve{\theta}^k - \theta^k$ to get:

$$\breve{\theta}^k - \theta^k = \frac{y^k - \theta^k}{Y - y^k + 1} \tag{10}$$

which is non-negative only if $y^k \geq \theta^k$. If however $y^k < \theta^k$ then $\breve{\theta}^k < \theta^k$. On the contrary, the ZSG-DEA efficiency scores implied by (6) are *never* less than their conventional DEA efficiency scores. This can be verified by substituting (6) into $\hat{\theta}^k - \theta^k$ to get:

$$\hat{\theta}^k - \theta^k = \frac{y^k(1 - \theta^k)}{Y} \tag{11}$$

that is *always* non-negative since by definition $0 < \theta^k \leq 1$.

Second, consider the case that (5) may imply ZSG-DEA efficiency scores that are greater than one, i.e., $\breve{\theta}^k > 1$, which may occur if either $\theta^k = 1$ or $\theta^k < 1$. This is inconsistent with the definition of efficiency. Nevertheless, for $\theta^k = 1$, (5) implies:

$$\breve{\theta}^k = \frac{Y}{Y - y^k + 1} \tag{12}$$

which differs from one unless $y^k = 1$. If however $y^k > 1$ then $\breve{\theta}^k > 1$. On the other hand, in the case of $\theta^k < 1$, for (5) to imply $\breve{\theta}^k < 1$ it is necessary that:

$$(Y - y^k)(\theta^k - 1) + (y^k - 1) < 0 \tag{13}$$

As the first term in (13) is negative for $\theta^k < 1$, a sufficient condition for the above inequality to hold is that $y^k \leq 1$. If however $y^k > 1$ then it is possible for the second term in (13) to be greater than the absolute value of the first term and thus, for (5) to

[5] The left-hand side term in (4) is equal to $x^k c'$ in Figure 1, the first right-hand side term in (4) is equal to $x^k c$, and thus $RC^k = \frac{x^k c'}{x^k c}$ corresponds to the vertical distance between T_{DEA} and $T_{ZSG\text{-}DEA}$ at x^k.

imply $\breve{\theta}^k > 1$. On the contrary, the ZSG-DEA efficiency scores implied by (6) are *never* greater-than-one. Indeed, in terms of (6), $\hat{\theta}^k > 1$ requires that:

$$\theta^k(Y - y^k) - (Y - y^k) > 0 \tag{14}$$

which is impossible since by definition $0 < \theta^k \leq 1$.

Third, consider the case that (5) may imply $\breve{\theta}^k < 1$ even though $\theta^k = 1$. From (12), we can see that this occurs if $y^k < 1$. On the other hand, if $y^k > 1$, then it is possible for the second term in (13) to be equal to the absolute value of the first term and thus, for (5) to imply $\breve{\theta}^k = 1$ even though $\theta^k < 1$. As we have seen, both of these are inconsistent with the BCET, which postulates that $\hat{\theta}^k = 1$ as long as $\theta^k = 1$ and $\hat{\theta}^k < 1$ as long as $\theta^k < 1$.[6] On the contrary, if $\theta^k = 1$ then (6) implies that $\hat{\theta}^k = \frac{Y - y^k + y^k}{Y} = 1$. On the other hand, if $\theta^k < 1$ then (6) implies that $\hat{\theta}^k < 1$. This can be verified by considering that, in terms of (6), $\hat{\theta}^k < 1$ requires that:

$$(Y - y^k)(\theta^k - 1) < 0 \tag{15}$$

which clearly holds for $\theta^k < 1$.

3 Empirical Results

To further demonstrate that (5) may provide results that are inconsistent with the main postulates of the ZSG-DEA model, we provide some empirical evidence using three different data sets. First, we closely examine the conventional DEA and the ZSG-DEA efficiency scores reported in Table B1 of Hu and Fang (2010), who evaluated the performance of a sample of securities firms operated in Taiwan from 2001 to 2005 considering three inputs (fixed assets, financial capital and expenses) and a single output (market share). Descriptive statistics of these data are given in Table 1. Nevertheless, as Hu and Fang (2010) do not report the raw data, we cannot compute $\hat{\theta}^k$ directly from θ^k. For this reason, in Table 2 we report only θ^k and $\breve{\theta}^k$ implied by (5).

As it follows from Table 3, 31.5 to 46% (depending on the year under consideration) of the results based on (5) are counterintuitive. This means that 16 to 23

[6] In terms of Figure 1, this means that DMUs a and b are on both the conventional DEA and the ZSG-DEA frontiers while DMU k is inefficient with respect to both frontiers.

firms have inappropriate ZSG-DEA efficiency scores. Specifically, 89.5 to 100% (depending on the year under consideration) of the counterintuitive results (or in other words 29.4 to 44% of all ZSG-DEA efficiency scores) belong to the case where $\breve{\theta}^k < \theta^k$ for $\theta^k < 1$ and $y^k < \theta^k$. The reason that most of the counterintuitive results belong to this case is that the average level of firms' market share ranged from 1.64 to 2.00% through years and its minimum level from 0.01 to 0.05%; see Table 1 in Hu and Fang (2010). Consequently, there were several firms whose actual market share was smaller than their conventional DEA efficiency score. For 17 firms, in particular, this was the case for all their yearly observations (see firms #2, #3, #6, #15, #16, #18, #22, #24, #34, #37, #38, #42, #49, #55, #60, #64 and #65 in Table 2). On the other hand, there are no counterintuitive results belonging to either the case where $\breve{\theta}^k > 1$ for $\theta^k = 1$ and $y^k > 1$ or the case where $\breve{\theta}^k < 1$ even though $\theta^k = 1$ because $y^k < 1$. Therefore, all firms deemed efficient by the conventional DEA model had a ZSG-DEA efficiency score that is equal to one. This implies in turn that their market share was very close to 1% since, in any other case, their $\breve{\theta}$'s would differ from one (see (12)). Similarly, there are no counterintuitive results belonging to the case where $\breve{\theta}^k > 1$ for $\theta^k < 1$ and $y^k > 1$, while 0 to 10.5% (depending on the year under consideration) of the counterintuitive results (or in other words 0 to 4% of all ZSG-DEA efficiency scores) belong to the case where $\breve{\theta}^k = 1$ even though $\theta^k < 1$ because $y^k > 1$.

The second data set refers to Sydney 2000 Olympic Games and it is used to estimate efficiency with countries' population and Gross Domestic Product (GDP) as inputs and their medal index as a single output based on an output-oriented VRS conventional DEA model. The medal index is a weighted average of each country's gold, silver and bronze medals won computed for robustness purposes by means of five alternative weighting schemes, the first of which was proposed by Lins et al. (2003) and the other four by Churilov and Flitman (2006). The resulted model variables are reported in Table 4.

From Table 5, where for each alternative medal index we report the conventional DEA efficiency scores and the ZSG-DEA efficiency scores implied by (5) and (6), we can see that the ZSG-DEA efficiency scores implied by (5) are greater (less) than those implied by (6) for values of medal indices greater (less) than one, as required by the models given in section 2. In addition, we can see that for all values of medal indices, (6) implies ZSG-DEA efficiency scores that are: (*i*) greater-than-or-equal-to their conventional DEA efficiency scores; (*ii*) between zero and one; and (*iii*) equal to (less than) one for countries deemed efficient (inefficient) by the conventional DEA model.

On the contrary, as it follows from Table 6, 15 to 18% (depending on the medal index considered) of the results implied by (5) are counterintuitive. This means that 12 to 14 countries have inappropriate ZSG-DEA efficiency scores. Specifically, 23.1 to 30.8% (depending on the medal index considered) of the counterintuitive results (or in other words 3.8 to 5.1% of all ZSG-DEA efficiency scores implied by (5)) belong to the case where $\breve{\theta}^k < \theta^k$ for $\theta^k < 1$ and $y^k < \theta^k$. Another 41.7 to 50% (depending on the medal index considered) of the counterintuitive results (or in other words 6.3 to 8.9% of all ZSG-DEA efficiency scores implied by (5)) belong to the cases where $\breve{\theta}^k > 1$ for either $\theta^k = 1$ or $\theta^k < 1$ and $y^k > 1$. Finally, an additional 21.4 to 33.3% (depending on the medal index considered) of the counterintuitive results (or in other words 3.8 to 5.1% of all ZSG-DEA efficiency scores implied by (5)) belong to the case where $\breve{\theta}^k < 1$ even though $\theta^k = 1$ because $y^k < 1$. Notice that we found no results belonging to the case where $\breve{\theta}^k = 1$ even though $\theta^k < 1$ because $y^k > 1$. This is not surprising as it is rather rare for the second term in (13) to be exactly equal to the absolute value of the first term.

The last data set refers to the 32 teams participated in the regular season of the 2009 National Football League (NFL) and it was taken from Collier et al. (2011), where an output-oriented VRS conventional DEA model is used to estimate teams' efficiency scores with the number of their total wins as the single output (reported in the second column of Table 7) and three indices capturing teams' skills in offense and defense as inputs (i.e., offensive yards per play to defensive yards per play, offensive third-down conversion success to defensive third-down conversion success and defensive penalty yards to offensive penalty yards). These conventional DEA efficiency scores along with their ZSG-DEA efficiency scores implied by (5) and (6) are reported in the last three columns of Table 7. From this table, we can see that (with the exception of team #29 whose actual output is equal to one) the ZSG-DEA efficiency scores implied by (5) are greater than those implied by (6) as each team won more than one games in the league season under consideration. In addition, we can see that the efficiency scores implied by (6) satisfy the postulates of the ZSG-DEA model. On the contrary, as it follows from Table 8, 40.5% of the results implied by (5) are counterintuitive and 13 teams have inappropriate ZSG-DEA efficiency scores (see the efficiency scores of teams #2, #3, #4, #7, #8, #14, #16, #17, #18, #20, #23, #24 and #26 in Table 7). In particular, all counterintuitive results belong to the cases where $\breve{\theta}^k > 1$ for either $\theta^k = 1$ or $\theta^k < 1$ and $y^k > 1$.

4 Concluding Remarks

In this note, we have provided both theoretical and empirical evidence for choosing between the two alternative relations used to compute VRS ZSG-DEA efficiency scores under the proportional output reduction strategy with a single output. Both types of evidence are in favor of (6) rather than (5) as the latter fails in several occasions to fulfill either the postulates of the ZSG-DEA model (the TAT and the BCET) or the very definition of efficiency. We also provided an alternative to (6) by means of (9) where the VRS ZSG-DEA efficiency score of each DMU under evaluation is equal to its VRS conventional DEA efficiency score plus its output share minus their product.

Empirical results from three different data sets indicate that, for the data at hand, most of the counterintuitive ZSG-DEA efficiency scores implied by (5) fall into the cases where (*i*) $\breve{\theta}^k < \theta^k$ for $\theta^k < 1$ and $y^k < \theta^k$, (*ii*) $\breve{\theta}^k > 1$ for $\theta^k = 1$ and $y^k > 1$, and (*iii*) $\breve{\theta}^k < 1$ even though $\theta^k = 1$ because $y^k < 1$. This does not mean that the other two cases, where either $\breve{\theta}^k > 1$ for $\theta^k < 1$ and $y^k > 1$ or $\breve{\theta}^k = 1$ even though $\theta^k < 1$ because $y^k > 1$, are less important as they may account for the majority of the counterintuitive results in some other data sets. However, despite the fact that more empirical analysis is always welcome, it is our belief that the empirical and theoretical evidence presented in this note is sufficient to warn researchers working with the ZSG-DEA model.

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Figure 1: The conventional DEA and the ZSG-DEA frontiers for the proportional output reduction strategy

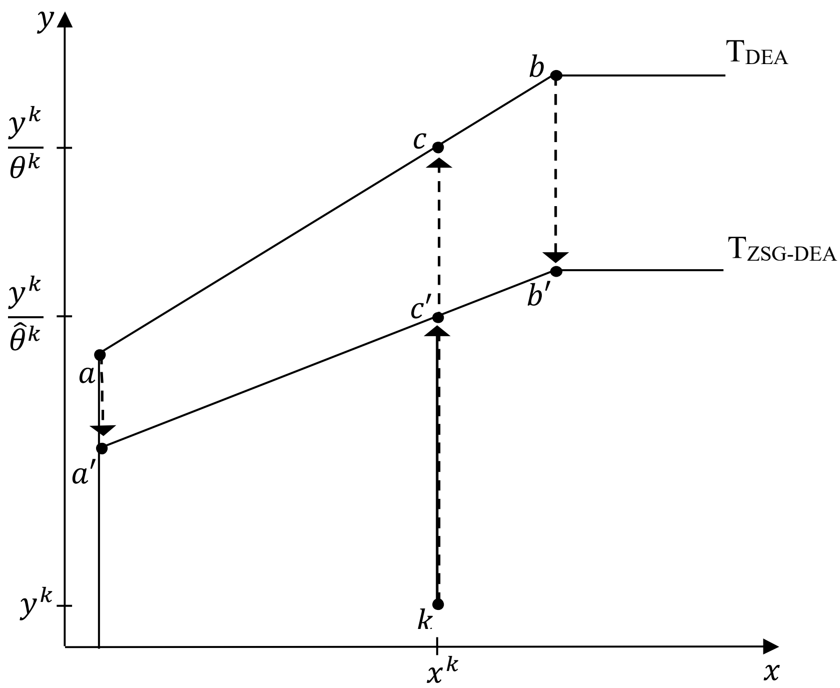

Table 1: Descriptive Statistics of Model's Variables, Securities Firms in Taiwan

2001	Fixed Assets NT$ (000,000,000s)	Financial Capital NT$ (000,000,000s)	Expenses NT$ (000,000,000s)	Market Share (%)
Max	4,455.02	7,815.06	22,315.20	8.66
Min	5.74	48.70	150.00	0.05
Average	932.13	1,637.57	4,413.48	1.64
Standard Deviation	1,074.12	1,733.19	4,730.23	1.89
2002				
Max	4,306.02	24,689.52	10,560.52	10.48
Min	1.84	151.29	10.64	0.02
Average	1,006.18	5,112.88	1,935.13	1.85
Standard Deviation	1,132.26	5,439.46	2,323.28	2.42
2003				
Max	4,413.71	25,382.95	8,587.78	11.01
Min	0.00	154.58	11.12	0.01
Average	1,067.66	5,594.04	2,126.09	2.02
Standard Deviation	1,259.69	5,958.09	2,555.90	2.53
2004				
Max	6,203.25	31,988.93	14,008.10	9.39
Min	0.00	156.84	21.18	0.02
Average	1,135.49	6,163.88	3,010.77	2.00
Standard Deviation	1,439.44	6,822.10	3,757.83	2.51
2005				
Max	6,692.11	33,559.95	12,772.28	7.63
Min	0.00	157.81	26.80	0.02
Average	1,141.64	6,284.13	3,213.83	2.00
Standard Deviation	1,482.82	7,069.14	3,682.73	2.31

Source: Table 1 in Hu and Fang (2010).

Table 2: Estimated Efficiency Scores, Securities Firms in Taiwan

Securities Firm	2001		2002		2003		2004		2005	
	θ^k	$\breve{\theta}^k$	θ^k	$\breve{\theta}^k$	θ^k	$\breve{\theta}^k$	θ^k	$\breve{\theta}^k$	θ^k	$\breve{\theta}^k$
1. Jih Sun	0.9510	0.9893	1.0000	1.0000	0.7630	0.8105	0.8840	0.9307	1.0000	1.0000
2. Jen Hsin	0.5970	0.5958								
3. First	0.5960	0.5954	0.8540	0.8512	0.6260	0.6247	0.6420	0.6387	0.4300	0.4275
4. Asia	0.7480	0.7524	0.5840	0.5872	0.5640	0.5664				
5. Tingkong	0.6530	0.6555								
6. Entrust	0.5520	0.5506								
7. Horizon	0.4520	0.4601	0.3760	0.3789	0.4780	0.4818	0.3850	0.3865	0.4670	0.4666
8. Macquarie	0.5010	0.4986	0.6230	0.6198	1.0000	1.0000	1.0000	1.0000	0.8980	0.8936
9. ABN Amro	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
10. Merrill Lynch	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
11. Nomura (HK)	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.6710	0.6657
12. Societe Generale (HK)	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.6720	0.6657	0.8750	0.8740
13. Goldman Sachs (Asia)	1.0000	1.0000	1.0000	1.0000	0.4990	0.4944	1.0000	1.0000	1.0000	1.0000
14. Oriental	0.9800	0.9865	0.4900	0.4924	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
15. First Taiwan	0.4800	0.4787	0.4420	0.4402						
16. Tachan	0.7350	0.7337	0.7810	0.7781	0.5340	0.5333	0.9930	0.9881	0.7650	0.7637
17. Hua Nan	0.8090	0.8272	0.9570	0.9797	0.7860	0.8049	0.9500	0.9654	0.8600	0.8796
18. Full Long	0.6250	0.6235	0.6480	0.6463	0.3140	0.3130	0.5470	0.5433	0.1860	0.1855
19. Pacific	0.8270	0.8310	0.6140	0.6166	0.5160	0.5177	0.4600	0.4602	0.4650	0.4648
20. Ta Ching	0.8440	0.8439	0.7940	0.7936	0.6910	0.6912	0.7770	0.7743	0.6860	0.6836
21. Capital	0.8870	0.9273	0.8120	0.8588	0.7280	0.7707	1.0000	1.0000	0.9200	0.9666
22. Chung Hsing	0.6230	0.6224	0.5580	0.5571						
23. First Taisec	0.9080	0.9088	0.7980	0.7994	0.8130	0.8176	0.7510	0.7589	0.9250	0.9428
24. Forwin	0.4460	0.4451	0.3530	0.3518	0.3840	0.3826	0.4920	0.4884	0.1730	0.1716
25. Sinopac	0.8990	0.9329	0.8540	0.9073	0.8760	0.9387	0.8760	0.9225	0.9650	1.0000
26. Taiwan	1.0000	1.0000	0.8990	0.9453	0.8460	0.9000	1.0000	1.0000	1.0000	1.0000
27. Taiyu	0.6590	0.6617	0.6690	0.6739						
28. KGI	1.0000	1.0000	0.8330	0.8740	0.7590	0.8044	0.9950	1.0000	1.0000	1.0000
29. IBT	0.9770	0.9765	0.6720	0.6730	1.0000	1.0000	1.0000	1.0000	0.9210	0.9318

30. Grand Cathay	0.8310	0.8691	0.6970	0.7332	0.7800	0.8183	0.9480	1.0000	1.0000	1.0000
31. Taiwan Intl.	0.8810	0.9040	0.8500	0.8777	0.6870	0.7113	0.7140	0.7360	0.7850	0.8117
32. President	0.9280	0.9667	0.9580	1.0000	0.8920	0.9358	0.9450	0.9814	0.9460	0.9811
33. Masterlink	0.8580	0.8902	0.7560	0.7910	0.8090	0.8520	0.9130	0.9506	0.8980	0.9296
34. Primasia	0.4820	0.4816	0.6510	0.6482	0.6070	0.6050	0.7990	0.7970	0.4870	0.4845
35. Chinatrust	1.0000	1.0000	0.5110	0.5159	0.7770	0.7859	0.9090	0.9188	1.0000	1.0000
36. Barits	1.0000	1.0000	0.7120	0.7182						
37. Grand Fortune	0.5830	0.5824	0.5300	0.5282	0.6010	0.5989	0.4120	0.4087	0.5630	0.5579
38. Ta Chong	0.9370	0.9353	0.6320	0.6312	0.5320	0.5314	0.7760	0.7729	0.6460	0.6445
39. Reliance	0.7720	0.7699	1.0000	1.0000	0.4060	0.4087	0.5890	0.5856	0.5810	0.5770
40. Mega	0.6210	0.6277	0.5690	0.5743	0.8630	0.9044	0.7570	0.7864	0.8840	0.9121
41. Concord Intl.	0.6530	0.6532	0.9050	0.9024	0.5780	0.5767	1.0000	1.0000	0.4770	0.4760
42. Jinhwa	0.6050	0.6013								
43. Waterland	0.6230	0.6277	0.6070	0.6160	0.5290	0.5387	0.6960	0.7026	0.5560	0.5620
44. Hsinbao	0.9950	0.9991								
45. J.P. Morgan	0.6570	0.6575	0.6880	0.6871	0.6190	0.6178	0.6010	0.5997	0.7960	0.7940
46. Concord	0.8320	0.8403	0.6480	0.6558	0.6280	0.6361	0.7970	0.8021	0.7240	0.7328
Continue										
47. Concourse	0.6780	0.6781								
48. Sinopac (Old)	0.7870	0.7928								
49. Grand Orient	0.6110	0.6105								
50. Shinkong	0.6140	0.6144	0.2590	0.2598	0.7680	0.7656	0.6780	0.6745	0.7280	0.7275
51. Citibank	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
52. Fu Hwa	0.8310	0.8498	0.8790	0.9052	0.8280	0.8540	0.8800	0.9034	0.7030	0.7259
53. Sun-Fund	1.0000	1.0000	0.4980	0.4955	0.3740	0.3721	0.4090	0.4057	0.3210	0.3182
54. Ho Tung	1.0000	1.0000	0.5030	0.4990	0.8230	0.8165	0.6330	0.6279		
55. E. Sun	0.8070	0.8013	0.7050	0.7004	0.4420	0.4414	0.6180	0.6160	0.6550	0.6539
56. Daiwa	1.0000	1.0000	0.9430	0.9363	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
57. Fubon	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
58. Polaris	0.9530	0.9944	0.9750	1.0000	0.9950	1.0000	1.0000	1.0000	1.0000	1.0000
59. Yuanta	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
60. Far Eastern	0.6600	0.6559	0.6090	0.6042	0.4430	0.4398	0.5540	0.5493	0.2520	0.2503
61. Yuan Li	0.9160	0.9133	0.7430	0.7401	1.0000	1.0000				

62. Deutsche (Asia)			1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
63. Lehman Brothers					1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
64. HSBC (HK)									0.4670	0.4630
65. Cathay							0.3840	0.3807	0.6910	0.6866
Average	0.8010	0.8068	0.7489	0.7564	0.7482	0.7581	0.8087	0.8144	0.7673	0.7721

Source: Table B1 in Hu and Fang (2010).

Table 3: Counterintuitive Results Implied by (5), Securities Firms in Taiwan

Cases	2001	2002	2003	2004	2005
$\breve{\theta}^k < \theta^k$ for $\theta^k < 1$ and $y^k < \theta^k$	20	19	15	17	22
$\breve{\theta}^k > 1$ for $\theta^k = 1$ and $y^k > 1$	-	-	-	-	-
$\breve{\theta}^k > 1$ for $\theta^k < 1$ and $y^k > 1$	-	-	-	-	-
$\breve{\theta}^k < 1$ even though $\theta^k = 1$ because $y^k < 1$	-	-	-	-	-
$\breve{\theta}^k = 1$ even though $\theta^k < 1$ because $y^k > 1$	-	2	1	2	1
Percentage (%) of the total	32.8	38.9	31.4	38.0	46.0

Table 4: Model's Variables, Sydney 2000 Olympic Games

	Country	Population (000s)	GDP US$ 1998 (000,000s)	Medal Index A	Medal Index B	Medal Index C	Medal Index D	Medal Index E
1.	Algeria	31,471	53,155	1.3498	1.2106	1.2500	1.1250	1.6665
2.	Argentina	37,032	305,773	0.8372	0.9474	0.7500	0.7500	1.3332
3.	Armenia	3,520	1,876	0.1749	0.1053	0.1250	0.0625	0.3333
4.	Australia	18,886	380,081	18.3682	19.4209	18.3750	18.8750	19.3314
5.	Austria	8,211	212,755	1.4065	1.4210	1.5000	1.5625	0.9999
6.	Azerbaijan	7,734	4,153	1.3377	1.1579	1.3750	1.3125	0.9999
7.	Bahamas	307	3,498	0.8251	0.8947	0.8750	0.9375	0.6666
8.	Barbados	270	2,354	0.1749	0.1053	0.1250	0.0625	0.3333
9.	Belarus	10,236	13,921	4.3992	3.8424	4.0000	3.5000	5.6661
10.	Belgium	10,161	250,895	1.0121	1.0527	0.8750	0.8125	1.6665
11.	Brazil	170,115	794,947	2.5116	2.8422	2.2500	2.2500	3.9996
12.	Britain	58,830	1,408,037	10.0567	10.2104	10.2500	10.4375	9.3324
13.	Bulgaria	8,225	12,091	4.7190	5.0525	4.8750	5.1250	4.3329
14.	Cameroon	15,085	10,590	0.5814	0.5263	0.6250	0.6250	0.3333
15.	Canada	31,147	605,467	3.8745	3.5265	3.6250	3.3125	4.6662
16.	Chile	15,211	74,853	0.1749	0.1053	0.1250	0.0625	0.3333
17.	China	1,255,445	975,481	22.8019	22.2103	23.3750	23.4375	19.6647
18.	Colombia	42,321	106,437	0.5814	0.5263	0.6250	0.6250	0.3333
19.	Costa Rica	4,023	11,236	0.3498	0.2106	0.2500	0.1250	0.6666
20.	Croatia	4,473	21,283	0.7563	0.6316	0.7500	0.6875	0.6666
21.	Cuba	11,201	24,575	10.3004	10.5788	10.5000	10.7500	9.6657
22.	Czech Rep.	10,244	56,199	2.4186	2.4737	2.3750	2.3750	2.6664
23.	Denmark	5,293	175,119	2.0688	2.2631	2.1250	2.2500	1.9998
24.	Estonia	1,396	5,140	0.9312	0.7369	0.8750	0.7500	0.9999
25.	Ethiopia	62,565	6,694	3.0940	2.7895	3.1250	3.0000	2.6664
26.	Finland	5,176	129,058	1.5814	1.5263	1.6250	1.6250	1.3332
27.	France	59,080	1,461,580	12.8939	13.1578	13.0000	13.1875	12.6654
28.	FYROM	2,024	3,548	0.1749	0.1053	0.1250	0.0625	0.3333
29.	Georgia	4,968	4,839	1.0494	0.6318	0.7500	0.3750	1.9998
30.	Germany	82,220	2,152,766	16.8299	16.3688	16.2500	15.6875	18.9981
31.	Greece	10,645	122,024	4.3125	4.6315	4.3750	4.5625	4.3329
32.	Hungary	10,036	46,607	6.6381	6.7367	6.8750	7.0625	5.6661
33.	Iceland	281	8,415	0.1749	0.1053	0.1250	0.0625	0.3333
34.	India	1,013,662	427,765	0.1749	0.1053	0.1250	0.0625	0.3333
35.	Indonesia	212,107	101,387	1.6623	1.8421	1.6250	1.6875	1.9998
36.	Iran	67,702	192,951	1.9191	1.6842	2.0000	1.9375	1.3332
37.	Ireland	3,730	86,156	0.2437	0.3684	0.2500	0.3125	0.3333
38.	Israel	6,217	105,944	0.1749	0.1053	0.1250	0.0625	0.3333
39.	Italy	57,298	1,183,719	11.7815	11.1580	11.7500	11.4375	11.3322
40.	Jamaica	2,583	6,992	1.4995	1.7895	1.3750	1.4375	2.3331
41.	Japan	126,714	3,795,845	5.7311	6.1052	5.7500	5.9375	5.9994
42.	Kazakhstan	16,223	22,193	2.7190	3.0525	2.8750	3.1250	2.3331
43.	Kenya	30,080	11,220	2.2437	2.3684	2.2500	2.3125	2.3331
44.	Korea People's	24,039	10,337	0.7684	0.6843	0.6250	0.5000	1.3332
45.	Korea Rep.	46,844	325,847	8.7684	8.6843	8.6250	8.5000	9.3324
46.	Kuwait	1,972	27,561	0.1749	0.1053	0.1250	0.0625	0.3333
47.	Kyrgyzstan	4,699	1,720	0.1749	0.1053	0.1250	0.0625	0.3333
48.	Latvia	2,357	6,218	1.0000	1.0000	1.0000	1.0000	0.9999
49.	Lithuania	3,670	10,625	1.6875	1.3685	1.6250	1.4375	1.6665
50.	Mexico	98,881	427,561	1.5935	1.5790	1.5000	1.4375	1.9998
51.	Moldova	4,380	1,638	0.4186	0.4737	0.3750	0.3750	0.6666
52.	Morocco	28,351	36,913	0.9433	0.7896	0.7500	0.5625	1.6665
53.	Mozambique	19,680	1,811	0.5814	0.5263	0.6250	0.6250	0.3333
54.	Netherlands	15,786	393,955	9.8697	10.0524	10.2500	10.5625	8.3325
55.	New Zealand	3,862	54,010	1.1061	0.8422	1.0000	0.8125	1.3332
56.	Nigeria	111,506	128,566	0.7311	1.1052	0.7500	0.9375	0.9999
57.	Norway	4,465	148,251	3.5814	3.5263	3.6250	3.6250	3.3330
58.	Poland	38,765	158,781	5.2316	5.3157	5.3750	5.5000	4.6662
59.	Portugal	9,873	109,393	0.3498	0.2106	0.2500	0.1250	0.6666
60.	Qatar	599	10,821	0.1749	0.1053	0.1250	0.0625	0.3333

Continue								
61.	Romania	22,327	37,911	9.4317	8.9474	9.5000	9.3125	8.6658
62.	Russia	146,934	284,464	30.3256	30.1052	30.5000	30.5000	29.3304
63.	Saudi Arabia	21,607	156,845	0.4186	0.4737	0.3750	0.3750	0.6666
64.	Slovakia	5,387	20,401	1.4874	1.7368	1.5000	1.6250	1.6665
65.	Slovenia	1,989	19,488	1.1628	1.0526	1.2500	1.2500	0.6666
66.	South Africa	40,377	137,443	1.0121	1.0527	0.8750	0.8125	1.6665
67.	Spain	39,630	553,710	3.3498	3.2106	3.2500	3.1250	3.6663
68.	Sri Lanka	18,827	15,965	0.1749	0.1053	0.1250	0.0625	0.3333
69.	Sweden	8,910	238,699	4.0688	4.2631	4.1250	4.2500	3.9996
70.	Switzerland	7,386	265,231	2.3934	2.9473	2.3750	2.6250	2.9997
71.	Thailand	61,399	116,044	0.9312	0.7369	0.8750	0.7500	0.9999
72.	Trinidad-Tobago	1,295	5,985	0.4186	0.4737	0.3750	0.3750	0.6666
73.	Turkey	66,591	204,501	1.9191	1.6842	2.0000	1.9375	1.3332
74.	Ukraine	50,546	42,155	5.9302	6.3159	5.6250	5.6250	7.6659
75.	United States	278,357	8,645,490	34.5388	33.2106	34.7500	34.2500	32.3301
76.	Uruguay	3,337	21,133	0.2437	0.3684	0.2500	0.3125	0.3333
77.	Uzbekistan	24,318	11,429	1.1749	1.1053	1.1250	1.0625	1.3332
78.	Vietnam	79,832	26,824	0.2437	0.3684	0.2500	0.3125	0.3333
79.	Yugoslavia	10,640	11,959	1.0000	1.0000	1.0000	1.0000	0.9999

Note: Medal index A was computed with the use of a weighting scheme that employs the following values (0.5814, 0.2437, 0.1749) as weights respectively assigned to the number of gold, silver and bronze medals. On the other hand, medal indices B, C, D and E were computed with the use of weighting schemes that respectively use the following values (0.5263, 0.3684, 0.1053), (0.6250, 0.2500, 0.1250), (0.6250, 0.3125, 0.0625) and (0.3333, 0.3333, 0.3333) as weights.

Source: The data in the first three columns were taken from Churilov and Flitman (2006).
The data in the last five columns come from authors' calculations.

Table 5: Estimated Efficiency Scores, Sydney 2000 Olympic Games

		Medal Index A			Medal Index B			Medal Index C			Medal Index D			Medal Index E		
	Country	θ^k	$\breve{\theta}^k$	$\hat{\theta}^k$	θ^k	$\breve{\theta}^k$	$\hat{\theta}^k$	θ^k	$\breve{\theta}^k$	$\hat{\theta}^k$	θ^k	$\breve{\theta}^k$	$\hat{\theta}^k$	θ^k	$\breve{\theta}^k$	$\hat{\theta}^k$
1.	Algeria	0.1080	0.1121	0.1120	0.0950	0.0987	0.0986	0.0980	0.1018	0.1017	0.0870	0.0904	0.0904	0.1410	0.1460	0.1457
2.	Argentina	0.0440	0.0466	0.0466	0.0480	0.0510	0.0510	0.0400	0.0423	0.0424	0.0390	0.0414	0.0414	0.0690	0.0731	0.0730
3.	Armenia	0.4190	0.4182	0.4193	0.2300	0.2296	0.2303	0.3290	0.3283	0.3293	0.1670	0.1667	0.1672	0.5270	0.5264	0.5275
4.	Australia	1.0000	1.0605	1.0000	1.0000	1.0649	1.0000	1.0000	1.0608	1.0000	1.0000	1.0630	1.0000	1.0000	1.0634	1.0000
5.	Austria	0.1700	0.1741	0.1738	0.1620	0.1662	0.1659	0.1800	0.1844	0.1841	0.1820	0.1866	0.1862	0.1160	0.1189	0.1189
6.	Azerbaijan	0.8770	0.8785	0.8775	0.7300	0.7314	0.7310	0.9080	0.9095	0.9084	0.8500	0.8515	0.8507	0.5830	0.5844	0.5844
7.	Bahamas	1.0000	0.9994	1.0000	1.0000	0.9997	1.0000	1.0000	0.9996	1.0000	1.0000	0.9998	1.0000	1.0000	0.9989	1.0000
8.	Barbados	1.0000	0.9973	1.0000	1.0000	0.9970	1.0000	1.0000	0.9971	1.0000	1.0000	0.9969	1.0000	1.0000	0.9978	1.0000
9.	Belarus	0.7680	0.7801	0.7714	0.6520	0.6627	0.6564	0.6870	0.6980	0.6911	0.5870	0.5967	0.5918	1.0000	1.0154	1.0000
10.	Belgium	0.1000	0.1030	0.1030	0.0980	0.1012	0.1011	0.0860	0.0886	0.0886	0.0780	0.0804	0.0805	0.1580	0.1629	0.1626
11.	Brazil	0.0820	0.0900	0.0896	0.0940	0.1031	0.1025	0.0730	0.0802	0.0799	0.0730	0.0803	0.0799	0.1360	0.1487	0.1472
12.	Britain	0.4550	0.4875	0.4730	0.4490	0.4823	0.4676	0.4630	0.4963	0.4812	0.4640	0.4981	0.4826	0.4160	0.4458	0.4337
13.	Bulgaria	0.9580	0.9705	0.9587	0.9950	1.0086	0.9951	0.9750	0.9880	0.9754	1.0000	1.0139	1.0000	0.8890	0.9003	0.8906
14.	Cameroon	0.1340	0.1355	0.1357	0.1190	0.1203	0.1205	0.1410	0.1426	0.1428	0.1380	0.1396	0.1398	0.0770	0.0778	0.0780
15.	Canada	0.1990	0.2112	0.2092	0.1730	0.1842	0.1826	0.1860	0.1974	0.1957	0.1660	0.1765	0.1752	0.2300	0.2446	0.2417
16.	Chile	0.0150	0.0155	0.0156	0.0090	0.0093	0.0093	0.0100	0.0104	0.0104	0.0050	0.0052	0.0052	0.0290	0.0300	0.0301
17.	China	0.7430	0.8211	0.7623	0.7320	0.8084	0.7517	0.7580	0.8386	0.7767	0.7610	0.8423	0.7796	0.6650	0.7307	0.6864
18.	Colombia	0.0380	0.0398	0.0398	0.0340	0.0356	0.0357	0.0400	0.0419	0.0420	0.0400	0.0419	0.0420	0.0230	0.0240	0.0241
19.	Costa Rica	0.0860	0.0869	0.0871	0.0500	0.0505	0.0507	0.0600	0.0606	0.0608	0.0290	0.0293	0.0294	0.1780	0.1796	0.1798
20.	Croatia	0.1680	0.1699	0.1701	0.1360	0.1376	0.1378	0.1640	0.1659	0.1661	0.1460	0.1478	0.1479	0.1590	0.1606	0.1608
21.	Cuba	1.0000	1.0315	1.0000	1.0000	1.0327	1.0000	1.0000	1.0324	1.0000	1.0000	1.0334	1.0000	1.0000	1.0290	1.0000
22.	Czech Rep.	0.2520	0.2592	0.2579	0.2490	0.2564	0.2551	0.2430	0.2501	0.2489	0.2370	0.2441	0.2430	0.2900	0.2978	0.2962
23.	Denmark	0.3740	0.3796	0.3783	0.3860	0.3922	0.3906	0.3810	0.3868	0.3853	0.3910	0.3972	0.3955	0.3520	0.3574	0.3562
24.	Estonia	0.5880	0.5891	0.5893	0.4410	0.4420	0.4424	0.5330	0.5341	0.5343	0.4370	0.4380	0.4384	0.6870	0.6880	0.6880
25.	Ethiopia	1.0000	1.0069	1.0000	1.0000	1.0060	1.0000	1.0000	1.0071	1.0000	1.0000	1.0067	1.0000	0.9700	0.9755	0.9703
26.	Finland	0.2920	0.2962	0.2957	0.2650	0.2692	0.2687	0.2980	0.3024	0.3018	0.2880	0.2924	0.2918	0.2400	0.2436	0.2433
27.	France	0.5830	0.6251	0.6007	0.5780	0.6214	0.5964	0.5860	0.6286	0.6038	0.5860	0.6296	0.6041	0.5640	0.6049	0.5820
28.	FYROM	0.1760	0.1760	0.1765	0.0990	0.0990	0.0993	0.1230	0.1230	0.1234	0.0590	0.0590	0.0592	0.3180	0.3180	0.3187
29.	Georgia	0.5940	0.5955	0.5954	0.3410	0.3420	0.3424	0.4260	0.4271	0.4274	0.2080	0.2086	0.2090	1.0000	1.0033	1.0000
30.	Germany	0.6930	0.7489	0.7100	0.6630	0.7177	0.6812	0.6670	0.7211	0.6849	0.6370	0.6895	0.6559	0.7830	0.8459	0.7964
31.	Greece	0.4220	0.4349	0.4302	0.4340	0.4481	0.4427	0.4240	0.4372	0.4323	0.4310	0.4449	0.4396	0.4260	0.4388	0.4341

Continue

32.	Hungary	0.7070	0.7269	0.7134	0.6950	0.7154	0.7018	0.7200	0.7407	0.7264	0.7220	0.7435	0.7285	0.6340	0.6506	0.6407
33.	Iceland	0.4750	0.4740	0.4753	0.3090	0.3083	0.3092	0.3590	0.3582	0.3593	0.1950	0.1946	0.1952	0.7700	0.7686	0.7702
34.	India	0.0060	0.0066	0.0066	0.0030	0.0033	0.0033	0.0040	0.0044	0.0044	0.0020	0.0022	0.0022	0.0110	0.0120	0.0121
35.	Indonesia	0.1020	0.1071	0.1069	0.1130	0.1187	0.1184	0.0990	0.1040	0.1038	0.1020	0.1073	0.1070	0.1290	0.1351	0.1347
36.	Iran	0.0970	0.1030	0.1027	0.0840	0.0893	0.0891	0.1000	0.1063	0.1059	0.0960	0.1021	0.1018	0.0690	0.0731	0.0730
37.	Ireland	0.0600	0.0606	0.0608	0.0850	0.0859	0.0861	0.0610	0.0616	0.0618	0.0740	0.0748	0.0750	0.0810	0.0818	0.0820
38.	Israel	0.0280	0.0285	0.0286	0.0160	0.0163	0.0163	0.0200	0.0203	0.0204	0.0100	0.0102	0.0102	0.0520	0.0529	0.0530
39.	Italy	0.5370	0.5753	0.5549	0.4930	0.5295	0.5117	0.5340	0.5724	0.5521	0.5110	0.5485	0.5296	0.5070	0.5434	0.5252
40.	Jamaica	0.6160	0.6189	0.6179	0.7040	0.7076	0.7058	0.5490	0.5517	0.5510	0.5550	0.5579	0.5571	1.0000	1.0044	1.0000
41.	Japan	0.2020	0.2205	0.2170	0.2150	0.2348	0.2309	0.2010	0.2196	0.2162	0.2070	0.2263	0.2226	0.2160	0.2351	0.2313
42.	Kazakhstan	0.2920	0.3000	0.2983	0.3200	0.3291	0.3269	0.3030	0.3115	0.3096	0.3220	0.3314	0.3290	0.2660	0.2727	0.2716
43.	Kenya	0.4730	0.4788	0.4769	0.5010	0.5072	0.5049	0.4660	0.4719	0.4700	0.4750	0.4811	0.4790	0.5100	0.5160	0.5137
44.	Korea People's	0.1780	0.1799	0.1801	0.1580	0.1597	0.1599	0.1420	0.1436	0.1438	0.1120	0.1133	0.1135	0.3160	0.3193	0.3190
45.	Korea Rep.	0.4320	0.4601	0.4484	0.4140	0.4421	0.4308	0.4240	0.4518	0.4404	0.4100	0.4375	0.4266	0.4510	0.4807	0.4677
46.	Kuwait	0.0740	0.0743	0.0745	0.0420	0.0422	0.0423	0.0520	0.0522	0.0524	0.0250	0.0251	0.0252	0.1480	0.1486	0.1489
47.	Kyrgyzstan	0.3830	0.3823	0.3834	0.2060	0.2057	0.2063	0.3020	0.3014	0.3023	0.1520	0.1517	0.1522	0.4750	0.4745	0.4756
48.	Latvia	0.4760	0.4777	0.4777	0.4550	0.4568	0.4568	0.4630	0.4648	0.4648	0.4460	0.4478	0.4478	0.4930	0.4946	0.4946
49.	Lithuania	0.4500	0.4541	0.4531	0.3520	0.3554	0.3549	0.4220	0.4260	0.4251	0.3620	0.3656	0.3650	0.4830	0.4869	0.4858
50.	Mexico	0.0620	0.0670	0.0669	0.0610	0.0660	0.0659	0.0580	0.0628	0.0627	0.0550	0.0596	0.0595	0.0780	0.0843	0.0840
51.	Moldova	1.0000	0.9981	1.0000	1.0000	0.9983	1.0000	1.0000	0.9979	1.0000	1.0000	0.9979	1.0000	1.0000	0.9989	1.0000
52.	Morocco	0.0840	0.0868	0.0868	0.0690	0.0714	0.0714	0.0660	0.0683	0.0683	0.0480	0.0497	0.0498	0.1570	0.1619	0.1616
53.	Mozambique	1.0000	0.9986	1.0000	0.9510	0.9496	0.9511	1.0000	0.9988	1.0000	1.0000	0.9988	1.0000	0.4510	0.4506	0.4516
54.	Netherlands	0.6390	0.6702	0.6507	0.6160	0.6482	0.6288	0.6630	0.6956	0.6744	0.6650	0.6989	0.6767	0.5140	0.5400	0.5272
55.	New Zealand	0.2700	0.2727	0.2727	0.1950	0.1971	0.1972	0.2410	0.2435	0.2435	0.1890	0.1911	0.1912	0.3310	0.3343	0.3339
56.	Nigeria	0.0400	0.0423	0.0423	0.0600	0.0635	0.0634	0.0410	0.0433	0.0434	0.0500	0.0529	0.0530	0.0570	0.0601	0.0601
57.	Norway	0.7540	0.7634	0.7569	0.6990	0.7084	0.7025	0.7570	0.7665	0.7599	0.7320	0.7417	0.7352	0.6880	0.6967	0.6914
58.	Poland	0.3270	0.3433	0.3386	0.3240	0.3408	0.3359	0.3320	0.3489	0.3438	0.3350	0.3524	0.3471	0.2970	0.3114	0.3077
59.	Portugal	0.0370	0.0380	0.0381	0.0210	0.0216	0.0217	0.0260	0.0267	0.0268	0.0130	0.0134	0.0134	0.0710	0.0729	0.0730
60.	Qatar	0.1590	0.1591	0.1595	0.0890	0.0891	0.0893	0.1090	0.1091	0.1094	0.0520	0.0520	0.0522	0.3470	0.3470	0.3477
61.	Romania	0.8330	0.8621	0.8382	0.7730	0.8008	0.7797	0.8240	0.8534	0.8295	0.7920	0.8211	0.7984	0.8120	0.8382	0.8173
62.	Russia	1.0000	1.1067	1.0000	1.0000	1.1065	1.0000	1.0000	1.1078	1.0000	1.0000	1.1085	1.0000	1.0000	1.1014	1.0000
63.	Saudi Arabia	0.0300	0.0313	0.0313	0.0320	0.0335	0.0335	0.0260	0.0271	0.0272	0.0260	0.0272	0.0272	0.0480	0.0500	0.0501
64.	Slovakia	0.2820	0.2860	0.2855	0.3190	0.3237	0.3229	0.2790	0.2830	0.2826	0.2930	0.2974	0.2968	0.3380	0.3423	0.3416
65.	Slovenia	0.4970	0.4992	0.4989	0.4270	0.4291	0.4290	0.5200	0.5224	0.5220	0.5000	0.5025	0.5021	0.3060	0.3072	0.3075
66.	South Africa	0.0640	0.0671	0.0671	0.0660	0.0693	0.0693	0.0550	0.0577	0.0577	0.0500	0.0525	0.0526	0.1090	0.1141	0.1138
67.	Spain	0.1650	0.1755	0.1742	0.1520	0.1622	0.1610	0.1600	0.1703	0.1690	0.1510	0.1609	0.1598	0.1750	0.1864	0.1848
68.	Sri Lanka	0.0260	0.0265	0.0266	0.0150	0.0153	0.0153	0.0180	0.0184	0.0184	0.0090	0.0092	0.0092	0.0520	0.0529	0.0530
69.	Sweden	0.4550	0.4670	0.4623	0.4500	0.4628	0.4578	0.4590	0.4712	0.4664	0.4600	0.4727	0.4676	0.4300	0.4417	0.4374

Continue

70.	Switzerland	0.3190	0.3258	0.3244	0.3710	0.3796	0.3771	0.3150	0.3218	0.3204	0.3380	0.3456	0.3438	0.3860	0.3946	0.3920
71.	Thailand	0.0540	0.0569	0.0569	0.0420	0.0443	0.0443	0.0500	0.0527	0.0527	0.0420	0.0443	0.0444	0.0600	0.0631	0.0631
72.	Trinidad-Tobago	0.2480	0.2486	0.2490	0.2670	0.2677	0.2681	0.2140	0.2145	0.2150	0.2050	0.2056	0.2060	0.4480	0.4487	0.4492
73.	Turkey	0.0960	0.1020	0.1017	0.0830	0.0883	0.0881	0.0990	0.1053	0.1049	0.0950	0.1011	0.1008	0.0680	0.0721	0.0720
74.	Ukraine	0.5090	0.5271	0.5186	0.5310	0.5505	0.5408	0.4750	0.4923	0.4847	0.4650	0.4824	0.4750	0.6970	0.7202	0.7046
75.	United States	1.0000	1.1239	1.0000	1.0000	1.1193	1.0000	1.0000	1.1253	1.0000	1.0000	1.1240	1.0000	1.0000	1.1134	1.0000
76.	Uruguay	0.0700	0.0706	0.0707	0.1010	0.1019	0.1021	0.0700	0.0706	0.0708	0.0840	0.0848	0.0849	0.1010	0.1018	0.1020
77.	Uzbekistan	0.2450	0.2481	0.2479	0.2300	0.2329	0.2328	0.2310	0.2340	0.2339	0.2150	0.2178	0.2178	0.2860	0.2894	0.2891
78.	Vietnam	0.0230	0.0237	0.0238	0.0340	0.0351	0.0352	0.0230	0.0237	0.0238	0.0290	0.0299	0.0300	0.0340	0.0350	0.0350
79.	Yugoslavia	0.2040	0.2066	0.2066	0.1990	0.2016	0.2016	0.2010	0.2036	0.2036	0.1970	0.1997	0.1997	0.2050	0.2076	0.2076
	Max	1.0000	1.1239	1.0000	1.0000	1.1193	1.0000	1.0000	1.1253	1.0000	1.0000	1.1240	1.0000	1.0000	1.1134	1.0000
	Min	0.0060	0.0066	0.0066	0.0030	0.0033	0.0033	0.0040	0.0044	0.0044	0.0020	0.0022	0.0022	0.0110	0.0120	0.0121
	Average	0.3850	0.3970	0.3893	0.3621	0.3741	0.3664	0.3723	0.3843	0.3765	0.3538	0.3659	0.3581	0.4074	0.4192	0.4117
	Standard Deviation	0.3249	0.3364	0.3251	0.3210	0.3331	0.3214	0.3267	0.3387	0.3270	0.3293	0.3419	0.3299	0.3216	0.3318	0.3212

Table 6: Counterintuitive Results Implied by (5), Sydney 2000 Olympic Games

Cases	Medal Index A	Medal Index B	Medal Index C	Medal Index D	Medal Index E
$\breve{\theta}^k < \theta^k$ for $\theta^k < 1$ and $y^k < \theta^k$	3	4	3	3	4
$\breve{\theta}^k > 1$ for $\theta^k = 1$ and $y^k > 1$	5	5	5	6	7
$\breve{\theta}^k > 1$ for $\theta^k < 1$ and $y^k > 1$	-	1	-	-	-
$\breve{\theta}^k < 1$ even though $\theta^k = 1$ because $y^k < 1$	4	3	4	4	3
$\breve{\theta}^k = 1$ even though $\theta^k < 1$ because $y^k > 1$	-	-	-	-	-
Percentage (%) of the total	15.2	16.5	15.2	16.5	17.7

Table 7: Model's Variables and Estimated Efficiency Scores, 2009 NFL

Team	Wins	Yards	Third-Down	Penalty	θ^k	$\breve{\theta}^k$	$\hat{\theta}^k$
1. Arizona Cardinals	10.000	1.048	1.027	0.948	0.950	0.987	0.952
2. Atlanta Falcons	9.000	0.928	0.929	1.343	1.000	1.032	1.000
3. Baltimore Ravens	9.000	1.142	1.133	0.677	1.000	1.032	1.000
4. Buffalo Bills	6.000	0.959	0.637	1.075	1.000	1.020	1.000
5. Carolina Panthers	8.000	1.000	1.049	0.905	0.795	0.824	0.801
6. Chicago Bears	7.000	0.977	0.907	0.890	0.853	0.878	0.857
7. Cincinnati Bengals	10.000	0.994	1.053	0.889	1.000	1.036	1.000
8. Cleveland Browns	5.000	0.738	0.838	1.198	1.000	1.016	1.000
9. Dallas Cowboys	11.000	1.214	1.160	0.861	0.911	0.952	0.915
10. Denver Broncos	8.000	1.058	0.976	0.908	0.828	0.857	0.833
11. Detroit Lions	2.000	0.757	0.887	1.229	0.370	0.376	0.375
12. Green Bay Packers	11.000	1.213	1.306	0.865	0.901	0.942	0.905
13. Houston Texans	9.000	1.101	1.023	0.874	0.889	0.922	0.893
14. Indianapolis Colts	14.000	1.184	1.093	1.628	1.000	1.053	1.000
15. Jacksonville Jaguars	7.000	0.929	1.003	0.919	0.815	0.840	0.820
16. Kansas City Chiefs	4.000	0.814	0.717	1.231	1.000	1.012	1.000
17. Miami Dolphins	7.000	0.860	1.406	0.920	1.000	1.024	1.000
18. Minnesota Vikings	12.000	1.108	1.300	1.192	0.957	1.002	0.959
19. New England Patriots	10.000	1.085	1.177	1.050	0.836	0.873	0.842
20. New Orleans Saints	13.000	1.142	1.177	0.911	1.000	1.049	1.000
21. New York Giants	8.000	1.056	1.108	0.845	0.755	0.784	0.763
22. New York Jets	9.000	1.177	1.177	1.004	0.685	0.719	0.696
23. Oakland Raiders	5.000	0.797	0.830	0.743	1.000	1.016	1.000
24. Philadelphia Eagles	11.000	1.185	1.097	0.830	1.000	1.041	1.000
25. Pittsburgh Steelers	9.000	1.160	0.932	1.174	0.866	0.899	0.871
26. San Diego Chargers	13.000	1.123	1.099	1.395	1.000	1.049	1.000
27. San Francisco 49ers	8.000	0.996	0.813	1.278	0.943	0.971	0.945
28. Seattle Seahawks	5.000	0.871	0.854	1.156	0.708	0.725	0.714
29. St. Louis Rams	1.000	0.763	0.742	0.735	1.000	1.000	1.000
30. Tampa Bay Buccaneers	3.000	0.850	0.810	1.050	0.491	0.501	0.497
31. Tennessee Titans	8.000	1.008	1.019	0.882	0.818	0.847	0.824
32. Washington Redskins	4.000	1.008	1.002	1.124	0.393	0.407	0.402
Max	14.000	1.214	1.406	1.628	1.000	1.053	1.000
Min	1.000	0.738	0.637	0.677	0.370	0.376	0.375
Average	8.000	1.008	1.009	1.023	0.868	0.896	0.871
Standard Deviation	3.223	0.144	0.178	0.214	0.175	0.181	0.173

Source: The data in the first six columns were taken from Collier et al. (2011).
The data in the last two columns come from authors' calculations.

Table 8: Counterintuitive Results Implied by (5), 2009 NFL

Cases	Frequency
$\breve{\theta}^k < \theta^k$ for $\theta^k < 1$ and $y^k < \theta^k$	-
$\breve{\theta}^k > 1$ for $\theta^k = 1$ and $y^k > 1$	12
$\breve{\theta}^k > 1$ for $\theta^k < 1$ and $y^k > 1$	1
$\breve{\theta}^k < 1$ even though $\theta^k = 1$ because $y^k < 1$	-
$\breve{\theta}^k = 1$ even though $\theta^k < 1$ because $y^k > 1$	-
Percentage (%) of the total	40.5