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ORIGINAL RESEARCH

Emergency decision support modeling under generalized spherical fuzzy Einstein aggregation information

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Abstract
Dominant emergency action should be adopted in the case of an emergency situation. Emergency is interpreted as limited time and information, harmfulness and uncertainty, and decision-makers are often critically bound by uncertainty and risk. This framework implements an emergency decision-making approach to address the emergency situation of COVID-19 in a spherical fuzzy environment. As the spherical fuzzy set (SFS) is a generalized framework of fuzzy structure to handle more uncertainty and ambiguity in decision-making problems (DMPs). Keeping in view the features of the SFSs, the purpose of this paper is to present some robust generalized operating laws in accordance with the Einstein norms. In addition, list of propose aggregation operators using Einstein operational laws under spherical fuzzy environment are developed. Furthermore, we design the algorithm based on the proposed aggregation operators to tackle the uncertainty in emergency decision making problems. Finally, numerical case study of COVID-19 as an emergency decision making is presented to demonstrate the applicability and validity of the proposed technique. Besides, the comparison of the existing and the proposed technique is established to show the effectiveness and validity of the established technique.

Keywords Spherical fuzzy sets · Generalized Einstein aggregation operators · Emergency decision making technique · COVID-19

1 Introduction

In the 21st century, with the rapid economic globalization growth and the acceleration of industrialization, environmental problems, people are facing with several disasters such as epidemic, extremist attacks, earthquakes, storms and other natural disasters that are quite sensitive to human beings. An epidemic of respirational virus triggered by a novel coronavirus disease (COVID-19) that was first spotted in Wuhan City, Hubei Province, China and which has now spotted in above 200 places globally, including cases in the United States, Italy, England, Germany, Iran and Pakistan (WHO COVID-19 Dashboard 2021). It has become dominated headline all over the world. The virus is new and rapidly scattering all over the world. There are various uncertainties regarding its origins, nature, and development. The number of people infected with the COVID-19 is increasing day by day. People are being isolated. Clinical masks and gloves, frequently used as a fence to epidemiologic spread, are selling out, however fitness authorities for example the World Health Organization (WHO) and the Centers for Disease Control and Prevention (Centers for Disease Control and Prevention 2019) recommend people that wearing masks and gloves are not beneficial or essential for avoiding infection in healthy people. The anxiety of COVID-19 is expected to be due to its novelty and the uncertainties about how bad the recent outbreak might become. The anxiety of COVID-19 is much larger than the anxiety of seasonal influenza, however, the latter has killed considerably more people. In

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such type of emergencies, decision makers must be split with some arrangements to lessen and release the infected lives. To date mostly affected countries (WHO COVID-19 Dashboard 2021) are from America, Europe and South-East Asia, because they make mistake of not locked down at early stage. Till date, it has been spread to most of the countries as per WHO report. Around a million population is infected by this disease in which around 1,305,164 deaths reported (Johns Hopkins CSSE 2020). The worst affected part of the world for this outbreak is America, where around 22,960,102 confirmed cases are reported. Among all countries around the world United States, Italy, Spain are in the top list of affected countries (WHO COVID-19 Dashboard 2021). The numbers of COVID-19 patients increasing very rapidly and till date it is going to touch the figure of a million, in which mostly affected countries are from United States, Europe and Asia. Although it started from china even though US and Europe are in top of the list of infected patients.

The COVID-19 is affecting 209 countries and territories around the globe and 2 international conveyances. As of November 15, 2020, Global update; more than 53,507,282 (Johns Hopkins CSSE 2020) cases have been confirmed globally. At least 1,305,164 (Pakistan COVID-19 Dashboard 2019) deaths have been attributed to the disease, most in mainland china, Italy, Iran, USA, UK etc. with more than 1000 of deaths in other countries. More than 36769792 (Pakistan COVID-19 Dashboard 2019) people have recovered. The risk of it spreading further is very high. The pandemic has resulted in travel restrictions and nationwide lockdowns in several countries.

In the case of an emergency, decision makers or disaster response departments should implement strategies or select an appropriate emergency strategy to avoid further escalation of the crisis. It is a matter for the evolving strategy sector to take rapid and effective decisions. Emergency decision-making as an integral part of the disaster response has been a significant task for many governments and a subject of discussion in academic circles. Under this scenario, when making decisions, people are usually bound logical rather than completely reasonable. It is therefore important to establish decision-making approaches that understand human actions in order to provide people with efficient means of responding to emergency situations. However, it is usually difficult to quantify the considered attributes in crisp values due to ambiguity, incompleteness, and uncertainty of the criterion information. To this end, fuzzy sets (FSs) were introduced to tackle multi criteria group decision making (MCGDM) and have become one of the most effective tools for quantification of the considered criteria/attributes. So far, many generalization of fuzzy sets (Zadeh 1965), such as intuitionistic FS (IFS) (Atanassov 1986), Pythagorean FS (PyFS) (Yager 2013), picture FS (PFS) (Cuong and Kreinovich 2013), and spherical FS (SFS) (Ashraf and Abdullah 2019a; Ashraf et al. 2019b), have been presented within academia. The SFSs, which was presented by Ashraf and Abdullah (2019a), is considered as one of the most recent and important type.

A SFS contains three degrees of membership grades, namely positive, neutral and negative, such that the square sum of membership grades is less than or equal to 1. For the aggregation of the criterion information to sort alternatives, traditional decision-making methods are used by many researcher (Ashraf et al. 2020b, c; Ashraf and Abdullah 2020d; Jin et al. 2019c; Qiyas et al. 2021) to contribute the spherical fuzzy set theory. The algebraic aggregation operators (Ashraf et al. 2019b) proposed by using the algebraic norms to address the problem of ambiguity in decision making (DM). Ashraf et al. (2020a) presented Dombi norm based some novel aggregation operators (AOp) for spherical fuzzy environment. Jin et al. (2019a) introduced the logarithmic operational laws based novel AOp and also give brief study of their implementation in real world DM problems. Ashraf et al. (2019c) developed the DM algorithm utilizing distance measure of SFSs and discussed their implementations in DM. Ashraf et al. (2019d) presented the idea of the representation of spherical fuzzy t-norms and t-conorms and explained the technique of TOPSIS to deal with uncertainty to aggregate the criteria for sorting the alternative among the list of alternatives. GRA method for spherical fuzzy data based on linguistic SFSs are given by Ashraf et al. (2018). Jan et al. (2021) presented the analysis of double domination by using the concept of spherical fuzzy information and discussed their application in decision making problems.

It is evident that the above-mentioned AOp is focused on the algebraic operating laws of the SFSs for the implementation of the combination process. Algebraic product and sum are not only fundamental SFS operations that describe the union and the intersection of any two SFSs. A general union and intersection under SF information can be developed from a generalized norm, i.e., instances of deferent-norms families may be used to execute the respective intersections and unions under SF environment. The Einstein product is a good replacement of the algebraic product for an intersection and is capable of delivering smooth estimates of the algebraic product. Equally, a good alternative to the algebraic sum for an intersection is the Einstein sum. However, there seems to be little work in the literature on aggregation approaches that use the Einstein operations on FSs and IFSs to aggregate the intuitionistic fuzzy values. Geometric interaction operators based on Einstein operations for IFS proposed by Garg (2016a). In (Garg 2016a, 2017) Garg presented the Einstein aggregation operators for PyFS and addressed their implementations in DM problems. In (Rahman et al. 2019a, b, 2020) proposed the generalized Einstein aggregation operators for interval values PyFS and addressed their implementations in DM. Khan et al. (2019) proposed picture fuzzy Einstein aggregation (PFEA) operators and discussed their implementation (application) in decision making. The main aim of this paper is therefore to establish some AOp based on Einstein's operational laws under spherical fuzzy environment.

From the above discussion we note that in many practical applications, various aggregation operators have been put forward and implemented. Although, in practical problems many existing AOp are not capable to address such specific cases. In some circumstances, many of these may result in unreasonable or counter-intuitive results. Certain new regulations built without a simple function may have a complicated description. But generalized aggregation operators for SFSs continue to be an open subject that attracts many researchers.

WHO is working with global expert networks and partnerships for laboratory, infection prevention and control, clinical management and mathematical modelling. In such cases, it is essential to provide an efficient way in emergency response for avoiding additional losses and to save the lives of the people. Preventive and mitigation measures play a vital role in both health care and community settings. In this research, a new way of aggregating the SF information is built. The novel generalized spherical fuzzy Einstein aggregation operators are established employing Einstein t-norm and t-conorm. This is an attempt to propose generalized Einstein aggregation operators from a different perspective. We note that, generalized aggregation operators for dealing the uncertainty in practical problems using novel and emerging spherical fuzzy sets do not exist in the literature. The properties of the proposed Einstein operators are discussed. The performance of the proposed generalized Einstein aggregation operators is illustrated by the emergency decision making for COVID-19, as a decision making problem, where the weights of the attributes are unknown. In this regards, spherical fuzzy entropy measure is used to evaluate the weight vector of the attributes.

The motivation of developed AOp is summarized as below.

(1) A very difficult MCGDM problem is the estimation of the supreme option in spherical fuzzy environment due to the involvement of several imprecise factors. Assessment of information in different MCGDM techniques is simply depicted through existing fuzzy numbers which may not consider all the data in a real-world problem.
(2) As a general theory, spherical fuzzy numbers describe efficient execution in the assessment process about uncertain, imprecise and vague information. Thus, spherical fuzzy set theory provides an excellent approach for the assessment of objects under multinary data.
(3) In view of the fact that generalized Einstein AOp are simple but provide a pioneering tool for solving MCDM problems when combine with other powerful mathematical tools, this article aims to develop Einstein AOs in spherical fuzzy environment to handle complex emergency problems.
(4) The Einstein AOp employed in the construction of spherical fuzzy Einstein AOp are more suitable than all other aggregation approaches to tackle the MCGDM situations as developed AOp have ability to consider all the information within the aggregation procedure.
(5) Einstein AOp make the optimal outcomes more accurate and definite when utilized in practical MCGDM problems under spherical fuzzy environment. However, the proposed AOp handle the drawbacks of AOp present in the literature.

Therefore, some generalized SF Einstein AOp are developed to choose the best option in different emergency decision-making situations like COVID-19. The developed operators have some advantages over other approaches which are given as below:

(1) Our proposed methods explain the problems more accurately which involve multiple attributes because they consider spherical fuzzy numbers.
(2) The developed AOp are more precise and efficient with single attribute.
(3) To solve practical problems by using generalized Einstein AOp with spherical fuzzy numbers is very significant.

The contributions of this paper are mainly reflected in the following aspects:

(a) To present some more advanced and generalized operational laws under spherical FSs based on Einstein t-norm and t-conorm.
(b) To present list of novel aggregation operators with the help of the defined generalized Einstein t-norm and t-conorm under spherical fuzzy numbers. Also, the several fundamental properties between the proposed aggregation operators are derived to show its significance.
(c) To present a novel MAGDM technique based on the proposed generalized Einstein aggregation operators to address the group decision making problems under spherical fuzzy environment.
(d) The consistency and effectiveness of the proposed method is demonstrated through a numerical illustration of emergency decision making/emergency response, as well as their detailed evaluations.
(e) The reliability and validity of the proposed technique is demonstrated with the help of sensitivity and comparison analysis.

The rest of this article is organized as set out below. Section 2 provides information concerning IFSs, FSs, PFSs, PyFSs, and SFSs. Section 3 describes the algebraic operations of SFSs. In Sect. 4, novel Einstein basic operational laws are proposed based on the Einstein norm, along with associated proof of its properties. The generalized AOp and their properties are proposed in Sect. 5, is the cornerstone of this work. Section 6 introduces the novel methodology for interacting with the ambiguity in DM problems in order to pick the finest alternative according to the list of attributes. Section 7 provides an overview of the evolved MCDM approach and a comparative analysis with some existing frameworks to MCDM. The article is concluded in Sect. 8 (Table 1).

2 Preliminaries

Let's recall the rudiments of picture FSs and spherical FSs in this section for a while. Upon review, these ideas will be used here.

Definition 1 (Cuong and Kreinovich 2013) A PFS ε in fixed set V is described as

$$\varepsilon = \left\{ \left\langle \hbar_{\text{Ↄ}}, \rho_{\text{Ↄ}}(\hbar_{\text{Ↄ}}), \daleth_{\text{Ↄ}}(\hbar_{\text{Ↄ}}), \tilde{n}_{\text{Ↄ}}(\hbar_{\text{Ↄ}}) \right\rangle | \hbar_{\text{Ↄ}} \in V \right\}, \tag{2.1}$$

for each $\hbar_{\text{Ↄ}} \in V$, the positive membership $\rho_{\text{Ↄ}} : V \to \Theta$, neutral membership $\daleth_{\text{Ↄ}} : V \to \Theta$ and the negative membership $\tilde{n}_{\text{Ↄ}} : V \to \Theta$ specifies the degree of positive, neutral and negative membership grades of the element $\hbar_{\text{Ↄ}}$ to the picture fuzzy set ε, respectively, where $\Theta = [0, 1]$ (is unit interval). In addition, it is important that $0 \le \rho_{\text{Ↄ}}(\hbar_{\text{Ↄ}}) + \daleth_{\text{Ↄ}}(\hbar_{\text{Ↄ}}) + \tilde{n}_{\text{Ↄ}}(\hbar_{\text{Ↄ}}) \le 1$, for each $\hbar_{\text{Ↄ}} \in V$.

Definition 2 (Ashraf and Abdullah 2019a) A SFS ε in fixed set V is described as

$$\varepsilon = \left\{ \left\langle \hbar_{\text{Ↄ}}, \rho_{\text{Ↄ}}(\hbar_{\text{Ↄ}}), \daleth_{\text{Ↄ}}(\hbar_{\text{Ↄ}}), \tilde{n}_{\text{Ↄ}}(\hbar_{\text{Ↄ}}) \right\rangle | \hbar_{\text{Ↄ}} \in V \right\}, \tag{2.2}$$

for each $\hbar_{\text{Ↄ}} \in V$, the positive membership $\rho_{\text{Ↄ}} : V \to \Theta$, neutral membership $\daleth_{\text{Ↄ}} : V \to \Theta$ and the negative membership $\tilde{n}_{\text{Ↄ}} : V \to \Theta$ specifies the degree of positive, neutral and negative membership grades of the element $\hbar_{\text{Ↄ}}$ to the spherical fuzzy set ε, respectively, where $\Theta = [0, 1]$. In addition, it is important that $0 \le \rho^2_{\text{Ↄ}}(\hbar_{\text{Ↄ}}) + \daleth^2_{\text{Ↄ}}(\hbar_{\text{Ↄ}}) + \tilde{n}^2_{\text{Ↄ}}(\hbar_{\text{Ↄ}}) \le 1$, for each $\hbar_{\text{Ↄ}} \in V$.

We shall signify, for convenience, the SFN by the triplet $\varepsilon = (\rho_{\text{Ↄ}}, \daleth_{\text{Ↄ}}, \tilde{n}_{\text{Ↄ}})$.

Let $\varepsilon_1, \varepsilon_2 \in \hat{S}_F\hat{S}(V)$. Ashraf (Ashraf and Abdullah 2019a) defined the following notions:

(1) $\varepsilon_1 \sqsubseteq \varepsilon_2$ if and only if $\rho_{\text{Ↄ}_1}(\hbar_{\text{Ↄ}}) \le \rho_{\text{Ↄ}_2}(\hbar_{\text{Ↄ}}), \daleth_{\text{Ↄ}_1}(\hbar_{\text{Ↄ}}) \le \daleth_{\text{Ↄ}_2}(\hbar_{\text{Ↄ}})$ and $\tilde{n}_{\text{Ↄ}_1}(\hbar_{\text{Ↄ}}) \ge \tilde{n}_{\text{Ↄ}_2}(\hbar_{\text{Ↄ}})$ for each $\hbar_{\text{Ↄ}} \in V$. Clearly $\varepsilon_1 = \varepsilon_2$ if $\varepsilon_1 \sqsubseteq \varepsilon_2$ and $\varepsilon_2 \sqsubseteq \varepsilon_1$.

(2) $\varepsilon_1 \sqcap \varepsilon_2 = \left\{ \min\left(\rho_{\text{Ↄ}_1}, \rho_{\text{Ↄ}_2}\right), \min\left(\daleth_{\text{Ↄ}_1}, \daleth_{\text{Ↄ}_2}\right), \max\left(\tilde{n}_{\text{Ↄ}_1}, \tilde{n}_{\text{Ↄ}_2}\right) \right\}$,

(3) $\varepsilon_1 \sqcup \varepsilon_2 = \left\{ \max\left(\rho_{\text{Ↄ}_1}, \rho_{\text{Ↄ}_2}\right), \min\left(\daleth_{\text{Ↄ}_1}, \daleth_{\text{Ↄ}_2}\right), \min\left(\tilde{n}_{\text{Ↄ}_1}, \tilde{n}_{\text{Ↄ}_2}\right) \right\}$,

(4) $\varepsilon_1^c = \left\{ \tilde{n}_{\text{Ↄ}_1}, \daleth_{\text{Ↄ}_1}, \rho_{\text{Ↄ}_1} \right\}$, where $\varepsilon_1, \varepsilon_2 \in \hat{S}_F\hat{S}(V)$.

Definition 3 (Ashraf and Abdullah 2019a) Let $\varepsilon_1 = \left\{ \rho_{\text{Ↄ}_1}, \daleth_{\text{Ↄ}_1}, \tilde{n}_{\text{Ↄ}_1} \right\}$ and $\varepsilon_2 = \left\{ \rho_{\text{Ↄ}_2}, \daleth_{\text{Ↄ}_2}, \tilde{n}_{\text{Ↄ}_2} \right\} \in \hat{S}_F N(V)$ with $\Psi > 0$. Then, the operational rules are as follows:

$$(1)\ \varepsilon_1 \otimes \varepsilon_2 = \left\{ \rho_{\text{Ↄ}_1}\rho_{\text{Ↄ}_2}, \daleth_{\text{Ↄ}_1}\daleth_{\text{Ↄ}_2}, \sqrt{\tilde{n}^2_{\text{Ↄ}_1} + \tilde{n}^2_{\text{Ↄ}_2} - \tilde{n}^2_{\text{Ↄ}_1}\tilde{n}^2_{\text{Ↄ}_2}} \right\};$$

$$(2)\ \varepsilon_1 \oplus \varepsilon_2 = \left\{ \sqrt{\rho^2_{\text{Ↄ}_1} + \rho^2_{\text{Ↄ}_2} - \rho^2_{\text{Ↄ}_1}\rho^2_{\text{Ↄ}_2}}, \daleth_{\text{Ↄ}_1}\daleth_{\text{Ↄ}_2}, \tilde{n}_{\text{Ↄ}_1}\tilde{n}_{\text{Ↄ}_2} \right\};$$

$$(3)\ \varepsilon_1^{\Psi} = \left\{ \left(\rho_{\text{Ↄ}_1}\right)^{\Psi}, \left(\daleth_{\text{Ↄ}_1}\right)^{\Psi}, \sqrt{1 - \left(1 - \tilde{n}^2_{\text{Ↄ}_1}\right)^{\Psi}} \right\};$$

$$(4)\ \Psi \cdot \varepsilon_1 = \left\{ \sqrt{1 - \left(1 - \rho^2_{\text{Ↄ}_1}\right)^{\Psi}}, \left(\daleth_{\text{Ↄ}_1}\right)^{\Psi}, \left(\tilde{n}_{\text{Ↄ}_1}\right)^{\Psi} \right\}.$$

3 Spherical fuzzy Einstein operators

In this segment, we shall be familiarized with generalized union and intersection for the spherical fuzzy numbers, which are as follows:

$$\varepsilon_1 \vee \varepsilon_2 = \left\{ T\left(\rho_{\text{Ↄ}_1}, \rho_{\text{Ↄ}_2}\right), S\left(\daleth_{\text{Ↄ}_1}, \daleth_{\text{Ↄ}_2}\right), S\left(\tilde{n}_{\text{Ↄ}_1}, \tilde{n}_{\text{Ↄ}_2}\right) \right\},$$
$$\varepsilon_1 \wedge \varepsilon_2 = \left\{ S\left(\rho_{\text{Ↄ}_1}, \rho_{\text{Ↄ}_2}\right), S\left(\daleth_{\text{Ↄ}_1}, \daleth_{\text{Ↄ}_2}\right), T\left(\tilde{n}_{\text{Ↄ}_1}, \tilde{n}_{\text{Ↄ}_2}\right) \right\}.$$

We can also write:

Table 1 List of abbreviation

Abbreviation	Description	Abbreviation	Description
COVID-19	Coronavirus disease	MCGDM	Multi criteria group decision making
DMPs	Decision making problems	WHO	World Health Organization
FSs	Fuzzy sets	IFSs	Intuitionistic fuzzy sets
PyFSs	Pythagorean fuzzy sets	PFSs	Picture fuzzy sets
SFSs	Spherical fuzzy sets	DM	Decision making
AOp	Aggregation operators	PPE	Personal protective equipment

$$\varepsilon_1 \vee \varepsilon_2 = \left\{ \max\left(\rho_{\mathfrak{D}_1}, \rho_{\mathfrak{D}_2}\right), \min\left(\daleth_{\mathfrak{D}_1}, \daleth_{\mathfrak{D}_2}\right), \min\left(\tilde{n}_{\mathfrak{D}_1}, \tilde{n}_{\mathfrak{D}_2}\right) \right\},$$
$$\varepsilon_1 \wedge \varepsilon_2 = \left\{ \min\left(\rho_{\mathfrak{D}_1}, \rho_{\mathfrak{D}_2}\right), \min\left(\daleth_{\mathfrak{D}_1}, \daleth_{\mathfrak{D}_2}\right), \max\left(\tilde{n}_{\mathfrak{D}_1}, \tilde{n}_{\mathfrak{D}_2}\right) \right\}.$$

In above equations, T and S represent the t-norm and s-norm respectively. As, we know that, t-norm (T) and s-norm (S) are the general terms including all types of operators and also contented the necessitate of conjunction and disjunction operators, respectively. Here, we enlist some types of t-norm and s-norm in the Table 2:

However, algebraic sum and algebraic product are obtained using algebraic norm. Algebraic operators can not be only norm operator which used to perform union and intersection. We have many families of norm operators, which can be used to perform corresponding union and intersection. Einstein t-norm and Einstein s-norm are one of the effective family member of norm operators. Einstein sum and product are suitable replacements, respectively, giving the same smooth approximation as the algebraic product and the sum. Einstein t-norm and s-norm for spherical fuzzy environment as follows:

Einstein t-norm Einstein s-norm

$$\check{T}_e(d,l) = \frac{dl}{\sqrt{1+(1-d^2)\cdot(1-l^2)}} \qquad \widehat{S}_E(d,l) = \frac{\sqrt{d^2+l^2}}{\sqrt{1+d^2\cdot l^2}}$$

Where $\check{T}_e(d,l)$ and $\widehat{S}_E(d,l)$ are said to be Einstein t-norm and Einstein s-norm respectively. Also $\check{T}_e(d,l)$ satisfies the basic properties as follows:

For unite interval $v = [0,1]$, the mapping $\Omega : v \times v \to v$ is said to be t-norm iff

(1) Ω is commutative, monotonic and associative,
(2) $\Omega(d,1) = d$.

and

For unite interval $v = [0,1]$, the mapping $\Omega : v \times v \to v$ is said to be s-norm iff

(1) Ω is commutative, monotonic and associative,
(2) $\Omega(d,0) = d$.

Definition 4 Let $\varepsilon_1 = \left\{\rho_{\mathfrak{D}_1}, \daleth_{\mathfrak{D}_1}, \tilde{n}_{\mathfrak{D}_1}\right\}$ and $\varepsilon_2 = \left\{\rho_{\mathfrak{D}_2}, \daleth_{\mathfrak{D}_2}, \tilde{n}_{\mathfrak{D}_2}\right\} \in \hat{S}_F N(V)$ and $\Psi \geq 0$. Then the Einstein operations for spherical fuzzy numbers are follows as:

(1) $$\varepsilon_1 \oplus \varepsilon_2 = \left(\frac{\sqrt{\rho^2_{\mathfrak{D}_1} + \rho^2_{\mathfrak{D}_2}}}{\sqrt{1+\rho^2_{\mathfrak{D}_1} \cdot \rho^2_{\mathfrak{D}_2}}}, \frac{\daleth_{\mathfrak{D}_1} \cdot \daleth_{\mathfrak{D}_2}}{\sqrt{1+\left(1-\daleth^2_{\mathfrak{D}_1}\right)\cdot\left(1-\daleth^2_{\mathfrak{D}_2}\right)}}, \frac{\tilde{n}_{\mathfrak{D}_1} \cdot \tilde{n}_{\mathfrak{D}_2}}{\sqrt{1+\left(1-\tilde{n}^2_{\mathfrak{D}_1}\right)\cdot\left(1-\tilde{n}^2_{\mathfrak{D}_2}\right)}} \right);$$

(2) $$\varepsilon_1 \otimes \varepsilon_2 = \left(\frac{\rho_{\mathfrak{D}_1} \cdot \rho_{\mathfrak{D}_2}}{\sqrt{1+\left(1-\rho^2_{\mathfrak{D}_1}\right)\cdot\left(1-\rho^2_{\mathfrak{D}_2}\right)}}, \frac{\daleth_{\mathfrak{D}_1} \cdot \daleth_{\mathfrak{D}_2}}{\sqrt{1+\left(1-\daleth^2_{\mathfrak{D}_1}\right)\cdot\left(1-\daleth^2_{\mathfrak{D}_2}\right)}}, \frac{\sqrt{\tilde{n}^2_{\mathfrak{D}_1} + \tilde{n}^2_{\mathfrak{D}_2}}}{\sqrt{1+\tilde{n}^2_{\mathfrak{D}_1} \cdot \tilde{n}^2_{\mathfrak{D}_2}}} \right);$$

(3) $$\Psi \cdot \varepsilon_1 = \left(\frac{\sqrt{\left(1+\rho^2_{\mathfrak{D}_1}\right)^{\Psi} - \left(1-\rho^2_{\mathfrak{D}_1}\right)^{\Psi}}}{\sqrt{\left(1+\rho^2_{\mathfrak{D}_1}\right)^{\Psi} + \left(1-\rho^2_{\mathfrak{D}_1}\right)^{\Psi}}}, \frac{\sqrt{2}\left(\daleth_{\mathfrak{D}_1}\right)^{\Psi}}{\sqrt{\left(2-\daleth^2_{\mathfrak{D}_1}\right)^{\Psi} + \left(\daleth^2_{\mathfrak{D}_1}\right)^{\Psi}}}, \frac{\sqrt{2}\left(\tilde{n}_{\mathfrak{D}_1}\right)^{\Psi}}{\sqrt{\left(2-\tilde{n}^2_{\mathfrak{D}_1}\right)^{\Psi} + \left(\tilde{n}^2_{\mathfrak{D}_1}\right)^{\Psi}}} \right);$$

(4) $$\varepsilon_1^{\Psi} = \left(\frac{\sqrt{2}\left(\rho_{\mathfrak{D}_1}\right)^{\Psi}}{\sqrt{\left(2-\rho^2_{\mathfrak{D}_1}\right)^{\Psi} + \left(\rho^2_{\mathfrak{D}_1}\right)^{\Psi}}}, \frac{\sqrt{2}\left(\daleth_{\mathfrak{D}_1}\right)^{\Psi}}{\sqrt{\left(2-\daleth^2_{\mathfrak{D}_1}\right)^{\Psi} + \left(\daleth^2_{\mathfrak{D}_1}\right)^{\Psi}}}, \frac{\sqrt{\left(1+\tilde{n}^2_{\mathfrak{D}_1}\right)^{\Psi} - \left(1-\tilde{n}^2_{\mathfrak{D}_1}\right)^{\Psi}}}{\sqrt{\left(1+\tilde{n}^2_{\mathfrak{D}_1}\right)^{\Psi} + \left(1-\tilde{n}^2_{\mathfrak{D}_1}\right)^{\Psi}}} \right);$$

3.1 Comparison technique for spherical fuzzy sets

In this subsection, we define the score and accuracy values of the spherical fuzzy sets (SFSs). On the basis of score and accuracy values, we can find that which spherical fuzzy set is better than other one.

Definition 5 (Ashraf and Abdullah 2019a) Let $\varepsilon_g = \left\{\rho_{\mathfrak{D}_g}, \daleth_{\mathfrak{D}_g}, \tilde{n}_{\mathfrak{D}_g}\right\} \in \hat{S}_F N(V)\,(g \in \mathbb{N})$. Then

(a) $\check{s}\check{c}(\varepsilon_g) = \frac{1}{3}(2 + \rho_{\mathfrak{D}_g} - \daleth_{\mathfrak{D}_g} - \tilde{n}_{\mathfrak{D}_g}) \in [0,1]$ is called score.
(b) $\tilde{a}\check{c}(\varepsilon_g) = \left(\rho_{\mathfrak{D}_g} - \tilde{n}_{\mathfrak{D}_g}\right) \in [0,1]$ is called the accuracy.

Definition 6 (Ashraf and Abdullah 2019a) Let $\varepsilon_1 = \left\{\rho_{\mathfrak{D}_1}, \daleth_{\mathfrak{D}_1}, \tilde{n}_{\mathfrak{D}_1}\right\}$ and $\varepsilon_2 = \left\{\rho_{\mathfrak{D}_2}, \daleth_{\mathfrak{D}_2}, \tilde{n}_{\mathfrak{D}_2}\right\} \in \hat{S}_F N(V)$. Then

(1) If $\check{s}\check{c}(\varepsilon_1) < \check{s}\check{c}(\varepsilon_2)$, then $\varepsilon_1 < \varepsilon_2$,
(2) If $\check{s}\check{c}(\varepsilon_1) > \check{s}\check{c}(\varepsilon_2)$, then $\varepsilon_1 > \varepsilon_2$,
(3) If $\check{s}\check{c}(\varepsilon_1) = \check{s}\check{c}(\varepsilon_2)$, then
(a) $\tilde{a}\check{c}(\varepsilon_1) < \tilde{a}\check{c}(\varepsilon_2)$, then $\varepsilon_1 < \varepsilon_2$,
(b) $\tilde{a}\check{c}(\varepsilon_1) > \tilde{a}\check{c}(\varepsilon_2)$, then $\varepsilon_1 > \varepsilon_2$,
(c) $\tilde{a}\check{c}(\varepsilon_1) = \tilde{a}\check{c}(\varepsilon_2)$, then $\varepsilon_1 = \varepsilon_2$.

Table 2 Different types of t-norm and s-norm

Name	t-norm	s-norm
Algebraic	$\check{T}_a(d,l) = dl$	$\widehat{S}_A(d,l) = d + l - dl$
Einstein	$\check{T}_e(d,l) = \frac{dl}{1+(1-d)(1-l)}$	$\widehat{S}_E(d,l) = \frac{d+l}{1+dl}$
Hamacher	$\check{T}_h(d,l) = \frac{dl}{\gamma+(1-\gamma)(d+l-dl)}, \gamma > 0$	$\widehat{S}_H(d,l) = \frac{d+l-dl-(1-\gamma)dl}{1-(1-\gamma)dl}, \gamma > 0$
Frank	$\check{T}_f(d,l) = \log_\gamma\left(1 + \frac{(\gamma^d-1)(\gamma^l-1)}{\gamma-1}\right)$	$\widehat{S}_F(d,l) = 1 - \log_\gamma\left(1 + \frac{(\gamma^{1-d}-1)(\gamma^{1-l}-1)}{\gamma-1}\right)$

4 Spherical fuzzy Einstein aggregation (SFEA) operators

In this section, we propose the novel aggregation operators based on Spherical Einstein t-norm and Spherical Einstein s-norm under spherical fuzzy environments.

4.1 Generalized Einstein averaging aggregation operators

In this part of the section, we propose the Einstein weighted averaging and ordered weighted averaging aggregation operators under spherical fuzzy environments.

Definition 7 Consider $\varepsilon_g = \left\{ \rho_{\eth_g}, \daleth_{\eth_g}, \tilde{n}_{\eth_g} \right\} \in \hat{S}_F N(V)$ $(g \in \mathbb{N})$. Then the Einstein averaging aggregation operator for $\hat{S}_F N(V)$ is denoted by *SFEWA* and defined as follows:

$$SFEWA\left(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{\tilde{n}}\right) = \sum_{g=1}^{\tilde{n}} \kappa_g \varepsilon_g, \tag{4.1}$$

where the weights of $\varepsilon_g (g \in \mathbb{N})$ with $\kappa_g \geq 0$, $\sum_{g=1}^{\tilde{n}} \kappa_g = 1$ is $\kappa_g (g \in \mathbb{N})$.

Theorem 1 *Suppose* $\varepsilon_g = \left\{ \rho_{\eth_g}, \daleth_{\eth_g}, \tilde{n}_{\eth_g} \right\} \in \hat{S}_F N(V) (g \in \mathbb{N})$ *and weight vectors of* $\varepsilon_g (g \in \mathbb{N})$ *be* $\kappa = \left(\kappa_1, \kappa_2, \ldots, \kappa_{\tilde{n}}\right)^T$ *related to limit* $\sum_{g=1}^{\tilde{n}} \kappa_\daleth = 1$. *Then the operator of the SFEWA is a mapping of the* $G^{\tilde{n}} \longrightarrow G$ *such that*

$$SFEWA\left(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{\tilde{n}}\right) = \sum_{g=1}^{\tilde{n}} \kappa_g \varepsilon_g = \left(\begin{array}{c} \dfrac{\sqrt{\prod_{g=1}^{\tilde{n}} \left(1+\rho_{\eth_g}^2\right)^{\kappa_g} - \prod_{g=1}^{\tilde{n}} \left(1-\rho_{\eth_g}^2\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}} \left(1+\rho_{\eth_g}^2\right)^{\kappa_g} + \prod_{g=1}^{\tilde{n}} \left(1-\rho_{\eth_g}^2\right)^{\kappa_g}}}, \\ \dfrac{\sqrt{2} \prod_{g=1}^{\tilde{n}} \left(\daleth_{\eth_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{\tilde{n}} \left(2-\daleth_{\eth_g}^2\right)^{\kappa_g} + \prod_{g=1}^{\tilde{n}} \left(\daleth_{\eth_g}^2\right)^{\kappa_g}}}, \\ \dfrac{\sqrt{2} \prod_{g=1}^{\tilde{n}} \left(\tilde{n}_{\eth_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{\tilde{n}} \left(2-\tilde{n}_{\eth_g}^2\right)^{\kappa_g} + \prod_{g=1}^{\tilde{n}} \left(\tilde{n}_{\eth_g}^2\right)^{\kappa_g}}} \end{array} \right). \tag{4.2}$$

Proof By using Mathematical induction on $\tilde{n}$ to prove the Equation (4.2).

When $\tilde{n} = 2$,

$$SFEWA\left(\varepsilon_1, \varepsilon_2\right) = \sum_{g=1}^{2} \kappa_g \varepsilon_g = \kappa_1 \varepsilon_1 + \kappa_2 \varepsilon_2$$

According to Definition 4, we have

$$\kappa_1 \cdot \varepsilon_1 = \left(\begin{array}{c} \dfrac{\sqrt{\left(1+\rho_{\eth_1}^2\right)^{\kappa_1} - \left(1-\rho_{\eth_1}^2\right)^{\kappa_1}}}{\sqrt{\left(1+\rho_{\eth_1}^2\right)^{\kappa_1} + \left(1-\rho_{\eth_1}^2\right)^{\kappa_1}}}, \\ \dfrac{\sqrt{2}\left(\daleth_{\eth_1}\right)^{\kappa_1}}{\sqrt{\left(2-\daleth_{\eth_1}^2\right)^{\kappa_1} + \left(\daleth_{\eth_1}^2\right)^{\kappa_1}}}, \dfrac{\sqrt{2}\left(\tilde{n}_{\eth_1}\right)^{\kappa_1}}{\sqrt{\left(2-\tilde{n}_{\eth_1}^2\right)^{\kappa_1} + \left(\tilde{n}_{\eth_1}^2\right)^{\kappa_1}}} \end{array} \right)$$

$$\kappa_2 \cdot \varepsilon_2 = \left(\begin{array}{c} \dfrac{\sqrt{\left(1+\rho_{\eth_2}^2\right)^{\kappa_2} - \left(1-\rho_{\eth_2}^2\right)^{\kappa_2}}}{\sqrt{\left(1+\rho_{\eth_2}^2\right)^{\kappa_2} + \left(1-\rho_{\eth_2}^2\right)^{\kappa_2}}}, \\ \dfrac{\sqrt{2}\left(\daleth_{\eth_2}\right)^{\kappa_2}}{\sqrt{\left(2-\daleth_{\eth_2}^2\right)^{\kappa_2} + \left(\daleth_{\eth_2}^2\right)^{\kappa_2}}}, \dfrac{\sqrt{2}\left(\tilde{n}_{\eth_2}\right)^{\kappa_2}}{\sqrt{\left(2-\tilde{n}_{\eth_2}^2\right)^{\kappa_2} + \left(\tilde{n}_{\eth_2}^2\right)^{\kappa_2}}} \end{array} \right)$$

Then,

$$SFEWA\left(\varepsilon_1, \varepsilon_2\right) = \kappa_1 \varepsilon_1 + \kappa_2 \varepsilon_2$$

$$= \left(\begin{array}{c} \dfrac{\sqrt{\dfrac{\left(1+\rho_{\eth_1}^2\right)^{\kappa_1} - \left(1-\rho_{\eth_1}^2\right)^{\kappa_1}}{\left(1+\rho_{\eth_1}^2\right)^{\kappa_1} + \left(1-\rho_{\eth_1}^2\right)^{\kappa_1}} + \dfrac{\left(1+\rho_{\eth_2}^2\right)^{\kappa_2} - \left(1-\rho_{\eth_2}^2\right)^{\kappa_2}}{\left(1+\rho_{\eth_2}^2\right)^{\kappa_2} + \left(1-\rho_{\eth_2}^2\right)^{\kappa_2}}}}{\sqrt{1+\dfrac{\left(1+\rho_{\eth_1}^2\right)^{\kappa_1} - \left(1-\rho_{\eth_1}^2\right)^{\kappa_1}}{\left(1+\rho_{\eth_1}^2\right)^{\kappa_1} + \left(1-\rho_{\eth_1}^2\right)^{\kappa_1}} \cdot \dfrac{\left(1+\rho_{\eth_2}^2\right)^{\kappa_2} - \left(1-\rho_{\eth_2}^2\right)^{\kappa_2}}{\left(1+\rho_{\eth_2}^2\right)^{\kappa_2} + \left(1-\rho_{\eth_2}^2\right)^{\kappa_2}}}}, \\ \dfrac{\dfrac{\sqrt{2}\left(\daleth_{\eth_1}\right)^{\kappa_1}}{\sqrt{\left(2-\daleth_{\eth_1}^2\right)^{\kappa_1} + \left(\daleth_{\eth_1}^2\right)^{\kappa_1}}} \cdot \dfrac{\sqrt{2}\left(\daleth_{\eth_2}\right)^{\kappa_2}}{\sqrt{\left(2-\daleth_{\eth_2}^2\right)^{\kappa_2} + \left(\daleth_{\eth_2}^2\right)^{\kappa_2}}}}{\sqrt{1+\left(1-\left(\dfrac{\sqrt{2}\left(\daleth_{\eth_1}\right)^{\kappa_1}}{\sqrt{\left(2-\daleth_{\eth_1}^2\right)^{\kappa_1} + \left(\daleth_{\eth_1}^2\right)^{\kappa_1}}}\right)^2\right) \cdot \left(1-\left(\dfrac{\sqrt{2}\left(\daleth_{\eth_2}\right)^{\kappa_2}}{\sqrt{\left(2-\daleth_{\eth_2}^2\right)^{\kappa_2} + \left(\daleth_{\eth_2}^2\right)^{\kappa_2}}}\right)^2\right)}}, \\ \dfrac{\dfrac{\sqrt{2}\left(\tilde{n}_{\eth_1}\right)^{\kappa_1}}{\sqrt{\left(2-\tilde{n}_{\eth_1}^2\right)^{\kappa_1} + \left(\tilde{n}_{\eth_1}^2\right)^{\kappa_1}}} \cdot \dfrac{\sqrt{2}\left(\tilde{n}_{\eth_2}\right)^{\kappa_2}}{\sqrt{\left(2-\tilde{n}_{\eth_2}^2\right)^{\kappa_2} + \left(\tilde{n}_{\eth_2}^2\right)^{\kappa_2}}}}{\sqrt{1+\left(1-\left(\dfrac{\sqrt{2}\left(\tilde{n}_{\eth_1}\right)^{\kappa_1}}{\sqrt{\left(2-\tilde{n}_{\eth_1}^2\right)^{\kappa_1} + \left(\tilde{n}_{\eth_1}^2\right)^{\kappa_1}}}\right)^2\right) \cdot \left(1-\left(\dfrac{\sqrt{2}\left(\tilde{n}_{\eth_2}\right)^{\kappa_2}}{\sqrt{\left(2-\tilde{n}_{\eth_2}^2\right)^{\kappa_2} + \left(\tilde{n}_{\eth_2}^2\right)^{\kappa_2}}}\right)^2\right)}} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt{\dfrac{\left(1+\rho_{\eth_1}^2\right)^{\kappa_1} \cdot \left(1+\rho_{\eth_2}^2\right)^{\kappa_2} - \left(1-\rho_{\eth_1}^2\right)^{\kappa_1} \cdot \left(1-\rho_{\eth_2}^2\right)^{\kappa_2}}{\left(1+\rho_{\eth_1}^2\right)^{\kappa_1} \cdot \left(1+\rho_{\eth_2}^2\right)^{\kappa_2} + \left(1-\rho_{\eth_1}^2\right)^{\kappa_1} \cdot \left(1-\rho_{\eth_2}^2\right)^{\kappa_2}}}, \\ \dfrac{\sqrt{2}\left(\daleth_{\eth_1} . \daleth_{\eth_2}\right)}{\sqrt{\left(2-\daleth_{\eth_1}^2\right)^{\kappa_1} . \left(2-\daleth_{\eth_2}^2\right)^{\kappa_2} + \left(\daleth_{\eth_1}^2\right)^{\kappa_1} . \left(\daleth_{\eth_2}^2\right)^{\kappa_2}}}, \\ \dfrac{\sqrt{2}\left(\tilde{n}_{\eth_1} . \tilde{n}_{\eth_2}\right)}{\sqrt{\left(2-\tilde{n}_{\eth_1}^2\right)^{\kappa_1} . \left(2-\tilde{n}_{\eth_2}^2\right)^{\kappa_2} + \left(\tilde{n}_{\eth_1}^2\right)^{\kappa_1} . \left(\tilde{n}_{\eth_2}^2\right)^{\kappa_2}}} \end{array} \right).$$

Thus the Equation (4.2), is true for $\tilde{n} = 2$.

Suppose that Equation 4.2, is true for $\tilde{n} = z$, we have

$$SFEWA\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_z\right)=\begin{pmatrix}\frac{\sqrt{\prod_{g=1}^{z}\left(1+\rho^2_{Ɗ_g}\right)^{\kappa_g}-\prod_{g=1}^{z}\left(1-\rho^2_{Ɗ_g}\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{z}\left(1+\rho^2_{Ɗ_g}\right)^{\kappa_g}+\prod_{g=1}^{z}\left(1-\rho^2_{Ɗ_g}\right)^{\kappa_g}}},\\ \frac{\sqrt{2}\prod_{g=1}^{z}\left(\daleth_{Ɗ_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{z}\left(2-\daleth^2_{Ɗ_g}\right)^{\kappa_g}+\prod_{g=1}^{z}\left(\daleth^2_{Ɗ_g}\right)^{\kappa_g}}},\\ \frac{\sqrt{2}\prod_{g=1}^{z}\left(\tilde{n}_{Ɗ_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{z}\left(2-\tilde{n}^2_{Ɗ_g}\right)^{\kappa_g}+\prod_{g=1}^{z}\left(\tilde{n}^2_{Ɗ_g}\right)^{\kappa_g}}}\end{pmatrix}.$$

Next we have to prove that Equation (4.2) is true for $\tilde{n}=z+1$. For this we have

$$\begin{aligned}&SFEWA\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_z,\varepsilon_{z+1}\right)\\&=\sum_{g=1}^{z}\kappa_g\varepsilon_g+\kappa_{z+1}\varepsilon_{z+1}\\&=\begin{pmatrix}\frac{\sqrt{\prod_{g=1}^{z}\left(1+\rho^2_{Ɗ_g}\right)^{\kappa_g}-\prod_{g=1}^{z}\left(1-\rho^2_{Ɗ_g}\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{z}\left(1+\rho^2_{Ɗ_g}\right)^{\kappa_g}+\prod_{g=1}^{z}\left(1-\rho^2_{Ɗ_g}\right)^{\kappa_g}}},\\ \frac{\sqrt{2}\prod_{g=1}^{z}\left(\daleth_{Ɗ_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{z}\left(2-\daleth^2_{Ɗ_g}\right)^{\kappa_g}+\prod_{g=1}^{z}\left(\daleth^2_{Ɗ_g}\right)^{\kappa_g}}},\\ \frac{\sqrt{2}\prod_{g=1}^{z}\left(\tilde{n}_{Ɗ_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{z}\left(2-\tilde{n}^2_{Ɗ_g}\right)^{\kappa_g}+\prod_{g=1}^{z}\left(\tilde{n}^2_{Ɗ_g}\right)^{\kappa_g}}}\end{pmatrix}\\&+\begin{pmatrix}\frac{\sqrt{\left(1+\rho^2_{\sigma_{z+1}}\right)^{\kappa_{z+1}}-\left(1-\rho^2_{\sigma_{z+1}}\right)^{\kappa_{z+1}}}}{\sqrt{\left(1+\rho^2_{\sigma_{z+1}}\right)^{\kappa_{z+1}}+\left(1-\rho^2_{\sigma_{z+1}}\right)^{\kappa_{z+1}}}},\\ \frac{\sqrt{2}\left(\daleth_{\sigma_{z+1}}\right)^{\kappa_{z+1}}}{\sqrt{\left(2-\daleth^2_{\sigma_{z+1}}\right)^{\kappa_{z+1}}+\left(\daleth^2_{\sigma_{z+1}}\right)^{\kappa_{z+1}}}},\\ \frac{\sqrt{2}\left(\tilde{n}_{\sigma_{z+1}}\right)^{\kappa_{z+1}}}{\sqrt{\left(2-\tilde{n}^2_{\sigma_{z+1}}\right)^{\kappa_{z+1}}+\left(\tilde{n}^2_{\sigma_{z+1}}\right)^{\kappa_{z+1}}}}\end{pmatrix}\\&=\begin{pmatrix}\frac{\sqrt{\prod_{g=1}^{z+1}\left(1+\rho^2_{Ɗ_g}\right)^{\kappa_g}-\prod_{g=1}^{z+1}\left(1-\rho^2_{Ɗ_g}\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{z+1}\left(1+\rho^2_{Ɗ_g}\right)^{\kappa_g}+\prod_{g=1}^{z+1}\left(1-\rho^2_{Ɗ_g}\right)^{\kappa_g}}},\\ \frac{\sqrt{2}\prod_{g=1}^{z+1}\left(\daleth_{Ɗ_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{z+1}\left(2-\daleth^2_{Ɗ_g}\right)^{\kappa_g}+\prod_{g=1}^{z+1}\left(\daleth^2_{Ɗ_g}\right)^{\kappa_g}}},\frac{\sqrt{2}\prod_{g=1}^{z+1}\left(\tilde{n}_{Ɗ_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{z+1}\left(2-\tilde{n}^2_{Ɗ_g}\right)^{\kappa_g}+\prod_{g=1}^{z+1}\left(\tilde{n}^2_{Ɗ_g}\right)^{\kappa_g}}}\end{pmatrix}\end{aligned}$$

Hence Equation (4.2) is true for $\tilde{n}=z+1$. Therefore, by the principal of mathematical induction the result is true for all $\tilde{n}$. □

The following properties of SFEWA operator can be proved using definitons.

Theorem 2 1) *Let* $\varepsilon_g=\left\{\rho_{Ɗ_g},\daleth_{Ɗ_g},\tilde{n}_{Ɗ_g}\right\}\in\hat{S}_FN(V)$ $(g=1,2,3,\ldots,\tilde{n})$, *if* $\varepsilon_1=\varepsilon_2=\cdots\varepsilon_{\tilde{n}-1}=\varepsilon_{\tilde{n}}=\varepsilon$, *then* $SFEWA\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{\tilde{n}}\right)=\varepsilon$.

2) *Let* $\varepsilon_g=\left\{\rho_{Ɗ_g},\daleth_{Ɗ_g},\tilde{n}_{Ɗ_g}\right\}\in\hat{S}_FN(V)\,(g=1,2,3,\ldots,\tilde{n})$ *and* $\varepsilon^-=\min_g\varepsilon_g$, $\varepsilon^+=\max_g\varepsilon_g$. *Then*, $\varepsilon^-\le SFEWA\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{\tilde{n}}\right)\le\varepsilon^+$.

3) *Let* $\varepsilon_g=\left\{\rho_{Ɗ_g},\daleth_{Ɗ_g},\tilde{n}_{Ɗ_g}\right\}$ *and* $\varepsilon_{\tilde{g}}=\left\{\rho_{\sigma_{\tilde{g}}},\daleth_{\sigma_{\tilde{g}}},\tilde{n}_{\sigma_{\tilde{g}}}\right\}$ $\in\hat{S}_FN(V)\left(g,\tilde{g}\in\mathbb{N}\right)$ *i.e.* $\varepsilon_g\le\varepsilon_{\tilde{g}}$ *for all g. Then*

$SFEWA\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{\tilde{n}}\right)\le SFEWA\left(\varepsilon_{\tilde{1}},\varepsilon_{\tilde{2}},\varepsilon_{\tilde{3}},\ldots,\varepsilon_{\tilde{\tilde{n}}}\right)$.

Definition 8 Let $\varepsilon_g=\left\{\rho_{Ɗ_g},\daleth_{Ɗ_g},\tilde{n}_{Ɗ_g}\right\}\in\hat{S}_FN(V)$ $(g=1,2,3,\ldots,\tilde{n})$. Then the Generalized Einstein averaging aggregation operator for $\hat{S}_FN(V)$ is denoted by *GSFEWA* and defined as follows:

$$GSFEWA\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{\tilde{n}}\right)=\left(\sum_{g=1}^{\tilde{n}}\kappa_g\varepsilon_g^{\Upsilon}\right)^{\frac{1}{\Upsilon}},\tag{4.3}$$

where the weights of ε_g $(g\in\mathbb{N})$ with $\kappa_g\ge 0$, $\sum_{g=1}^{\tilde{n}}\kappa_g=1$ is $\kappa_g(g\in\mathbb{N})$.

Theorem 3 *Let* $\varepsilon_g=\left\{\rho_{Ɗ_g},\daleth_{Ɗ_g},\tilde{n}_{Ɗ_g}\right\}\in\hat{S}_FN(V)\,(g\in\mathbb{N})$ *and weights of* $\varepsilon_g(g\in\mathbb{N})$ *subject to* $\sum_{g=1}^{\tilde{n}}\kappa_g=1$ *be denoted by* $\kappa=\left(\kappa_1,\kappa_2,\ldots,\kappa_{\tilde{n}}\right)^T$. *The GSFEWA operator is a mapping* $G^{\tilde{n}}\longrightarrow G$ *such that*

$$
GSFEWA\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{\tilde{n}}\right)
= \left(
\begin{array}{c}
\dfrac{\sqrt{2\left\{\prod_{g=1}^{\tilde{n}}\left\{\left(2-\rho_{\beth_g}^2\right)^{\Upsilon}+3\left(\rho_{\beth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left\{\left(2-\rho_{\beth_g}^2\right)^{\Upsilon}-\left(\rho_{\beth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right\}^{\frac{1}{\Upsilon}}}}{\sqrt{\begin{array}{c}\left(\prod_{g=1}^{\tilde{n}}\left\{\left(2-\rho_{\beth_g}^2\right)^{\Upsilon}+3\left(\rho_{\beth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}+3\prod_{g=1}^{\tilde{n}}\left\{\left(2-\rho_{\beth_g}^2\right)^{\Upsilon}-\left(\rho_{\beth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}+\\ \left(\prod_{g=1}^{\tilde{n}}\left\{\left(2-\rho_{\beth_g}^2\right)^{\Upsilon}+3\left(\rho_{\beth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left\{\left(2-\rho_{\beth_g}^2\right)^{\Upsilon}-\left(\rho_{\beth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}\end{array}}}, \\[2ex]
\sqrt{\dfrac{2\left(\left(\prod_{g=1}^{\tilde{n}}\left(\daleth_{\beth_g}^2\right)^{\Upsilon}\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}{\left(\left(\prod_{g=1}^{\tilde{n}}\left(2-\daleth_{\beth_g}^2\right)^{\Upsilon}\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}+\left(\left(\prod_{g=1}^{\tilde{n}}\left(\daleth_{\beth_g}^2\right)^{\Upsilon}\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}, \\[2ex]
\sqrt{\dfrac{\begin{array}{c}\left(\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\beth_g}^2\right)^{\Upsilon}+3\left(1-\tilde{n}_{\beth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}+3\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\beth_g}^2\right)^{\Upsilon}-\left(1-\tilde{n}_{\beth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}\\ -\left(\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\beth_g}^2\right)^{\Upsilon}+3\left(1-\tilde{n}_{\beth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\beth_g}^2\right)^{\Upsilon}-\left(1-\tilde{n}_{\beth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}\end{array}}{\begin{array}{c}\left(\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\beth_g}^2\right)^{\Upsilon}+3\left(1-\tilde{n}_{\beth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}+3\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\beth_g}^2\right)^{\Upsilon}-\left(1-\tilde{n}_{\beth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}\\ +\left(\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\beth_g}^2\right)^{\Upsilon}+3\left(1-\tilde{n}_{\beth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\beth_g}^2\right)^{\Upsilon}-\left(1-\tilde{n}_{\beth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}\end{array}}}
\end{array}
\right) \tag{4.4}
$$

Proof Using Definition (4), we have

$$\varepsilon_g^{\Upsilon}=\left(\frac{\sqrt{2}\left(\rho_{\eth_g}\right)^{\Upsilon}}{\sqrt{\left(2-\rho_{\eth_g}^2\right)^{\Upsilon}+\left(\rho_{\eth_g}^2\right)^{\Upsilon}}},\frac{\sqrt{2}\left(\daleth_{\eth_g}\right)^{\Upsilon}}{\sqrt{\left(2-\daleth_{\eth_g}^2\right)^{\Upsilon}+\left(\daleth_{\eth_g}^2\right)^{\Upsilon}}},\frac{\sqrt{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}-\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}}}{\sqrt{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}+\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}}}\right)$$

$$\Rightarrow \sum_{g=1}^{\tilde{n}}\kappa_g\varepsilon_g^{\Upsilon}$$

$$=\left(\frac{\sqrt{\prod_{g=1}^{\tilde{n}}\left(1+\frac{2\left(\rho_{\eth_g}\right)^{2\Upsilon}}{\left(2-\rho_{\eth_g}^2\right)^{\Upsilon}+\left(\rho_{\eth_g}^2\right)^{\Upsilon}}\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(1-\frac{2\left(\rho_{\eth_g}\right)^{2\Upsilon}}{\left(2-\rho_{\eth_g}^2\right)^{\Upsilon}+\left(\rho_{\eth_g}^2\right)^{\Upsilon}}\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(1+\frac{2\left(\rho_{\eth_g}\right)^{2\Upsilon}}{\left(2-\rho_{\eth_g}^2\right)^{\Upsilon}+\left(\rho_{\eth_g}^2\right)^{\Upsilon}}\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(1-\frac{2\left(\rho_{\eth_g}\right)^{2\Upsilon}}{\left(2-\rho_{\eth_g}^2\right)^{\Upsilon}+\left(\rho_{\eth_g}^2\right)^{\Upsilon}}\right)^{\kappa_g}}},\frac{\sqrt{2\prod_{g=1}^{\tilde{n}}\left(\frac{2\left(\daleth_{\eth_g}^2\right)^{\Upsilon}}{\left(2-\daleth_{\eth_g}^2\right)^{\Upsilon}+\left(\daleth_{\eth_g}^2\right)^{\Upsilon}}\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(2-\frac{2\left(\daleth_{\eth_g}^2\right)^{\Upsilon}}{\left(2-\daleth_{\eth_g}^2\right)^{\Upsilon}+\left(\daleth_{\eth_g}^2\right)^{\Upsilon}}\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\frac{2\left(\daleth_{\eth_g}^2\right)^{\Upsilon}}{\left(2-\daleth_{\eth_g}^2\right)^{\Upsilon}+\left(\daleth_{\eth_g}^2\right)^{\Upsilon}}\right)^{\kappa_g}}},\frac{\sqrt{2}\prod_{g=1}^{\tilde{n}}\left(\frac{\sqrt{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}-\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}}}{\sqrt{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}+\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}}}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(2-\frac{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}-\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}}{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}+\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}}\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\frac{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}-\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}}{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}+\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}}\right)^{\kappa_g}}}\right)$$

$$=\left(\frac{\sqrt{\prod_{g=1}^{\tilde{n}}\left\{\left(2-\rho_{\eth_g}^2\right)^{\Upsilon}+3\left(\rho_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left\{\left(2-\rho_{\eth_g}^2\right)^{\Upsilon}-\left(\rho_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left\{\left(2-\rho_{\eth_g}^2\right)^{\Upsilon}+3\left(\rho_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left\{\left(2-\rho_{\eth_g}^2\right)^{\Upsilon}-\left(\rho_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}}},\frac{\sqrt{2\prod_{g=1}^{\tilde{n}}\left(\left(\daleth_{\eth_g}^2\right)^{\Upsilon}\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(\left(2-\daleth_{\eth_g}^2\right)^{\Upsilon}\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\left(\daleth_{\eth_g}^2\right)^{\Upsilon}\right)^{\kappa_g}}},\frac{\sqrt{2\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}-\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}+3\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}-\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}}}\right)$$

Suppose that

$$\widetilde{a}_g=\left(2-\rho_{\eth_g}^2\right)^{\Upsilon}+3\left(\rho_{\eth_g}^2\right)^{\Upsilon},\quad \widetilde{b}_g=\left(2-\rho_{\eth_g}^2\right)^{\Upsilon}-\left(\rho_{\eth_g}^2\right)^{\Upsilon}$$
$$\widetilde{c}_g=\left(\daleth_{\eth_g}^2\right)^{\Upsilon}\quad \widetilde{d}_g=\left(2-\daleth_{\eth_g}^2\right)^{\Upsilon}$$
$$\widetilde{e}_g=\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}-\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}\quad \widetilde{f}_g=\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}+3\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}$$

$$\Rightarrow \sum_{g=1}^{\tilde{n}}\kappa_g\varepsilon_g^{\Upsilon}=\left(\frac{\sqrt{\prod_{g=1}^{\tilde{n}}\left(\widetilde{a}_g\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(\widetilde{b}_g\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(\widetilde{a}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{b}_g\right)^{\kappa_g}}},\frac{\sqrt{2\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(\widetilde{d}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}}},\frac{\sqrt{2\prod_{g=1}^{\tilde{n}}\left(\widetilde{e}_g\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(\widetilde{f}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{e}_g\right)^{\kappa_g}}}\right)$$

Therefore,

$$\left(\sum_{g=1}^{\tilde{n}}\kappa_g\varepsilon_g^{\Upsilon}\right)^{\frac{1}{\Upsilon}}$$

$$=\left(\frac{\sqrt{2\left(\frac{\prod_{g=1}^{\tilde{n}}\left(\widetilde{a}_g\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(\widetilde{b}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{a}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{b}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}}}{\sqrt{\left(2-\frac{\prod_{g=1}^{\tilde{n}}\left(\widetilde{a}_g\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(\widetilde{b}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{a}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{b}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}+\left(\frac{\prod_{g=1}^{\tilde{n}}\left(\widetilde{a}_g\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(\widetilde{b}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{a}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{b}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}}},\frac{\sqrt{2\left(\frac{2\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{d}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}}}{\sqrt{\left(2-\frac{2\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{d}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}+\left(\frac{2\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{d}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}}},\frac{\sqrt{\left(1+\frac{2\prod_{g=1}^{\tilde{n}}\left(\widetilde{e}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{f}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{e}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}-\left(1-\frac{2\prod_{g=1}^{\tilde{n}}\left(\widetilde{e}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{f}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{e}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}}}{\sqrt{\left(1+\frac{2\prod_{g=1}^{\tilde{n}}\left(\widetilde{e}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{f}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{e}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}+\left(1-\frac{2\prod_{g=1}^{\tilde{n}}\left(\widetilde{e}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{f}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{e}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}}}\right)$$

$$=\left(\frac{\sqrt{2\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{a}_g\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(\widetilde{b}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}{\sqrt{\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{a}_g\right)^{\kappa_g}+3\prod_{g=1}^{\tilde{n}}\left(\widetilde{b}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}+\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{a}_g\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(\widetilde{b}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}},\frac{\sqrt{2\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}{\sqrt{\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{d}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}+\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}},\frac{\sqrt{\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{e}_g\right)^{\kappa_g}+3\prod_{g=1}^{\tilde{n}}\left(\widetilde{f}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}-\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{e}_g\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(\widetilde{f}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}{\sqrt{\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{e}_g\right)^{\kappa_g}+3\prod_{g=1}^{\tilde{n}}\left(\widetilde{f}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}+\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{e}_g\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(\widetilde{f}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}\right)$$

By back substitution, we have

$$
= \left(\begin{array}{c}
\dfrac{\sqrt{2\left(\prod_{g=1}^{\tilde{n}}\left\{\left(2-\rho_{\eth_g}^2\right)^{\Upsilon}+3\left(\rho_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left\{\left(2-\rho_{\eth_g}^2\right)^{\Upsilon}-\left(\rho_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}{\sqrt{\begin{array}{c}\left(\prod_{g=1}^{\tilde{n}}\left\{\left(2-\rho_{\eth_g}^2\right)^{\Upsilon}+3\left(\rho_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}+3\prod_{g=1}^{\tilde{n}}\left\{\left(2-\rho_{\eth_g}^2\right)^{\Upsilon}-\left(\rho_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}+\\ \left(\prod_{g=1}^{\tilde{n}}\left\{\left(2-\rho_{\eth_g}^2\right)^{\Upsilon}+3\left(\rho_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left\{\left(2-\rho_{\eth_g}^2\right)^{\Upsilon}-\left(\rho_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}\end{array}}}, \\[2ex]
\dfrac{\sqrt{2\left(\prod_{g=1}^{\tilde{n}}\left(\left(\daleth_{\eth_g}^2\right)^{\Upsilon}\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}{\sqrt{\left(\prod_{g=1}^{\tilde{n}}\left(\left(2-\daleth_{\eth_g}^2\right)^{\Upsilon}\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}+\left(\prod_{g=1}^{\tilde{n}}\left(\left(\daleth_{\eth_g}^2\right)^{\Upsilon}\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}, \\[2ex]
\dfrac{\sqrt{\begin{array}{c}\left(\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}-\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}+3\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}+3\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}-\\ \left(\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}-\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}+3\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}\end{array}}}{\sqrt{\begin{array}{c}\left(\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}-\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}+3\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}+3\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}+\\ \left(\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}-\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left\{\left(1+\tilde{n}_{\eth_g}^2\right)^{\Upsilon}+3\left(1-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}\end{array}}}
\end{array} \right)
$$

Hence, proved. □

In procedure of aggregating the spherical fuzzy information parameter Υ plays the vital role. When we fixed the parameter to special number, then the GSFEWA operator can be reduced. Like, when we take $\Upsilon = 1$, GSFEWA operator is reduced to SFEWA.

$$
SFEWA\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{\tilde{n}}\right) = \left(\begin{array}{c}
\dfrac{\sqrt{\prod_{g=1}^{\tilde{n}}\left(1+\rho_{\eth_g}^2\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(1-\rho_{\eth_g}^2\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(1+\rho_{\eth_g}^2\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(1-\rho_{\eth_g}^2\right)^{\kappa_g}}}, \dfrac{\sqrt{2}\prod_{g=1}^{\tilde{n}}\left(\daleth_{\eth_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(2-\daleth_{\eth_g}^2\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\daleth_{\eth_g}^2\right)^{\kappa_g}}}, \\
\dfrac{\sqrt{2}\prod_{g=1}^{\tilde{n}}\left(\tilde{n}_{\eth_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(2-\tilde{n}_{\eth_g}^2\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\tilde{n}_{\eth_g}^2\right)^{\kappa_g}}}
\end{array} \right)
$$

The following properties of GSFEWA operator can be simply proved.

Theorem 4 1) *Let* $\varepsilon_g = \left\{\rho_{\eth_g}, \daleth_{\eth_g}, \tilde{n}_{\eth_g}\right\} \in \hat{S}_{F}N(V)$ $(g = 1,2,3,\ldots,\tilde{n})$, *if* $\varepsilon_1 = \varepsilon_2 = \cdots \varepsilon_{\tilde{n}-1} = \varepsilon_{\tilde{n}} = \varepsilon$, *then* $GSFEWA\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{\tilde{n}}\right) = \varepsilon$.

2) *Let* $\varepsilon_g = \left\{\rho_{\eth_g}, \daleth_{\eth_g}, \tilde{n}_{\eth_g}\right\} \in \hat{S}_{F}N(V)$ $(g = 1,2,3,\ldots,\tilde{n})$ *and* $\varepsilon^- = \min_g \varepsilon_g$, $\varepsilon^+ = \max_g \varepsilon_g$. *Then,* $\varepsilon^- \le GSFEWA\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{\tilde{n}}\right) \le \varepsilon^+$.

3) *Let* $\varepsilon_g = \left\{\rho_{\eth_g}, \daleth_{\eth_g}, \tilde{n}_{\eth_g}\right\}$ *and* $\varepsilon_{\tilde{g}} = \left\{\rho_{\sigma_{\tilde{g}}}, \daleth_{\sigma_{\tilde{g}}}, \tilde{n}_{\sigma_{\tilde{g}}}\right\} \in \hat{S}_{F}N(V)\left(g,\tilde{g} \in \mathbb{N}\right)$ *such that* $\varepsilon_g \le \varepsilon_{\tilde{g}}$ *for all g. Then*

$$GSFEWA\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{\tilde{n}}\right) \le GSFEWA\left(\varepsilon_{\tilde{1}},\varepsilon_{\tilde{2}},\varepsilon_{\tilde{3}},\ldots,\varepsilon_{\tilde{n}}\right).$$

Definition 9 Let $\varepsilon_g = \left\{\rho_{\eth_g}, \daleth_{\eth_g}, \tilde{n}_{\eth_g}\right\} \in \hat{S}_{F}N(V)$ $(g = 1,2,3,\ldots,\tilde{n})$. Then, the Generalized Einstein ordered averaging aggregation operator for $\hat{S}_{F}N(V)$ is denoted by *GSFEOWA* and defined as follows:

$$GSFEOWA\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{\tilde{n}}\right) = \left(\sum_{g=1}^{\tilde{n}} \kappa_g \varepsilon_{\Omega(g)}^{\Upsilon}\right)^{\frac{1}{\Upsilon}}, \tag{4.5}$$

where weights of ε_g $(g \in \mathbb{N})$, with $\kappa_g \geq 0$, $\sum_{g=1}^{\tilde{n}} \kappa_g = 1$ are $\kappa_g (g \in \mathbb{N})$. Where Υ is the real number greater than zero, $\Omega : (1, 2, \ldots, \tilde{n}) - \rightarrow (1, 2, \ldots, \tilde{n})$, SFN $\varepsilon_{\Omega(g)}$ is the gth largest of SFN ε_g.

Theorem 5 *Let* $\varepsilon_g = \left\{ \rho_{\text{Ↄ}_g}(\hbar_{\text{Ↄ}}), \daleth_{\text{Ↄ}_g}(\hbar_{\text{Ↄ}}), \tilde{n}_{\text{Ↄ}_g}(\hbar_{\text{Ↄ}}) \right\} \in \hat{S}_F N(V)$ $(g \in \mathbb{N})$ *and weights of* $\varepsilon_g (g \in \mathbb{N})$ *subject to* $\sum_{g=1}^{\tilde{n}} \kappa_g = 1$ *be* $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_{\tilde{n}})^T$. *The GSFEOWA operator is a mapping* $G^{\tilde{n}} \longrightarrow G$ *such that* $GSFEOWA(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{\tilde{n}}) =$

$$\left(\begin{array}{c} \dfrac{\sqrt{2\left\{ \prod_{g=1}^{\tilde{n}} \left\{ \left(2-\rho^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} + 3\left(\rho^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} \right\}^{\kappa_g} - \prod_{g=1}^{\tilde{n}} \left\{ \left(2-\rho^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} - \left(\rho^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} \right\}^{\kappa_g} \right\}^{\frac{1}{\Upsilon}}}}{\sqrt{\begin{array}{c} \left(\prod_{g=1}^{\tilde{n}} \left\{ \left(2-\rho^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} + 3\left(\rho^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} \right\}^{\kappa_g} + 3\prod_{g=1}^{\tilde{n}} \left\{ \left(2-\rho^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} - \left(\rho^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} \right\}^{\kappa_g} \right)^{\frac{1}{\Upsilon}} + \\ \left(\prod_{g=1}^{\tilde{n}} \left\{ \left(2-\rho^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} + 3\left(\rho^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} \right\}^{\kappa_g} - \prod_{g=1}^{\tilde{n}} \left\{ \left(2-\rho^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} - \left(\rho^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} \right\}^{\kappa_g} \right)^{\frac{1}{\Upsilon}} \end{array}}}, \\ \sqrt{\dfrac{2\left(\left(\prod_{g=1}^{\tilde{n}} \left(\daleth^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon}\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}{\left(\left(\prod_{g=1}^{\tilde{n}} \left(2-\daleth^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon}\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}} + \left(\left(\prod_{g=1}^{\tilde{n}} \left(\daleth^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon}\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}, \\ \sqrt{\dfrac{\begin{array}{c} \left(\prod_{g=1}^{\tilde{n}} \left\{ \left(1+\tilde{n}^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} + 3\left(1-\tilde{n}^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} \right\}^{\kappa_g} + 3\prod_{g=1}^{\tilde{n}} \left\{ \left(1+\tilde{n}^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} - \left(1-\tilde{n}^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} \right\}^{\kappa_g} \right)^{\frac{1}{\Upsilon}} \\ -\left(\prod_{g=1}^{\tilde{n}} \left\{ \left(1+\tilde{n}^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} + 3\left(1-\tilde{n}^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} \right\}^{\kappa_g} - \prod_{g=1}^{\tilde{n}} \left\{ \left(1+\tilde{n}^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} - \left(1-\tilde{n}^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} \right\}^{\kappa_g} \right)^{\frac{1}{\Upsilon}} \end{array}}{\begin{array}{c} \left(\prod_{g=1}^{\tilde{n}} \left\{ \left(1+\tilde{n}^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} + 3\left(1-\tilde{n}^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} \right\}^{\kappa_g} + 3\prod_{g=1}^{\tilde{n}} \left\{ \left(1+\tilde{n}^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} - \left(1-\tilde{n}^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} \right\}^{\kappa_g} \right)^{\frac{1}{\Upsilon}} \\ +\left(\prod_{g=1}^{\tilde{n}} \left\{ \left(1+\tilde{n}^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} + 3\left(1-\tilde{n}^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} \right\}^{\kappa_g} - \prod_{g=1}^{\tilde{n}} \left\{ \left(1+\tilde{n}^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} - \left(1-\tilde{n}^2_{\text{Ↄ}_{\Omega(g)}}\right)^{\Upsilon} \right\}^{\kappa_g} \right)^{\frac{1}{\Upsilon}} \end{array}}} \end{array} \right)$$

Proof The proof of this theorem is similar to that of Theorem 3 and hence it is omitted here. □

4.2 Generalized Einstein geometric aggregation operators

In this part of the section, we propose the Einstein weighted geometric and ordered weighted geometric aggregation operators under SF environments.

Definition 10 Let $\varepsilon_g = \left\{ \rho_{\text{Ↄ}_g}, \daleth_{\text{Ↄ}_g}, \tilde{n}_{\text{Ↄ}_g} \right\} \in \hat{S}_F N(V)$ $(g \in \mathbb{N})$. Then, the Einstein weighted geometric aggregation operator for $\hat{S}_F N(V)$ is represented by $SFEWG$ and defined as follows:

$$SFEWG(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{\tilde{n}}) = \prod_{g=1}^{\tilde{n}} (\varepsilon_g)^{\kappa_g}, \tag{4.6}$$

where weights of ε_g $(g \in \mathbb{N})$ with $\kappa_g \geq 0$, $\sum_{g=1}^{\tilde{n}} \kappa_g = 1$ is $\kappa_g (g \in \mathbb{N})$.

Theorem 6 *Let* $\varepsilon_g = \left\{ \rho_{\text{Ↄ}_g}, \daleth_{\text{Ↄ}_g}, \tilde{n}_{\text{Ↄ}_g} \right\} \in \hat{S}_F N(V)$ $(g \in \mathbb{N})$ *and weights of* $\varepsilon_g (g \in \mathbb{N})$ *subject to* $\sum_{g=1}^{\tilde{n}} \kappa_{\daleth} = 1$ *be denoted by* $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_{\tilde{n}})^T$. *The SFEWG operator is a mapping* $G^{\tilde{n}} \longrightarrow G$ *such that*

$$SFEWG\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{\tilde{n}}\right)=\prod_{g=1}^{\tilde{n}}\left(\varepsilon_g\right)^{\kappa_g}$$

$$=\left(\begin{array}{c}\dfrac{\sqrt{2}\prod_{g=1}^{\tilde{n}}\left(\rho_{\eth_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(2-\rho^2_{\eth_g}\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(P2_{\eth_g}\right)^{\kappa_g}}},\\ \dfrac{\sqrt{2}\prod_{g=1}^{\tilde{n}}\left(\daleth_{\eth_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(2-\daleth^2_{\eth_g}\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\daleth^2_{\eth_g}\right)^{\kappa_g}}},\\ \dfrac{\sqrt{\prod_{g=1}^{\tilde{n}}\left(1+\tilde{n}^2_{\eth_g}\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(1-\tilde{n}^2_{\eth_g}\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(1+\tilde{n}^2_{\eth_g}\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(1-\tilde{n}^2_{\eth_g}\right)^{\kappa_g}}}\end{array}\right). \tag{4.7}$$

Proof By using Mathematical induction on $\tilde{n}$ to prove the Equation 4.7 .

When $\tilde{n}=2$,

$$SFEWG\left(\varepsilon_1,\varepsilon_2\right)=\prod_{g=1}^{2}\left(\varepsilon_g\right)^{\kappa_g}=\left(\varepsilon_1\right)^{\kappa_1}+\left(\varepsilon_2\right)^{\kappa_2}$$

According to Definition 4, we have

$$\left(\varepsilon_1\right)^{\kappa_1}=\left(\begin{array}{c}\dfrac{\sqrt{2}\left(\rho_{\eth_1}\right)^{\kappa_1}}{\sqrt{\left(2-\rho^2_{\eth_1}\right)^{\kappa_1}+\left(\rho^2_{\eth_1}\right)^{\kappa_1}}},\\ \dfrac{\sqrt{2}\left(\daleth_{\eth_1}\right)^{\kappa_1}}{\sqrt{\left(2-\daleth^2_{\eth_1}\right)^{\kappa_1}+\left(\daleth^2_{\eth_1}\right)^{\kappa_1}}},\dfrac{\sqrt{\left(1+\tilde{n}^2_{\eth_1}\right)^{\kappa_1}-\left(1-\tilde{n}^2_{\eth_1}\right)^{\kappa_1}}}{\sqrt{\left(1+\tilde{n}^2_{\eth_1}\right)^{\kappa_1}+\left(1-\tilde{n}^2_{\eth_1}\right)^{\kappa_1}}}\end{array}\right)$$

$$\left(\varepsilon_2\right)^{\kappa_2}=\left(\begin{array}{c}\dfrac{\sqrt{2}\left(\rho_{\eth_2}\right)^{\kappa_2}}{\sqrt{\left(2-\rho^2_{\eth_2}\right)^{\kappa_2}+\left(\rho^2_{\eth_2}\right)^{\kappa_2}}},\\ \dfrac{\sqrt{2}\left(\daleth_{\eth_2}\right)^{\kappa_2}}{\sqrt{\left(2-\daleth^2_{\eth_2}\right)^{\kappa_2}+\left(\daleth^2_{\eth_2}\right)^{\kappa_2}}},\dfrac{\sqrt{\left(1+\tilde{n}^2_{\eth_2}\right)^{\kappa_2}-\left(1-\tilde{n}^2_{\eth_2}\right)^{\kappa_2}}}{\sqrt{\left(1+\tilde{n}^2_{\eth_2}\right)^{\kappa_2}+\left(1-\tilde{n}^2_{\eth_2}\right)^{\kappa_2}}}\end{array}\right)$$

Then,

$$SFEWG\left(\varepsilon_1,\varepsilon_2\right)=\left(\varepsilon_1\right)^{\kappa_1}+\left(\varepsilon_2\right)^{\kappa_2}$$

$$=\left(\begin{array}{c}\dfrac{\dfrac{\sqrt{2}\left(\rho_{\eth_1}\right)^{\kappa_1}}{\sqrt{\left(2-\rho^2_{\eth_1}\right)^{\kappa_1}+\left(\rho^2_{\eth_1}\right)^{\kappa_1}}}\cdot\dfrac{\sqrt{2}\left(\rho_{\eth_2}\right)^{\kappa_2}}{\sqrt{\left(2-\rho^2_{\eth_2}\right)^{\kappa_2}+\left(\rho^2_{\eth_2}\right)^{\kappa_2}}}}{\sqrt{1+\left(1-\left(\dfrac{\sqrt{2}\left(\rho_{\eth_1}\right)^{\kappa_1}}{\sqrt{\left(2-\rho^2_{\eth_1}\right)^{\kappa_1}+\left(\rho^2_{\eth_1}\right)^{\kappa_1}}}\right)^2\right)\cdot\left(1-\left(\dfrac{\sqrt{2}\left(\rho_{\eth_2}\right)^{\kappa_2}}{\sqrt{\left(2-\rho^2_{\eth_2}\right)^{\kappa_2}+\left(\rho^2_{\eth_2}\right)^{\kappa_2}}}\right)^2\right)}},\\ \dfrac{\dfrac{\sqrt{2}\left(\daleth_{\eth_1}\right)^{\kappa_1}}{\sqrt{\left(2-\daleth^2_{\eth_1}\right)^{\kappa_1}+\left(\daleth^2_{\eth_1}\right)^{\kappa_1}}}\cdot\dfrac{\sqrt{2}\left(\daleth_{\eth_2}\right)^{\kappa_2}}{\sqrt{\left(2-\daleth^2_{\eth_2}\right)^{\kappa_2}+\left(\daleth^2_{\eth_2}\right)^{\kappa_2}}}}{\sqrt{1+\left(1-\left(\dfrac{\sqrt{2}\left(\daleth_{\eth_1}\right)^{\kappa_1}}{\sqrt{\left(2-\daleth^2_{\eth_1}\right)^{\kappa_1}+\left(\daleth^2_{\eth_1}\right)^{\kappa_1}}}\right)^2\right)\cdot\left(1-\left(\dfrac{\sqrt{2}\left(\daleth_{\eth_2}\right)^{\kappa_2}}{\sqrt{\left(2-\daleth^2_{\eth_2}\right)^{\kappa_2}+\left(\daleth^2_{\eth_2}\right)^{\kappa_2}}}\right)^2\right)}},\\ \dfrac{\sqrt{\dfrac{\left(1+\tilde{n}^2_{\eth_1}\right)^{\kappa_1}-\left(1-\tilde{n}^2_{\eth_1}\right)^{\kappa_1}}{\left(1+\tilde{n}^2_{\eth_1}\right)^{\kappa_1}+\left(1-\tilde{n}^2_{\eth_1}\right)^{\kappa_1}}+\dfrac{\left(1+\tilde{n}^2_{\eth_2}\right)^{\kappa_2}-\left(1-\tilde{n}^2_{\eth_2}\right)^{\kappa_2}}{\left(1+\tilde{n}^2_{\eth_2}\right)^{\kappa_2}+\left(1-\tilde{n}^2_{\eth_2}\right)^{\kappa_2}}}}{\sqrt{1+\dfrac{\left(1+\tilde{n}^2_{\eth_1}\right)^{\kappa_1}-\left(1-\tilde{n}^2_{\eth_1}\right)^{\kappa_1}}{\left(1+\tilde{n}^2_{\eth_1}\right)^{\kappa_1}+\left(1-\tilde{n}^2_{\eth_1}\right)^{\kappa_1}}\cdot\dfrac{\left(1+\tilde{n}^2_{\eth_2}\right)^{\kappa_2}-\left(1-\tilde{n}^2_{\eth_2}\right)^{\kappa_2}}{\left(1+\tilde{n}^2_{\eth_2}\right)^{\kappa_2}+\left(1-\tilde{n}^2_{\eth_2}\right)^{\kappa_2}}}}\end{array}\right)$$

$$=\left(\begin{array}{c}\dfrac{\sqrt{2}\left(\rho_{\eth_1}.\rho_{\eth_2}\right)}{\sqrt{\left(2-\rho^2_{\eth_1}\right)^{\kappa_1}.\left(2-\rho^2_{\eth_2}\right)^{\kappa_2}+\left(\rho^2_{\eth_1}\right)^{\kappa_1}.\left(\rho^2_{\eth_2}\right)^{\kappa_2}}},\\ \dfrac{\sqrt{2}\left(\daleth_{\eth_1}.\daleth_{\eth_2}\right)}{\sqrt{\left(2-\daleth^2_{\eth_1}\right)^{\kappa_1}.\left(2-\daleth^2_{\eth_2}\right)^{\kappa_2}+\left(\daleth^2_{\eth_1}\right)^{\kappa_1}.\left(\daleth^2_{\eth_2}\right)^{\kappa_2}}},\\ \sqrt{\dfrac{\left(1+\tilde{n}^2_{\eth_1}\right)^{\kappa_1}\cdot\left(1+\tilde{n}^2_{\eth_2}\right)^{\kappa_2}-\left(1-\tilde{n}^2_{\eth_1}\right)^{\kappa_1}\cdot\left(1-\tilde{n}^2_{\eth_2}\right)^{\kappa_2}}{\left(1+\tilde{n}^2_{\eth_1}\right)^{\kappa_1}\cdot\left(1+\tilde{n}^2_{\eth_2}\right)^{\kappa_2}+\left(1-\tilde{n}^2_{\eth_1}\right)^{\kappa_1}\cdot\left(1-\tilde{n}^2_{\eth_2}\right)^{\kappa_2}}}\end{array}\right)$$

Thus, Equation 4.7, is true for $\tilde{n}=2$.

Assume Equation 4.7, for $\tilde{n}=z$ is true, we have

$$SFEWG\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_z\right)=\left(\begin{array}{c}\dfrac{\sqrt{2}\prod_{g=1}^{z}\left(\rho_{\eth_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{z}\left(2-\rho^2_{\eth_g}\right)^{\kappa_g}+\prod_{g=1}^{z}\left(\rho^2_{\eth_g}\right)^{\kappa_g}}},\\ \dfrac{\sqrt{2}\prod_{g=1}^{z}\left(\daleth_{\eth_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{z}\left(2-\daleth^2_{\eth_g}\right)^{\kappa_g}+\prod_{g=1}^{z}\left(\daleth^2_{\eth_g}\right)^{\kappa_g}}},\\ \dfrac{\sqrt{\prod_{g=1}^{z}\left(1+\tilde{n}^2_{\eth_g}\right)^{\kappa_g}-\prod_{g=1}^{z}\left(1-\tilde{n}^2_{\eth_g}\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{z}\left(1+\tilde{n}^2_{\eth_g}\right)^{\kappa_g}+\prod_{g=1}^{z}\left(1-\tilde{n}^2_{\eth_g}\right)^{\kappa_g}}}\end{array}\right)$$

Then, we have to prove Equation 4.7 for $\tilde{n}=z+1$ is true, for this we have

$$
\begin{aligned}
& SFEWG\left(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_z, \varepsilon_{z+1}\right) \\
& = \prod_{g=1}^{z}\left(\varepsilon_g\right)^{\kappa_g} + \left(\varepsilon_{z+1}\right)^{\kappa_{z+1}} \\
& = \left(\begin{array}{c}
\frac{\sqrt{2}\prod_{g=1}^{z}\left(\rho_{\text{פ}_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{z}\left(2-\rho_{\text{פ}_g}^2\right)^{\kappa_g}+\prod_{g=1}^{z}\left(\rho_{\text{פ}_g}^2\right)^{\kappa_g}}}, \\
\frac{\sqrt{2}\prod_{g=1}^{z}\left(\daleth_{\text{פ}_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{z}\left(2-\daleth_{\text{פ}_g}^2\right)^{\kappa_g}+\prod_{g=1}^{z}\left(\daleth_{\text{פ}_g}^2\right)^{\kappa_g}}}, \\
\frac{\sqrt{\prod_{g=1}^{z}\left(1+\tilde{n}_{\text{פ}_g}^2\right)^{\kappa_g}-\prod_{g=1}^{z}\left(1-\tilde{n}_{\text{פ}_g}^2\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{z}\left(1+\tilde{n}_{\text{פ}_g}^2\right)^{\kappa_g}+\prod_{g=1}^{z}\left(1-\tilde{n}_{\text{פ}_g}^2\right)^{\kappa_g}}}
\end{array}\right) \\
& + \left(\begin{array}{c}
\frac{\sqrt{2}\left(\rho_{\sigma_{z+1}}\right)^{\kappa_{z+1}}}{\sqrt{\left(2-\rho_{\sigma_{z+1}}^2\right)^{\kappa_{z+1}}+\left(\rho_{\sigma_{z+1}}^2\right)^{\kappa_{z+1}}}}, \\
\frac{\sqrt{2}\left(\daleth_{\sigma_{z+1}}\right)^{\kappa_{z+1}}}{\sqrt{\left(2-\daleth_{\sigma_{z+1}}^2\right)^{\kappa_{z+1}}+\left(\daleth_{\sigma_{z+1}}^2\right)^{\kappa_{z+1}}}}, \\
\frac{\sqrt{\left(1+\tilde{n}_{\sigma_{z+1}}^2\right)^{\kappa_{z+1}}-\left(1-\tilde{n}_{\sigma_{z+1}}^2\right)^{\kappa_{z+1}}}}{\sqrt{\left(1+\tilde{n}_{\sigma_{z+1}}^2\right)^{\kappa_{z+1}}+\left(1-\tilde{n}_{\sigma_{z+1}}^2\right)^{\kappa_{z+1}}}}
\end{array}\right) \\
& = \left(\begin{array}{c}
\frac{\sqrt{2}\prod_{g=1}^{z+1}\left(\rho_{\text{פ}_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{z+1}\left(2-\rho_{\text{פ}_g}^2\right)^{\kappa_g}+\prod_{g=1}^{z+1}\left(\rho_{\text{פ}_g}^2\right)^{\kappa_g}}}, \\
\frac{\sqrt{2}\prod_{g=1}^{z+1}\left(\daleth_{\text{פ}_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{z+1}\left(2-\daleth_{\text{פ}_g}^2\right)^{\kappa_g}+\prod_{g=1}^{z+1}\left(\daleth_{\text{פ}_g}^2\right)^{\kappa_g}}}, \\
\frac{\sqrt{\prod_{g=1}^{z+1}\left(1+\tilde{n}_{\text{פ}_g}^2\right)^{\kappa_g}-\prod_{g=1}^{z+1}\left(1-\tilde{n}_{\text{פ}_g}^2\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{z+1}\left(1+\tilde{n}_{\text{פ}_g}^2\right)^{\kappa_g}+\prod_{g=1}^{z+1}\left(1-\tilde{n}_{\text{פ}_g}^2\right)^{\kappa_g}}}
\end{array}\right)
\end{aligned}
$$

that is, when $\tilde{n} = z + 1$, Equation 4.7 also holds.

Hence, Equation 4.7 holds for any $\tilde{n}$. The proof is completed. □

The following properties of SFEWG operator can be simply proved.

Theorem 7 1) *Let* $\varepsilon_g = \left\{\rho_{\text{פ}_g}, \daleth_{\text{פ}_g}, \tilde{n}_{\text{פ}_g}\right\} \in \hat{S}_F N(V)$ $(g = 1, 2, 3, \ldots, \tilde{n})$, *if* $\varepsilon_1 = \varepsilon_2 = \cdots \varepsilon_{\tilde{n}-1} = \varepsilon_{\tilde{n}} = \varepsilon$, *then* $SFEWG\left(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{\tilde{n}}\right) = \varepsilon$.

2) *Let* $\varepsilon_g = \left\{\rho_{\text{פ}_g}, \daleth_{\text{פ}_g}, \tilde{n}_{\text{פ}_g}\right\} \in \hat{S}_F N(V)$ $(g = 1, 2, 3, \ldots, \tilde{n})$ *and* $\varepsilon^- = \min_g \varepsilon_g$, $\varepsilon^+ = \max_g \varepsilon_g$. *Then*, $\varepsilon^- \leq SFEWG\left(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{\tilde{n}}\right) \leq \varepsilon^+$.

3) *Let* $\varepsilon_g = \left\{\rho_{\text{פ}_g}, \daleth_{\text{פ}_g}, \tilde{n}_{\text{פ}_g}\right\}$ *and* $\varepsilon_{\tilde{g}} = \left\{\rho_{\sigma_{\tilde{g}}}, \daleth_{\sigma_{\tilde{g}}}, \tilde{n}_{\sigma_{\tilde{g}}}\right\} \in \hat{S}_F N(V)\left(g, \tilde{g} \in \mathbb{N}\right)$ *such that* $\varepsilon_g \leq \varepsilon_{\tilde{g}}$ *for all* g. *Then*

$SFEWG\left(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{\tilde{n}}\right) \leq SFEWG\left(\varepsilon_{\tilde{1}}, \varepsilon_{\tilde{2}}, \varepsilon_{\tilde{3}}, \ldots, \varepsilon_{\tilde{\tilde{n}}}\right)$.

Definition 11 Let $\varepsilon_g = \left\{\rho_{\text{פ}_g}, \daleth_{\text{פ}_g}, \tilde{n}_{\text{פ}_g}\right\} \in \hat{S}_F N(V)$ $(g = 1, 2, 3, \ldots, \tilde{n})$. Then, the Generalized Einstein geometric aggregation operator for $\hat{S}_F N(V)$ is denoted by *GSFEWG* and defined as follows:

$$
GSFEWG\left(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{\tilde{n}}\right) = \frac{1}{\Upsilon}\left(\prod_{g=1}^{\tilde{n}}\left(\Upsilon \cdot \varepsilon_g\right)^{\kappa_g}\right), \quad (4.8)
$$

where weights of ε_g $(g \in \mathbb{N})$ with $\kappa_g \geq 0$, $\sum_{g=1}^{\tilde{n}} \kappa_g = 1$ is $\kappa_g (g \in \mathbb{N})$, and $\Upsilon \geq 0$.

Theorem 8 *Let* $\varepsilon_g = \left\{\rho_{\text{פ}_g}, \daleth_{\text{פ}_g}, \tilde{n}_{\text{פ}_g}\right\} \in \hat{S}_F N(V)\,(g \in \mathbb{N})$ *and weights of* $\varepsilon_g (g \in \mathbb{N})$ *subject to* $\sum_{g=1}^{\tilde{n}} \kappa_g = 1$ *with* $\Upsilon \geq 0$ *be denoted by* $\kappa = \left(\kappa_1, \kappa_2, \ldots, \kappa_{\tilde{n}}\right)^T$. *The GSFEWG operator is a mapping* $G^{\tilde{n}} \longrightarrow G$ *such that*

$$GSFEWG(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{\tilde{n}}) = \left(\begin{array}{c} \sqrt{\dfrac{\left(\prod_{g=1}^{\tilde{n}} \left\{ \left(1+\rho_{\eth_g}^2\right)^{\Upsilon} + 3\left(1-\rho_{\eth_g}^2\right)^{\Upsilon} \right\}^{\kappa_g} + 3\prod_{g=1}^{\tilde{n}} \left\{ \left(1+\rho_{\eth_g}^2\right)^{\Upsilon} - \left(1-\rho_{\eth_g}^2\right)^{\Upsilon} \right\}^{\kappa_g} \right)^{\frac{1}{\Upsilon}} - \left(\prod_{g=1}^{\tilde{n}} \left\{ \left(1+\rho_{\eth_g}^2\right)^{\Upsilon} + 3\left(1-\rho_{\eth_g}^2\right)^{\Upsilon} \right\}^{\kappa_g} - \prod_{g=1}^{\tilde{n}} \left\{ \left(1+\rho_{\eth_g}^2\right)^{\Upsilon} - \left(1-\rho_{\eth_g}^2\right)^{\Upsilon} \right\}^{\kappa_g} \right)^{\frac{1}{\Upsilon}}}{\left(\prod_{g=1}^{\tilde{n}} \left\{ \left(1+\rho_{\eth_g}^2\right)^{\Upsilon} + 3\left(1-\rho_{\eth_g}^2\right)^{\Upsilon} \right\}^{\kappa_g} + 3\prod_{g=1}^{\tilde{n}} \left\{ \left(1+\rho_{\eth_g}^2\right)^{\Upsilon} - \left(1-\rho_{\eth_g}^2\right)^{\Upsilon} \right\}^{\kappa_g} \right)^{\frac{1}{\Upsilon}} + \left(\prod_{g=1}^{\tilde{n}} \left\{ \left(1+\rho_{\eth_g}^2\right)^{\Upsilon} + 3\left(1-\rho_{\eth_g}^2\right)^{\Upsilon} \right\}^{\kappa_g} - \prod_{g=1}^{\tilde{n}} \left\{ \left(1+\rho_{\eth_g}^2\right)^{\Upsilon} - \left(1-\rho_{\eth_g}^2\right)^{\Upsilon} \right\}^{\kappa_g} \right)^{\frac{1}{\Upsilon}}}}, \\ \sqrt{\dfrac{2\left(\left(\prod_{g=1}^{\tilde{n}} \left(\daleth_{\eth_g}^2\right)^{\Upsilon}\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}{\left(\left(\prod_{g=1}^{\tilde{n}} \left(2-\daleth_{\eth_g}^2\right)^{\Upsilon}\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}} + \left(\left(\prod_{g=1}^{\tilde{n}} \left(\daleth_{\eth_g}^2\right)^{\Upsilon}\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}, \\ \dfrac{\sqrt{2\left\{\prod_{g=1}^{\tilde{n}} \left\{ \left(2-\tilde{n}_{\eth_g}^2\right)^{\Upsilon} + 3\left(\tilde{n}_{\eth_g}^2\right)^{\Upsilon} \right\}^{\kappa_g} - \prod_{g=1}^{\tilde{n}} \left\{ \left(2-\tilde{n}_{\eth_g}^2\right)^{\Upsilon} - \left(\tilde{n}_{\eth_g}^2\right)^{\Upsilon} \right\}^{\kappa_g} \right\}^{\frac{1}{\Upsilon}}}}{\sqrt{\left(\prod_{g=1}^{\tilde{n}} \left\{ \left(2-\tilde{n}_{\eth_g}^2\right)^{\Upsilon} + 3\left(\tilde{n}_{\eth_g}^2\right)^{\Upsilon} \right\}^{\kappa_g} + 3\prod_{g=1}^{\tilde{n}} \left\{ \left(2-\tilde{n}_{\eth_g}^2\right)^{\Upsilon} - \left(\tilde{n}_{\eth_g}^2\right)^{\Upsilon} \right\}^{\kappa_g} \right)^{\frac{1}{\Upsilon}} + \left(\prod_{g=1}^{\tilde{n}} \left\{ \left(2-\tilde{n}_{\eth_g}^2\right)^{\Upsilon} + 3\left(\tilde{n}_{\eth_g}^2\right)^{\Upsilon} \right\}^{\kappa_g} - \prod_{g=1}^{\tilde{n}} \left\{ \left(2-\tilde{n}_{\eth_g}^2\right)^{\Upsilon} - \left(\tilde{n}_{\eth_g}^2\right)^{\Upsilon} \right\}^{\kappa_g} \right)^{\frac{1}{\Upsilon}}}} \end{array} \right) \tag{4.9}$$

Proof Since, form Definition 4 we have

$$\Upsilon\cdot\varepsilon_g = \left(\frac{\sqrt{\left(1+\rho^2_{\eth_g}\right)^{\Upsilon} - \left(1-\rho^2_{\eth_g}\right)^{\Upsilon}}}{\sqrt{\left(1+\rho^2_{\eth_g}\right)^{\Upsilon} + \left(1-\rho^2_{\eth_g}\right)^{\Upsilon}}}, \frac{\sqrt{2}\left(\daleth_{\eth_g}\right)^{\Upsilon}}{\sqrt{\left(2-\daleth^2_{\eth_g}\right)^{\Upsilon}+\left(\daleth^2_{\eth_g}\right)^{\Upsilon}}}, \frac{\sqrt{2}\left(\tilde{n}_{\eth_g}\right)^{\Upsilon}}{\sqrt{\left(2-\tilde{n}^2_{\eth_g}\right)^{\Upsilon}+\left(\tilde{n}^2_{\eth_g}\right)^{\Upsilon}}} \right)$$

$$\Rightarrow \prod_{g=1}^{\tilde{n}} \left(\Upsilon\cdot\varepsilon_g\right)^{\kappa_g}$$

$$= \left(\frac{\sqrt{2}\prod_{g=1}^{\tilde{n}}\left(\frac{\sqrt{\left(1+\rho^2_{\eth_g}\right)^{\Upsilon}-\left(1-\rho^2_{\eth_g}\right)^{\Upsilon}}}{\sqrt{\left(1+\rho^2_{\eth_g}\right)^{\Upsilon}+\left(1-\rho^2_{\eth_g}\right)^{\Upsilon}}}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(2-\frac{\left(1+\rho^2_{\eth_g}\right)^{\Upsilon}-\left(1-\rho^2_{\eth_g}\right)^{\Upsilon}}{\left(1+\rho^2_{\eth_g}\right)^{\Upsilon}+\left(1-\rho^2_{\eth_g}\right)^{\Upsilon}}\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\frac{\left(1+\rho^2_{\eth_g}\right)^{\Upsilon}-\left(1-\rho^2_{\eth_g}\right)^{\Upsilon}}{\left(1+\rho^2_{\eth_g}\right)^{\Upsilon}+\left(1-\rho^2_{\eth_g}\right)^{\Upsilon}}\right)^{\kappa_g}}}, \frac{\sqrt{2\prod_{g=1}^{\tilde{n}}\left(\frac{2\left(\daleth^2_{\eth_g}\right)^{\Upsilon}}{\left(2-\daleth^2_{\eth_g}\right)^{\Upsilon}+\left(\daleth^2_{\eth_g}\right)^{\Upsilon}}\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(2-\frac{2\left(\daleth^2_{\eth_g}\right)^{\Upsilon}}{\left(2-\daleth^2_{\eth_g}\right)^{\Upsilon}+\left(\daleth^2_{\eth_g}\right)^{\Upsilon}}\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\frac{2\left(\daleth^2_{\eth_g}\right)^{\Upsilon}}{\left(2-\daleth^2_{\eth_g}\right)^{\Upsilon}+\left(\daleth^2_{\eth_g}\right)^{\Upsilon}}\right)^{\kappa_g}}}, \frac{\sqrt{\prod_{g=1}^{\tilde{n}}\left(1+\frac{2\left(\tilde{n}^2_{\eth_g}\right)^{\Upsilon}}{\left(2-\tilde{n}^2_{\eth_g}\right)^{\Upsilon}+\left(\tilde{n}^2_{\eth_g}\right)^{\Upsilon}}\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(1-\frac{2\left(\tilde{n}^2_{\eth_g}\right)^{\Upsilon}}{\left(2-\tilde{n}^2_{\eth_g}\right)^{\Upsilon}+\left(\tilde{n}^2_{\eth_g}\right)^{\Upsilon}}\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(1+\frac{2\left(\tilde{n}^2_{\eth_g}\right)^{\Upsilon}}{\left(2-\tilde{n}^2_{\eth_g}\right)^{\Upsilon}+\left(\tilde{n}^2_{\eth_g}\right)^{\Upsilon}}\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(1-\frac{2\left(\tilde{n}^2_{\eth_g}\right)^{\Upsilon}}{\left(2-\tilde{n}^2_{\eth_g}\right)^{\Upsilon}+\left(\tilde{n}^2_{\eth_g}\right)^{\Upsilon}}\right)^{\kappa_g}}} \right)$$

$$= \left(\frac{\sqrt{2\prod_{g=1}^{\tilde{n}}\left\{\left(1+\rho^2_{\eth_g}\right)^{\Upsilon}-\left(1-\rho^2_{\eth_g}\right)^{\Upsilon}\right\}^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left\{\left(1+\rho^2_{\eth_g}\right)^{\Upsilon}+3\left(1-\rho^2_{\eth_g}\right)^{\Upsilon}\right\}^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left\{\left(1+\rho^2_{\eth_g}\right)^{\Upsilon}-\left(1-\rho^2_{\eth_g}\right)^{\Upsilon}\right\}^{\kappa_g}}}, \frac{\sqrt{2\prod_{g=1}^{\tilde{n}}\left(\left(\daleth^2_{\eth_g}\right)^{\Upsilon}\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(\left(2-\daleth^2_{\eth_g}\right)^{\Upsilon}\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\left(\daleth^2_{\eth_g}\right)^{\Upsilon}\right)^{\kappa_g}}}, \frac{\sqrt{\prod_{g=1}^{\tilde{n}}\left\{\left(2-\tilde{n}^2_{\eth_g}\right)^{\Upsilon}+3\left(\tilde{n}^2_{\eth_g}\right)^{\Upsilon}\right\}^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left\{\left(2-\tilde{n}^2_{\eth_g}\right)^{\Upsilon}-\left(\tilde{n}^2_{\eth_g}\right)^{\Upsilon}\right\}^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left\{\left(2-\tilde{n}^2_{\eth_g}\right)^{\Upsilon}+3\left(\tilde{n}^2_{\eth_g}\right)^{\Upsilon}\right\}^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left\{\left(2-\tilde{n}^2_{\eth_g}\right)^{\Upsilon}-\left(\tilde{n}^2_{\eth_g}\right)^{\Upsilon}\right\}^{\kappa_g}}} \right)$$

Suppose that

$$\widetilde{\underline{a}}_g = \left(1+\rho^2_{\eth_g}\right)^{\Upsilon}-\left(1-\rho^2_{\eth_g}\right)^{\Upsilon},\ \widetilde{\underline{b}}_g = \left(1+\rho^2_{\eth_g}\right)^{\Upsilon}+3\left(1-\rho^2_{\eth_g}\right)^{\Upsilon},$$
$$\widetilde{c}_g = \left(\daleth^2_{\eth_g}\right)^{\Upsilon},\quad \widetilde{d}_g = \left(2-\daleth^2_{\eth_g}\right)^{\Upsilon},$$
$$\widetilde{\underline{e}}_g = \left(2-\tilde{n}^2_{\eth_g}\right)^{\Upsilon}+3\left(\tilde{n}^2_{\eth_g}\right)^{\Upsilon},\quad \widetilde{\underline{f}}_g = \left(2-\tilde{n}^2_{\eth_g}\right)^{\Upsilon}-\left(\tilde{n}^2_{\eth_g}\right)^{\Upsilon}.$$

$$\Rightarrow \prod_{g=1}^{\tilde{n}}\left(\Upsilon\cdot\varepsilon_g\right)^{\kappa_g} = \left(\frac{\sqrt{2\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{a}}_g\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{b}}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{a}}_g\right)^{\kappa_g}}}, \frac{\sqrt{2\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(\widetilde{d}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}}}, \frac{\sqrt{\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{e}}_g\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{f}}_g\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{e}}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{f}}_g\right)^{\kappa_g}}} \right)$$

Therefore,

$$\frac{1}{\Upsilon}\cdot\left(\prod_{g=1}^{\tilde{n}}\left(\Upsilon\cdot\varepsilon_g\right)^{\kappa_g}\right) = \left(\frac{\sqrt{\left(1+\frac{2\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{a}}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{b}}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{a}}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}-\left(1-\frac{2\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{a}}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{b}}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{a}}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}}}{\sqrt{\left(1+\frac{2\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{a}}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{b}}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{a}}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}+\left(1-\frac{2\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{a}}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{b}}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{a}}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}}}, \frac{\sqrt{2\left(\frac{2\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{d}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}}}{\sqrt{\left(2-\frac{2\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{d}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}+\left(\frac{2\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{d}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}}}, \frac{\sqrt{2\left(\frac{\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{e}}_g\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{f}}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{e}}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{f}}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}}}{\sqrt{\left(2-\frac{\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{e}}_g\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{f}}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{e}}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{f}}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}+\left(\frac{\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{e}}_g\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{f}}_g\right)^{\kappa_g}}{\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{e}}_g\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{f}}_g\right)^{\kappa_g}}\right)^{\frac{1}{\Upsilon}}}} \right)$$

$$= \left(\frac{\sqrt{\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{a}}_g\right)^{\kappa_g}+3\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{b}}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}-\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{a}}_g\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{b}}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}{\sqrt{\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{a}}_g\right)^{\kappa_g}+3\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{b}}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}+\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{a}}_g\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{b}}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}, \frac{\sqrt{2\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}{\sqrt{\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{d}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}+\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{c}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}, \frac{\sqrt{2\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{e}}_g\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{f}}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}{\sqrt{\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{e}}_g\right)^{\kappa_g}+3\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{f}}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}+\left(\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{e}}_g\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(\widetilde{\underline{f}}_g\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}} \right)$$

By back substitution, we have

Fig. 1 Algorithm flow chart

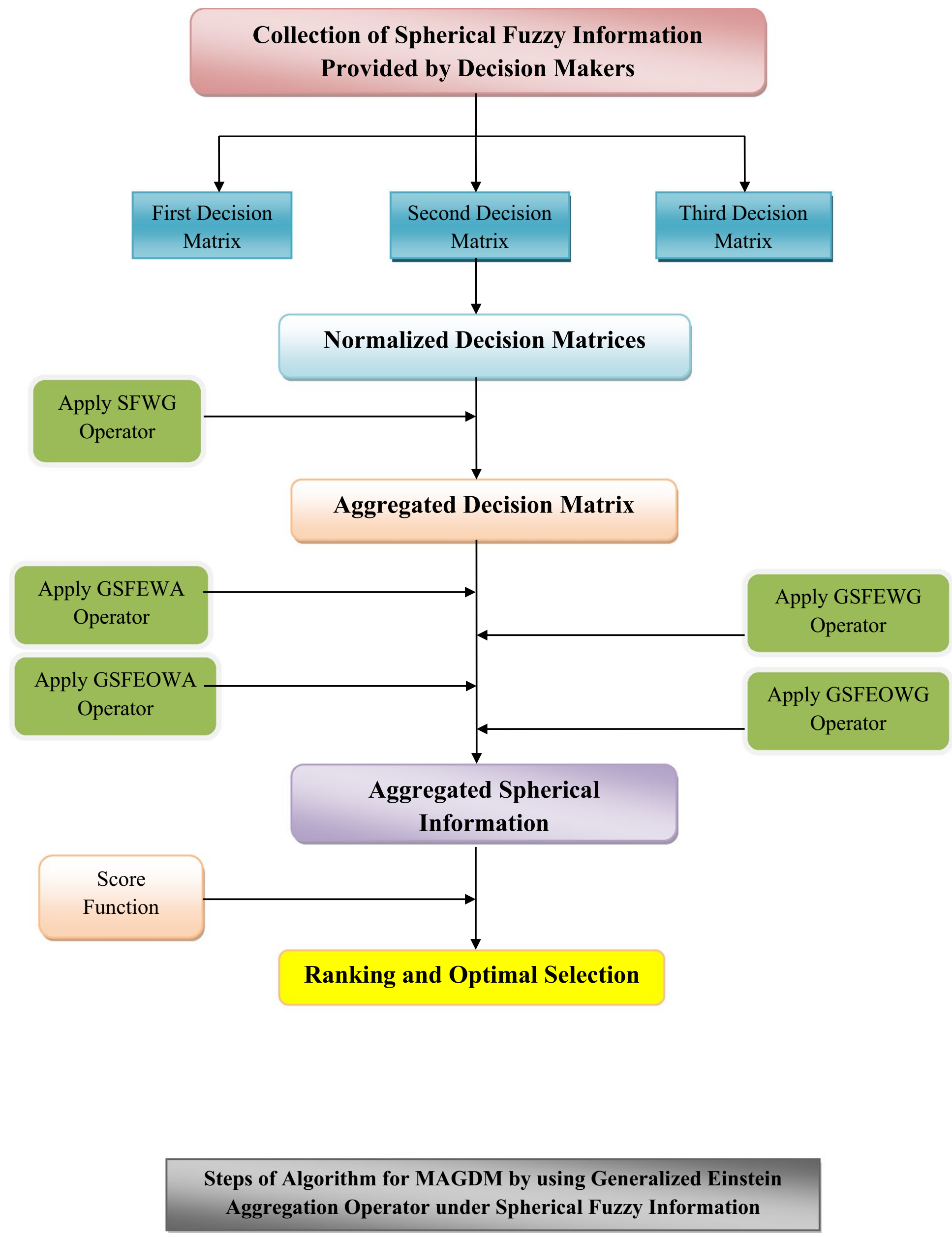

Table 3 Expert-1 information

	f_1	f_2	f_3	f_4
S_1	(0.84, 0.34, 0.40)	(0.43, 0.39, 0.78)	(0.67, 0.50, 0.30)	(0.31, 0.21, 0.71)
S_2	(0.60, 0.11, 0.53)	(0.23, 0.35, 0.59)	(0.72, 0.31, 0.41)	(0.11, 0.25, 0.82)
S_3	(0.79, 0.19, 0.39)	(0.11, 0.21, 0.91)	(0.71, 0.41, 0.13)	(0.34, 0.25, 0.51)
S_4	(0.63, 0.51, 0.13)	(0.49, 0.33, 0.42)	(0.61, 0.43, 0.45)	(0.49, 0.37, 0.59)
S_5	(0.57, 0.36, 0.29)	(0.50, 0.15, 0.60)	(0.70, 0.32, 0.40)	(0.33, 0.44, 0.65)

Table 4 Expert-2 information

	f_1	f_2	f_3	f_4
S_1	(0.61, 0.15, 0.53)	(0.16, 0.35, 0.62)	(0.61, 0.35, 0.47)	(0.55, 0.17, 0.74)
S_2	(0.66, 0.11, 0.51)	(0.43, 0.23, 0.77)	(0.93, 0.08, 0.09)	(0.02, 0.06, 0.99)
S_3	(0.88, 0.09, 0.07)	(0.05, 0.06, 0.89)	(0.56, 0.17, 0.44)	(0.43, 0.13, 0.61)
S_4	(0.59, 0.32, 0.34)	(0.24, 0.48, 0.51)	(0.68, 0.53, 0.39)	(0.34, 0.21, 0.61)
S_5	(0.71, 0.31, 0.24)	(0.35, 0.41, 0.69)	(0.73, 0.44, 0.21)	(0.22, 0.49, 0.74)

Table 5 Expert-3 information

	f_1	f_2	f_3	f_4
S_1	(0.85, 0.25.0.15)	(0.14, 0.23, 0.88)	(0.78, 0.38, 0.18)	(0.29, 0.39, 0.83)
S_2	(0.94, 0.04, 0.07)	(0.39, 0.19, 0.61)	(0.63, 0.18, 0.35)	(0.48, 0.49, 0.56)
S_3	(0.73, 0.13, 0.46)	(0.19, 0.39, 0.88)	(0.87, 0.35, 0.18)	(0.41, 0.13, 0.81)
S_4	(0.82, 0.12, 0.43)	(0.55, 0.21, 0.63)	(0.53, 0.33, 0.47)	(0.46, 0.23, 0.51)
S_5	(0.61, 0.33, 0.29)	(0.28, 0.41, 0.63)	(0.74, 0.34, 0.14)	(0.37, 0.32, 0.65)

Table 6 Expert-1 normalized information

	f_1	f_2	f_3	f_4
S_1	(0.84, 0.34, 0.40)	(0.78, 0.39, 0.43)	(0.67, 0.50, 0.30)	(0.71, 0.21, 0.31)
S_2	(0.60, 0.11, 0.53)	(0.59, 0.35, 0.23)	(0.72, 0.31, 0.41)	(0.82, 0.25, 0.11)
S_3	(0.79, 0.19, 0.39)	(0.91, 0.21, 0.11)	(0.71, 0.41, 0.13)	(0.51, 0.25, 0.34)
S_4	(0.63, 0.51, 0.13)	(0.42, 0.33, 0.49)	(0.61, 0.43, 0.45)	(0.59, 0.37, 0.49)
S_5	(0.57, 0.36, 0.29)	(0.60, 0.15, 0.50)	(0.70, 0.32, 0.40)	(0.65, 0.44, 0.33)

Table 7 Expert-2 normalized information

	f_1	f_2	f_3	f_4
S_1	(0.61, 0.15, 0.53)	(0.62, 0.35, 0.16)	(0.61, 0.35, 0.47)	(0.74, 0.17, 0.55)
S_2	(0.66, 0.11, 0.51)	(0.77, 0.23, 0.43)	(0.93, 0.08, 0.09)	(0.99, 0.06, 0.02)
S_3	(0.88, 0.09, 0.07)	(0.89, 0.06, 0.05)	(0.56, 0.17, 0.44)	(0.61, 0.13, 0.43)
S_4	(0.59, 0.32, 0.34)	(0.51, 0.48, 0.24)	(0.68, 0.53, 0.39)	(0.61, 0.21, 0.34)
S_5	(0.71, 0.31, 0.24)	(0.69, 0.41, 0.35)	(0.73, 0.44, 0.21)	(0.74, 0.49, 0.22)

Table 8 Expert-3 normalized information

	f_1	f_2	f_3	f_4
S_1	(0.85, 0.25.0.15)	(0.88, 0.23, 0.14)	(0.78, 0.38, 0.18)	(0.83, 0.39, 0.29)
S_2	(0.94, 0.04, 0.07)	(0.61, 0.19, 0.39)	(0.63, 0.18, 0.35)	(0.56, 0.49, 0.48)
S_3	(0.73, 0.13, 0.46)	(0.88, 0.39, 0.19)	(0.87, 0.35, 0.18)	(0.81, 0.13, 0.41)
S_4	(0.82, 0.12, 0.43)	(0.63, 0.21, 0.55)	(0.53, 0.33, 0.47)	(0.51, 0.23, 0.46)
S_5	(0.61, 0.33, 0.29)	(0.63, 0.41, 0.28)	(0.74, 0.34, 0.14)	(0.65, 0.32, 0.37)

Table 9 Aggregated SF information

	f_1	f_2	f_3	f_4
S_1	(0.788, 0.229, 0.319)	(0.785, 0.315, 0.208)	(0.696, 0.402, 0.297)	(0.767, 0.239, 0.371)
S_2	(0.807, 0.078, 0.279)	(0.674, 0.246, 0.342)	(0.818, 0.160, 0.227)	(0.919, 0.188, 0.097)
S_3	(0.814, 0.128, 0.223)	(0.893, 0.165, 0.099)	(0.748, 0.284, 0.223)	(0.677, 0.159, 0.393)
S_4	(0.702, 0.267, 0.271)	(0.533, 0.324, 0.395)	(0.615, 0.424, 0.433)	(0.573, 0.258, 0.421)
S_5	(0.639, 0.331, 0.271)	(0.644, 0.298, 0.363)	(0.724, 0.365, 0.224)	(0.685, 0.411, 0.296)

Table 10 Aggregate using SFEWA operator

S_1	(0.3094, 0.7936, 0.0979)
S_2	(0.3264, 0.7120, 0.0247)
S_3	(0.3192, 0.7261, 0.0214)
S_4	(0.2635, 0.8043, 0.1795)
S_5	(0.2834, 0.8195, 0.0879)

Table 11 Aggregate using SFEWG

S_1	(0.60005, 0.79366, 0.14615)
S_2	(0.66340, 0.71208, 0.12277)
S_3	(0.63264, 0.72614, 0.12485)
S_4	(0.41620, 0.80431, 0.18010)
S_5	(0.49192, 0.81953, 0.14080)

Table 12 Aggregate using GSFEWA

S_1	(0.44445, 0.79575, 0.11129)
S_2	(0.46454, 0.71761, 0.06174)
S_3	(0.45682, 0.72979, 0.05963)
S_4	(0.38654, 0.80591, 0.17035)
S_5	(0.41154, 0.82002, 0.10427)

Table 13 Aggregate using GSFEWG

S_1	(0.56223, 0.79575, 0.21632)
S_2	(0.60791, 0.71761, 0.18988)
S_3	(0.58129, 0.72979, 0.20267)
S_4	(0.37139, 0.80591, 0.26558)
S_5	(0.44870, 0.82002, 0.20767)

Table 14 Aggregate using GSFEOWA

S_1	(0.444296, 0.795641, 0.110545)
S_2	(0.464678, 0.716829, 0.061479)
S_3	(0.457115, 0.729603, 0.059262)
S_4	(0.386712, 0.805925, 0.170465)
S_5	(0.412008, 0.820026, 0.103837)

Table 15 Aggregate using GSFEOWG

S_1	(0.56141, 0.79564, 0.21541)
S_2	(0.60776, 0.71682, 0.18982)
S_3	(0.58205, 0.72960, 0.20252)
S_4	(0.37184, 0.80592, 0.26572)
S_5	(0.44997, 0.82002, 0.20715)

Table 16 Score and ranking of SFNs

Operators	Score					Ranking
	$\check{s}\check{c}(S_1)$	$\check{s}\check{c}(S_2)$	$\check{s}\check{c}(S_3)$	$\check{s}\check{c}(S_4)$	$\check{s}\check{c}(S_5)$	
SFEWA	0.472	0.529	0.523	0.426	0.458	$S_2 > S_3 > S_1 > S_5 > S_4$
SFEWG	0.553	0.609	0.593	0.477	0.510	$S_2 > S_3 > S_1 > S_5 > S_4$
GSFEWA	0.512	0.561	0.555	0.470	0.495	$S_2 > S_3 > S_1 > S_5 > S_4$
GSFEWG	0.516	0.566	0.549	0.433	0.473	$S_2 > S_3 > S_1 > S_5 > S_4$
GSFEOWA	0.512	0.562	0.556	0.470	0.496	$S_2 > S_3 > S_1 > S_5 > S_4$
GSFEOWG	0.516	0.567	0.549	0.433	0.474	$S_2 > S_3 > S_1 > S_5 > S_4$

$$= \left(\begin{array}{c} \dfrac{\sqrt{\left(\prod_{g=1}^{\tilde{n}}\left\{\left(1+\rho_{\eth_g}^2\right)^{\Upsilon}-\left(1-\rho_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}+3\prod_{g=1}^{\tilde{n}}\left\{\left(1+\rho_{\eth_g}^2\right)^{\Upsilon}+3\left(1-\rho_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}-\left(\prod_{g=1}^{\tilde{n}}\left\{\left(1+\rho_{\eth_g}^2\right)^{\Upsilon}-\left(1-\rho_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left\{\left(1+\rho_{\eth_g}^2\right)^{\Upsilon}+3\left(1-\rho_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}{\sqrt{\left(\prod_{g=1}^{\tilde{n}}\left\{\left(1+\rho_{\eth_g}^2\right)^{\Upsilon}-\left(1-\rho_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}+3\prod_{g=1}^{\tilde{n}}\left\{\left(1+\rho_{\eth_g}^2\right)^{\Upsilon}+3\left(1-\rho_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}+\left(\prod_{g=1}^{\tilde{n}}\left\{\left(1+\rho_{\eth_g}^2\right)^{\Upsilon}-\left(1-\rho_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left\{\left(1+\rho_{\eth_g}^2\right)^{\Upsilon}+3\left(1-\rho_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}, \\ \dfrac{\sqrt{2\left(\prod_{g=1}^{\tilde{n}}\left(\left(\daleth_{\eth_g}^2\right)^{\Upsilon}\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}{\sqrt{\left(\prod_{g=1}^{\tilde{n}}\left(\left(2-\daleth_{\eth_g}^2\right)^{\Upsilon}\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}+\left(\prod_{g=1}^{\tilde{n}}\left(\left(\daleth_{\eth_g}^2\right)^{\Upsilon}\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}, \\ \dfrac{\sqrt{2\left(\prod_{g=1}^{\tilde{n}}\left\{\left(2-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}+3\left(\tilde{n}_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left\{\left(2-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}-\left(\tilde{n}_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}{\sqrt{\left(\prod_{g=1}^{\tilde{n}}\left\{\left(2-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}+3\left(\tilde{n}_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}+3\prod_{g=1}^{\tilde{n}}\left\{\left(2-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}-\left(\tilde{n}_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}+\left(\prod_{g=1}^{\tilde{n}}\left\{\left(2-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}+3\left(\tilde{n}_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left\{\left(2-\tilde{n}_{\eth_g}^2\right)^{\Upsilon}-\left(\tilde{n}_{\eth_g}^2\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}} \end{array} \right)$$

Hence, proved. □

In procedure of aggregating the spherical fuzzy information parameter Υ plays the vital role. When we fixed the parameter to special number, then the GSFEWG operator can be reduced. Like, when we take $\Upsilon = 1$, GSFEWG operator is reduced to SFEWG.

$$SFEWG(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{\tilde{n}}) = \left(\begin{array}{c} \dfrac{\sqrt{2}\prod_{g=1}^{\tilde{n}}\left(\rho_{\eth_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(2-\rho_{\eth_g}^2\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\rho_{\eth_g}^2\right)^{\kappa_g}}}, \dfrac{\sqrt{2}\prod_{g=1}^{\tilde{n}}\left(\daleth_{\eth_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(2-\daleth_{\eth_g}^2\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(\daleth_{\eth_g}^2\right)^{\kappa_g}}}, \\ \dfrac{\sqrt{\prod_{g=1}^{\tilde{n}}\left(1+\tilde{n}_{\eth_g}^2\right)^{\kappa_g}-\prod_{g=1}^{\tilde{n}}\left(1-\tilde{n}_{\eth_g}^2\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(1+\tilde{n}_{\eth_g}^2\right)^{\kappa_g}+\prod_{g=1}^{\tilde{n}}\left(1-\tilde{n}_{\eth_g}^2\right)^{\kappa_g}}} \end{array} \right)$$

The following properties of GSFEWG operator can be simply proved.

Table 17 Sensitivity analysis on the different values of parameter Υ

Υ	Operators	Score					Ranking
		$\check{s}\check{c}(S_1)$	$\check{s}\check{c}(S_2)$	$\check{s}\check{c}(S_3)$	$\check{s}\check{c}(S_4)$	$\check{s}\check{c}(S_5)$	
$\to 0.2$	*GSFEWA*	0.546	0.630	0.626	0.471	0.543	$(2,3,1,5,4)$
	GSFEWG	0.628	0.687	0.668	0.521	0.565	
	GSFEOWA	0.546	0.629	0.626	0.471	0.543	
	GSFEOWG	0.628	0.687	0.669	0.521	0.565	
$\to 1$	*GSFEWA*	0.472	0.529	0.523	0.426	0.458	$(2,3,1,5,4)$
	GSFEWG	0.553	0.609	0.593	0.477	0.510	
	GSFEOWA	0.472	0.529	0.524	0.426	0.458	
	GSFEOWG	0.553	0.609	0.594	0.477	0.511	
$\to 2$	*GSFEWA*	0.512	0.561	0.555	0.470	0.495	$(2,3,1,5,4)$
	GSFEWG	0.516	0.566	0.549	0.433	0.473	
	GSFEOWA	0.512	0.562	0.556	0.470	0.496	
	GSFEOWG	0.516	0.567	0.549	0.433	0.474	
$\to 5$	*GSFEWA*	0.551	0.596	0.590	0.512	0.533	$(2,3,1,5,4)$
	GSFEWG	0.485	0.518	0.499	0.395	0.444	
	GSFEOWA	0.552	0.597	0.591	0.512	0.533	
	GSFEOWG	0.485	0.518	0.499	0.395	0.444	
$\to 10$	*GSFEWA*	0.566	0.612	0.604	0.531	0.550	$(2,3,1,5,4)$
	GSFEWG	0.464	0.488	0.470	0.376	0.429	
	GSFEOWA	0.567	0.613	0.605	0.532	0.550	
	GSFEOWG	0.465	0.489	0.470	0.377	0.429	
$\to 15$	*GSFEWA*	0.571	0.617	0.609	0.539	0.557	$(2,3,1,5,4)$
	GSFEWG	0.562	0.577	0.571	0.368	0.528	
	GSFEOWA	0.573	0.618	0.609	0.540	0.557	
	GSFEOWG	0.562	0.577	0.571	0.368	0.529	
$\to 30$	*GSFEWA*	0.578	0.624	0.615	0.548	0.564	$(2,3,1,5,4)$
	GSFEWG	0.553	0.568	0.561	0.486	0.524	
	GSFEOWA	0.579	0.625	0.615	0.550	0.565	
	GSFEOWG	0.554	0.568	0.561	0.487	0.525	

Table 18 Aggregated SF information matrix

	f_1	f_2	f_3	f_4
S_1	$(0.788, 0.229, 0.319)$	$(0.785, 0.315, 0.208)$	$(0.696, 0.402, 0.297)$	$(0.767, 0.239, 0.371)$
S_2	$(0.807, 0.078, 0.279)$	$(0.674, 0.246, 0.342)$	$(0.818, 0.160, 0.227)$	$(0.919, 0.188, 0.097)$
S_3	$(0.814, 0.128, 0.223)$	$(0.893, 0.165, 0.099)$	$(0.748, 0.284, 0.223)$	$(0.677, 0.159, 0.393)$
S_4	$(0.702, 0.267, 0.271)$	$(0.533, 0.324, 0.395)$	$(0.615, 0.424, 0.433)$	$(0.573, 0.258, 0.421)$
S_5	$(0.639, 0.331, 0.271)$	$(0.644, 0.298, 0.363)$	$(0.724, 0.365, 0.224)$	$(0.685, 0.411, 0.296)$

Theorem 9 1) *Let* $\varepsilon_g = \left\{\rho_{\eth_g}, \daleth_{\eth_g}, \tilde{n}_{\eth_g}\right\} \in \hat{S}_F N(V)$ $(g = 1, 2, 3, \ldots, \tilde{n})$, *if* $\varepsilon_1 = \varepsilon_2 = \cdots \varepsilon_{\tilde{n}-1} = \varepsilon_{\tilde{n}} = \varepsilon$, *then* $GSFEWG(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{\tilde{n}}) = \varepsilon$.

2) *Let* $\varepsilon_g = \left\{\rho_{\eth_g}, \daleth_{\eth_g}, \tilde{n}_{\eth_g}\right\} \in \hat{S}_F N(V)$ $(g = 1, 2, 3, \ldots, \tilde{n})$ *and* $\varepsilon^- = \min_g \varepsilon_g$, $\varepsilon^+ = \max_g \varepsilon_g$. *Then*, $\varepsilon^- \le GSFEWG(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{\tilde{n}}) \le \varepsilon^+$.

3) *Let* $\varepsilon_g = \left\{\rho_{\eth_g}, \daleth_{\eth_g}, \tilde{n}_{\eth_g}\right\}$ *and* $\varepsilon_{\widetilde{g}} = \left\{\rho_{\sigma_{\widetilde{g}}}, \daleth_{\sigma_{\widetilde{g}}}, \tilde{n}_{\sigma_{\widetilde{g}}}\right\} \in \hat{S}_F N(V)\left(g, \widetilde{g} \in \mathbb{N}\right)$ *such that* $\varepsilon_g \le \varepsilon_{\widetilde{g}}$ *for all* g. *Then*

$$GSFEWG(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{\tilde{n}}) \le GSFEWG(\varepsilon_{\widetilde{1}}, \varepsilon_{\widetilde{2}}, \varepsilon_{\widetilde{3}}, \ldots, \varepsilon_{\widetilde{\tilde{n}}}).$$

Definition 12 Let $\varepsilon_g = \left\{\rho_{\eth_g}, \daleth_{\eth_g}, \tilde{n}_{\eth_g}\right\} \in \hat{S}_F N(V)$ $(g = 1, 2, 3, \ldots, \tilde{n})$. Then, the Generalized Einstein ordered geometric aggregation operator for $\hat{S}_F N(V)$ is denoted by *GSFEOWG* and defined as follows:

$$GSFEOWG(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{\tilde{n}}) = \frac{1}{\Upsilon} \cdot \left(\prod_{g=1}^{\tilde{n}} \left(\Upsilon . \varepsilon_{\Omega(g)}\right)^{\kappa_g}\right), \tag{4.10}$$

Table 19 Score and ranking of SFNs

Log. Aggregation Operators (Jin et al. 2019a)	Score					Ranking
	$\check{s}\check{c}(S_1)$	$\check{s}\check{c}(S_2)$	$\check{s}\check{c}(S_3)$	$\check{s}\check{c}(S_4)$	$\check{s}\check{c}(S_5)$	
L-SFWA	0.982	0.998	0.984	0.737	0.934	$S_2 > S_3 > S_1 > S_5 > S_4$
L-SFOWA	0.980	0.993	0.987	0.613	0.903	$S_2 > S_3 > S_1 > S_5 > S_4$
L-SFHWA	0.9995	0.9999	0.9997	0.646	0.984	$S_2 > S_3 > S_1 > S_5 > S_4$
L-SFWG	0.979	0.995	0.972	0.622	0.926	$S_2 > S_1 > S_3 > S_5 > S_4$
L-SFOWG	0.976	0.979	0.973	0.330	0.892	$S_2 > S_1 > S_3 > S_5 > S_4$
L-SFHWG	0.9998	0.9999	0.9991	0.822	0.998	$S_2 > S_1 > S_3 > S_5 > S_4$
Proposed Einstein Aggregation Operators	Score					Ranking
	$\check{s}\check{c}(S_1)$	$\check{s}\check{c}(S_2)$	$\check{s}\check{c}(S_3)$	$\check{s}\check{c}(S_4)$	$\check{s}\check{c}(S_5)$	
SFEWA	0.472	0.529	0.523	0.426	0.458	$S_2 > S_3 > S_1 > S_5 > S_4$
SFEWG	0.553	0.609	0.593	0.477	0.510	$S_2 > S_3 > S_1 > S_5 > S_4$
GSFEWA	0.512	0.561	0.555	0.470	0.495	$S_2 > S_3 > S_1 > S_5 > S_4$
GSFEWG	0.516	0.566	0.549	0.433	0.473	$S_2 > S_3 > S_1 > S_5 > S_4$
GSFEOWA	0.512	0.562	0.556	0.470	0.496	$S_2 > S_3 > S_1 > S_5 > S_4$
GSFEOWG	0.516	0.567	0.549	0.433	0.474	$S_2 > S_3 > S_1 > S_5 > S_4$

Table 20 Aggregated SF information matrix

$$\begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{array}
\begin{array}{c}
\begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\
\begin{pmatrix}
(0.658, 0.427, 0.294) & (0.574, 0.361, 0.339) & (0.492, 0.548, 0.436) \\
(0.733, 0.489, 0.290) & (0.452, 0.677, 0.249) & (0.658, 0.307, 0.499) \\
(0.388, 0.663, 0.441) & (0.684, 0.276, 0.273) & (0.443, 0.266, 0.670) \\
(0.765, 0.332, 0.443) & (0.571, 0.564, 0.367) & (0.314, 0.349, 0.632)
\end{pmatrix}
\end{array}$$

Table 21 Score and ranking of SFNs

Algebraic Aggregation Operators (Ashraf and Abdullah 2019a)	Score				Ranking
	$\check{s}\check{c}(S_1)$	$\check{s}\check{c}(S_2)$	$\check{s}\check{c}(S_3)$	$\check{s}\check{c}(S_4)$	
textitSFWA	0.982	0.998	0.984	0.737	$S_2 > S_3 > S_1 > S_4$
SFOWA	0.980	0.993	0.987	0.613	$S_2 > S_3 > S_1 > S_4$
SFHWA	0.9995	0.9999	0.9997	0.646	$S_2 > S_3 > S_1 > S_4$
SFWG	0.979	0.995	0.972	0.622	$S_2 > S_1 > S_3 > S_4$
SFOWG	0.976	0.979	0.973	0.330	$S_2 > S_1 > S_3 > S_4$
SFHWG	0.9998	0.9999	0.9991	0.822	$S_2 > S_1 > S_3 > S_4$
Proposed Einstein Aggregation Operators	Score				Ranking
	$\check{s}\check{c}(S_1)$	$\check{s}\check{c}(S_2)$	$\check{s}\check{c}(S_3)$	$\check{s}\check{c}(S_4)$	
SFEWA	0.465	0.474	0.416	0.4294	$S_2 > S_1 > S_4 > S_3$
SFEWG	0.429	0.453	0.375	0.407	$S_2 > S_1 > S_4 > S_3$
GSFEWA	0.490	0.498	0.445	0.466	$S_2 > S_1 > S_4 > S_3$
GSFEWG	0.401	0.414	0.335	0.360	$S_2 > S_1 > S_4 > S_3$

Table 22 Spherical fuzzy information D^1 (Barukab et al. 2019)

$$\begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{array}
\begin{array}{c}
\begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\
\begin{pmatrix}
(0.84, 0.34, 0.40) & (0.43, 0.39, 0.78) & (0.67, 0.50, 0.30) & (0.31, 0.21, 0.71) \\
(0.60, 0.11, 0.53) & (0.23, 0.35, 0.59) & (0.72, 0.31, 0.41) & (0.11, 0.25, 0.82) \\
(0.79, 0.19, 0.39) & (0.11, 0.21, 0.91) & (0.71, 0.41, 0.13) & (0.34, 0.25, 0.51) \\
(0.63, 0.51, 0.13) & (0.49, 0.33, 0.42) & (0.61, 0.43, 0.45) & (0.49, 0.37, 0.59) \\
(0.57, 0.36, 0.29) & (0.50, 0.15, 0.60) & (0.70, 0.32, 0.40) & (0.33, 0.44, 0.65)
\end{pmatrix}
\end{array}$$

Table 23 Spherical fuzzy information D^2 (Barukab et al. 2019)

	f_1	f_2	f_3	f_4
S_1	(0.61, 0.15, 0.53)	(0.16, 0.35, 0.62)	(0.61, 0.35, 0.47)	(0.55, 0.17, 0.74)
S_2	(0.66, 0.11, 0.51)	(0.43, 0.23, 0.77)	(0.93, 0.08, 0.09)	(0.02, 0.06, 0.99)
S_3	(0.88, 0.09, 0.07)	(0.05, 0.06, 0.89)	(0.56, 0.17, 0.44)	(0.43, 0.13, 0.61)
S_4	(0.59, 0.32, 0.34)	(0.24, 0.48, 0.51)	(0.68, 0.53, 0.39)	(0.34, 0.21, 0.61)
S_5	(0.71, 0.31, 0.24)	(0.35, 0.41, 0.69)	(0.73, 0.44, 0.21)	(0.22, 0.49, 0.74)

Table 24 Spherical fuzzy information D^3 (Barukab et al. 2019)

	f_1	f_2	f_3	f_4
S_1	(0.85, 0.25.0.15)	(0.14, 0.23, 0.88)	(0.78, 0.38, 0.18)	(0.29, 0.39, 0.83)
S_2	(0.94, 0.04, 0.07)	(0.39, 0.19, 0.61)	(0.63, 0.18, 0.35)	(0.48, 0.49, 0.56)
S_3	(0.73, 0.13, 0.46)	(0.19, 0.39, 0.88)	(0.87, 0.35, 0.18)	(0.41, 0.13, 0.81)
S_4	(0.82, 0.12, 0.43)	(0.55, 0.21, 0.63)	(0.53, 0.33, 0.47)	(0.46, 0.23, 0.51)
S_5	(0.61, 0.33, 0.29)	(0.28, 0.41, 0.63)	(0.74, 0.34, 0.14)	(0.37, 0.32, 0.65)

Table 25 Collected spherical fuzzy information (Barukab et al. 2019)

	f_1	f_2	f_3	f_4
S_1	(0.79, 0.23, 0.31)	(0.78, 0.31, 0.21)	(0.69, 0.40, 0.29)	(0.76, 0.24, 0.36)
S_2	(0.80, 0.07, 0.27)	(0.67, 0.24, 0.34)	(0.81, 0.16, 0.23)	(0.91, 0.19, 0.10)
S_3	(0.81, 0.13, 0.23)	(0.89, 0.17, 0.10)	(0.75, 0.29, 0.21)	(0.67, 0.16, 0.39)
S_4	(0.70, 0.26, 0.26)	(0.53, 0.32, 0.40)	(0.61, 0.42, 0.43)	(0.57, 0.26, 0.42)
S_5	(0.63, 0.33, 0.27)	(0.64, 0.29, 0.36)	(0.72, 0.36, 0.22)	(0.68, 0.41, 0.29)

Table 26 Score and ranking of spherical fuzzy information

	Final revised closeness indices					Ranking
	S_1	S_2	S_3	S_4	S_5	
TOPSIS Method (Barukab et al. 2019)	0.4047	0.5641	0.5908	0.2576	0.3018	$S_3 > S_2 > S_1 > S_5 > S_4$
Proposed Einstein Aggregation Operators	Score					Ranking
	$\check{s}\check{c}(S_1)$	$\check{s}\check{c}(S_2)$	$\check{s}\check{c}(S_3)$	$\check{s}\check{c}(S_4)$	$\check{s}\check{c}(S_5)$	
SFEWA	0.47	0.52	0.53	0.45	0.46	$S_3 > S_2 > S_1 > S_5 > S_4$
SFEWG	0.55	0.59	0.61	0.48	0.51	$S_3 > S_2 > S_1 > S_5 > S_4$
GSFEWA	0.51	0.55	0.56	0.47	0.49	$S_3 > S_2 > S_1 > S_5 > S_4$
GSFEWG	0.52	0.55	0.57	0.43	0.47	$S_3 > S_2 > S_1 > S_5 > S_4$
GSFEOWA	0.51	0.55	0.56	0.47	0.50	$S_3 > S_2 > S_1 > S_5 > S_4$
GSFEOWG	0.51	0.55	0.57	0.43	0.47	$S_3 > S_2 > S_1 > S_5 > S_4$

where weights of $\varepsilon_g (g \in \mathbb{N})$ subject to $\kappa_g \geq 0$ and $\sum_{g=1}^{\tilde{n}} \kappa_g = 1$ is $\kappa_g (g \in \mathbb{N})$. Where Υ is the real number greater than zero, $\Omega: (1, 2, \ldots, \tilde{n}) - \rightarrow (1, 2, \ldots, \tilde{n})$, *SFN* $\varepsilon_{\Omega(g)}$ is the gth largest of *SFN* ε_g.

Theorem 10 *Let* $\varepsilon_g = \left\{ \rho_{\eth_g}, \daleth_{\eth_g}, \tilde{n}_{\eth_g} \right\} \in \hat{S}_F N(V) (g \in \mathbb{N})$ *and weights of* $\varepsilon_g (g \in \mathbb{N})$ *subject to* $\sum_{g=1}^{\tilde{n}} \kappa_g = 1$ *be represented by* $\kappa = \left(\kappa_1, \kappa_2, \ldots, \kappa_{\tilde{n}} \right)^T$. *The GSFEOWG operator is a mapping* $G^{\tilde{n}} \longrightarrow G$ *such that*

$$
\begin{aligned}
&GSFEOWG\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{\tilde n}\right)\\
&=\left(\sqrt{\frac{\left(\prod_{g=1}^{\tilde n}\left\{\left(1+\rho^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}+3\left(1-\rho^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}\right\}^{\kappa_g}+3\prod_{g=1}^{\tilde n}\left\{\left(1+\rho^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}-\left(1-\rho^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}-\left(\prod_{g=1}^{\tilde n}\left\{\left(1+\rho^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}+3\left(1-\rho^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}\right\}^{\kappa_g}-\prod_{g=1}^{\tilde n}\left\{\left(1+\rho^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}-\left(1-\rho^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}{\left(\prod_{g=1}^{\tilde n}\left\{\left(1+\rho^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}+3\left(1-\rho^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}\right\}^{\kappa_g}+3\prod_{g=1}^{\tilde n}\left\{\left(1+\rho^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}-\left(1-\rho^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}+\left(\prod_{g=1}^{\tilde n}\left\{\left(1+\rho^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}+3\left(1-\rho^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}\right\}^{\kappa_g}-\prod_{g=1}^{\tilde n}\left\{\left(1+\rho^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}-\left(1-\rho^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}},\right.\\
&\sqrt{\frac{2\left(\left(\prod_{g=1}^{\tilde n}\left(\daleth^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}{\left(\left(\prod_{g=1}^{\tilde n}\left(2-\daleth^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}+\left(\left(\prod_{g=1}^{\tilde n}\left(\daleth^2_{\varpi\Omega(g)}\right)^{\Upsilon}\right)^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}},\\
&\left.\frac{\sqrt{2\left\{\prod_{g=1}^{\tilde n}\left\{\left(2-\tilde n^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}+3\left(\tilde n^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}\right\}^{\kappa_g}-\prod_{g=1}^{\tilde n}\left\{\left(2-\tilde n^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}-\left(\tilde n^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}\right\}^{\kappa_g}\right\}^{\frac{1}{\Upsilon}}}}{\sqrt{\left(\prod_{g=1}^{\tilde n}\left\{\left(2-\tilde n^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}+3\left(\tilde n^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}\right\}^{\kappa_g}+3\prod_{g=1}^{\tilde n}\left\{\left(2-\tilde n^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}-\left(\tilde n^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}+\left(\prod_{g=1}^{\tilde n}\left\{\left(2-\tilde n^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}+3\left(\tilde n^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}\right\}^{\kappa_g}-\prod_{g=1}^{\tilde n}\left\{\left(2-\tilde n^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}-\left(\tilde n^2_{\eth_{\Omega(g)}}\right)^{\Upsilon}\right\}^{\kappa_g}\right)^{\frac{1}{\Upsilon}}}}\right)
\end{aligned}
$$

Proof The proof of this theorem is similar to that of Theorem 8 and hence it is omitted here. □

The following properties of GSFEOWG operator can be simply proved.

Theorem 11 1) *Let* $\varepsilon_g=\left\{\rho_{\eth_g},\daleth_{\eth_g},\tilde n_{\eth_g}\right\}\in\hat{S}_{F}N(V)$ $(g=1,2,3,\ldots,\tilde n)$, *if* $\varepsilon_1=\varepsilon_2=\cdots\varepsilon_{\tilde n-1}=\varepsilon_{\tilde n}=\varepsilon$, *then* $GSFEOWG\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{\tilde n}\right)=\varepsilon$.

2) *Let* $\varepsilon_g=\left\{\rho_{\eth_g},\daleth_{\eth_g},\tilde n_{\eth_g}\right\}\in\hat{S}_{F}N(V)\,(g=1,2,3,\ldots,\tilde n)$ *and* $\varepsilon^-=\min_g\varepsilon_g$, $\varepsilon^+=\max_g\varepsilon_g$. *Then*, $\varepsilon^-\le GSFEOWG\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{\tilde n}\right)\le\varepsilon^+$.

3) *Let* $\varepsilon_g=\left\{\rho_{\eth_g},\daleth_{\eth_g},\tilde n_{\eth_g}\right\}$ *and* $\varepsilon_{\tilde g}=\left\{\rho_{\sigma_{\tilde g}},\daleth_{\sigma_{\tilde g}},\tilde n_{\sigma_{\tilde g}}\right\}\in\hat{S}_{F}N(V)\left(g,\tilde g\in\mathbb{N}\right)$ *i.e.* $\varepsilon_g\le\varepsilon_{\tilde g}\;\forall\,g$. *Then*

$$GSFEOWG\left(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{\tilde n}\right)\le GSFEOWG\left(\varepsilon_{\tilde 1},\varepsilon_{\tilde 2},\ldots,\varepsilon_{\tilde n}\right).$$

5 Technique of solving MAGDM problem based on generalized Einstein aggregation operators

To tackle the MAGDM problems in a SF setting, we propose a generalized spherical fuzzy Einstein AOp-based methodology. MADM problems can be interpreted as a decision matrix (D), where columns show attributes collection and the alternatives are shown in the rows. Therefore, let consider a set of $\tilde n$ alternatives $\left\{S_1,S_2,S_3,\ldots,S_{\tilde n}\right\}$ for the decision matrix $D_{\tilde n\times m}$, and m attributes $\left\{f_1,f_2,f_3,\ldots,f_m\right\}$. $W=\left\{\rho_1,\rho_2,\rho_3,\ldots,\rho_m\right\}$ is presented the unknown weight of m attributes with $\rho_g\in[0,1]$ such that $\sum_{g=1}^{m}\rho_g=1$. Supposed the SF decision matrix (D) is denoted by $D=\left(\varepsilon_{ij}\right)_{\tilde n\times m}=\langle\rho_{ij},\daleth_{ij},\tilde n_{ij}\rangle_{\tilde n\times m}$, where ρ_{ij} represents the degree of the alternative gratifies the criteria f_j considered by decision maker (DM), $\daleth_{ij}$ represents the degree of the alternative is neutral for the criteria f_j considered by DM and $\tilde n_{ij}$ represents the degree of the alternative doesn't gratify the criteria f_j considered by DM. The following steps form the algorithm (Fig. 1);

Step-1 Data Collection: In the form of SF information, the decision maker provides the decision matrices as follows

$$D^t_{\tilde{n}\times m} = \begin{array}{c} \\ S_1 \\ S_2 \\ \vdots \\ S_{\tilde{n}} \end{array} \begin{pmatrix} f_1 & f_2 & & f_m \\ \left\langle \rho_{\beth_{11}}, \daleth_{\beth_{11}}, \tilde{n}_{\beth_{11}} \right\rangle & \left\langle \rho_{\beth_{12}}, \daleth_{\beth_{12}}, \tilde{n}_{\beth_{12}} \right\rangle & \cdots & \left\langle \rho_{\beth_{1m}}, \daleth_{\beth_{1m}}, \tilde{n}_{\beth_{1m}} \right\rangle \\ \left\langle \rho_{\beth_{21}}, \daleth_{\beth_{21}}, \tilde{n}_{\beth_{21}} \right\rangle & \left\langle \rho_{\beth_{22}}, \daleth_{\beth_{22}}, \tilde{n}_{\beth_{22}} \right\rangle & \cdots & \left\langle \rho_{\beth_{2m}}, \daleth_{\beth_{2m}}, \tilde{n}_{\beth_{2m}} \right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle \rho_{\beth_{n1}}, \daleth_{\beth_{n1}}, \tilde{n}_{\beth_{n1}} \right\rangle & \left\langle \rho_{\beth_{n2}}, \daleth_{\beth_{n2}}, \tilde{n}_{\beth_{n2}} \right\rangle & \cdots & \left\langle \rho_{\beth_{nm}}, \daleth_{\beth_{nm}}, \tilde{n}_{\beth_{nm}} \right\rangle \end{pmatrix}$$

Step-2 Normalization Data: Normalize the $D^t_{\tilde{n}\times m}$ matrix. There are usually two types of attributes/criteria in a MAGDM problem. They are benefit and cost type criteria, whose values have the opposite effects on decision making. To unify such effects, the values of cost criteria in $D^t_{\tilde{n}\times m}$ need to be normalized and a normalized matrix is obtained as follow:

$$D^t_{ij} = \begin{cases} \left(\rho_{\beth_{ij}}, \daleth_{\beth_{ij}}, \tilde{n}_{\beth_{ij}}\right) & if\ \ C_I \\ \left(\tilde{n}_{\beth_{ij}}, \daleth_{\beth_{ij}}, \rho_{\beth_{ij}}\right) & if\ \ C_{II} \end{cases} \tag{5.1}$$

where C_I refers to "if C_j is a benefit criterion" and C_{II} refers to "if C_j is a cost criterion".

Step-3 Aggregated information: The aggregated information of all decision-maker information is calculated in this step using the SFWA/SFWG operator, that is

$$SFEWA\left(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{\tilde{n}}\right) = \begin{pmatrix} \dfrac{\sqrt{\prod_{g=1}^{\tilde{n}}\left(1+\rho^2_{\beth_g}\right)^{\kappa_g} - \prod_{g=1}^{\tilde{n}}\left(1-\rho^2_{\beth_g}\right)^{\kappa_g}}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(1+\rho^2_{\beth_g}\right)^{\kappa_g} + \prod_{g=1}^{\tilde{n}}\left(1-\rho^2_{\beth_g}\right)^{\kappa_g}}}, \dfrac{\sqrt{2}\prod_{g=1}^{\tilde{n}}\left(\daleth_{\beth_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(2-\daleth^2_{\beth_g}\right)^{\kappa_g} + \prod_{g=1}^{\tilde{n}}\left(\daleth^2_{\beth_g}\right)^{\kappa_g}}}, \\ \dfrac{\sqrt{2}\prod_{g=1}^{\tilde{n}}\left(\tilde{n}_{\beth_g}\right)^{\kappa_g}}{\sqrt{\prod_{g=1}^{\tilde{n}}\left(2-\tilde{n}^2_{\beth_g}\right)^{\kappa_g} + \prod_{g=1}^{\tilde{n}}\left(\tilde{n}^2_{\beth_g}\right)^{\kappa_g}}} \end{pmatrix}.$$

Step-4 Unknown weight vector determined using SF-entropy measure: To determined the unknown weight vector of the attributes using spherical fuzzy entropy measure as following formula;

$$\kappa_j = \frac{1 + \frac{1}{\tilde{n}} \sum_{\daleth=1}^{\tilde{n}} \left(\rho_{\beth_{ij}} \log\left(\rho_{\beth_{ij}}\right) + \daleth_{\beth_{ij}} \log\left(\daleth_{\beth_{ij}}\right) + \tilde{n}_{\beth_{ij}} \log\left(\tilde{n}_{\beth_{ij}}\right)\right)}{\sum_{j=1}^{\tilde{n}} \left(1 + \frac{1}{\tilde{n}} \sum_{\daleth=1}^{\tilde{n}} \rho_{\beth_{ij}} \log\left(\rho_{\beth_{ij}}\right) + \daleth_{\beth_{ij}} \log\left(\daleth_{\beth_{ij}}\right) + \tilde{n}_{\beth_{ij}} \log\left(\tilde{n}_{\beth_{ij}}\right)\right)} \tag{5.2}$$

Step-5 Aggregate the information using entropy weight vector:

Step-5(a) Using SFEWA (operator) to combine (aggregate) the SF information.

Step-5(b) Using SFEWG to combine the SF information.

Step-5(c) Using GSFEWA to combine the SF information.

Step-5(d) Using GSFEWG to combine the SF information.

Step-5(e) Using GSFEOWA to combine the SF information.

Step-5(f) Using GSFEOWG to combine the SF information.

Step-6 Calculate the scores values $\check{s}\check{c}\left(\varepsilon_{\daleth}\right)$ of aggregated (combined) SFNs $\varepsilon_{\daleth}(\daleth = 1, 2, \ldots, \tilde{n})$ and rank by the maximum values for the score. If the score values are the same for two $\varepsilon_{\daleth}$ and ε_j, then we must take into consideration the accuracy degrees $\tilde{a}\check{c}\left(\varepsilon_{\daleth}\right)$ and $\tilde{a}\check{c}\left(\varepsilon_j\right)$, respectively, then we're going to rank the maximum degree of $\varepsilon_{\daleth}$ and ε_j.

Step-7 Choose the optimal alternative as per the highest score value or the accuracy degree.

6 Numerical application of the proposed technique

A numerical application regarding the emergency decision making for COVID-19 is firstly used in this section to demonstrate the designed MAGDM process. Then a comparison is made between the proposed AOp and the existing AOp of SFNs to demonstrate the feature and benefit of the generalized Einstein AOp.

6.1 Case study

To demonstrate the applicability and validity of the proposed methods, we extend a real case study about an emergency caused by an outbreak of novel coronavirus disease (COVID-19) pandemic that occurred in China.

Actions taken by governments and organizations: The spread was first observed around December 2019 in Wuhan, Hubei, China, and reported by the World Health Organization (WHO) on March 11, 2020 as an epidemic disease. In early 2020, the novel coronavirus pushed the Chinese government to initiate the largest lockdown in human history, threatening an estimated 45 million individuals. The name of the "Novel Coronavirus (COVID-19)" virus was announced by the WHO. On January 30, 2020, the WHO Director-General announced that the outbreak was triggering a global health emergency.

The risk that it will spread further is quite high. WHO defined the outbreak as a public health emergency of international concern. This disease is undoubtedly caused by enormous economic losses, environmental pollution, lack of personal protective equipment (PPE), PPE consists of respiratory/surgical masks, gloves, face protection. The potential for an expansion of the supply of EPP is limited and it is not possible to meet the current requirements for respirators and masks, especially if the widespread and improper use of EPP continues. The WHO collaborates with public health experts and laboratory partnerships, prevention and monitoring of diseases, clinical management and mathematical modeling.

In such a situation, it is important to provide an effective means of emergency response to prevent further losses and to save the lives of individuals. In both health care and community environments, preventive and mitigation measures are key. As a result of such an emergency decision, health professionals must respond immediately, rescue them urgently to control the situation effectively and prevent more fatalities. There are three decision makers to provied their information using spherical fuzzy information to deal the uncertainty in emergency response and their weights are $(0.314, 0.355, 0.331)^T$.

There are five emergency alternatives to overcome the COVID-2019 which are given by

(1) Control vector (S_1);
(2) Food facility at Home, Lockdown (S_2);
(3) Educate the People (S_3);
(4) Disease and surveillance (S_4);
(5) Improving nutrition (S_5).

And also four criteria. which are given by

(1) Space (f_1);
(2) People (f_2);
(3) Time (f_3);(4) Rescue people/Test (f_4).

Step-1 The expert evaluation results are listed in the Tables 3, 4 and 5:

Step-2 The attributes f_1 and f_3 are benefits type, f_2 and f_4 are cost attributes according to the experts. Normalized matrix calculated as the formula given 5.1, and outcomes are seen in Tables 6, 7 and 8:

Step-3 In Table 9 aggregated SF information is calculated using the SFWA operator.

Step-4 To computed the unknown weight vector of the attributes using spherical fuzzy entropy measure 5.2 as follows

$$W = \{\kappa_1 = 0.256, \kappa_2 = 0.248, \kappa_3 = 0.245, \kappa_4 = 0.251\}^T$$

Step-5(a) Used SFEWA operator to combine (aggregate) the SF information as given in Table 10:

Step-5(b) Used SFEWG operator to combine the SF information as given in Table 11:

Step-5(c) Used GSFEWA operator to combine the SF information with $\Upsilon = 2$, as given in Table 12:

Step-5(d) Used GSFEWG operator to combine the SF information with $\Upsilon = 2$, as given in Table 13:

Step-5(e) Used GSFEOWA operator to combine the SF information with $\Upsilon = 2$, as given in Table 14:

Step-5(f) Used GSFEOWG operator to combine the SF information with $\Upsilon = 2$, as given in Table 15:

Step-6 Calculated the scores values $\check{s}\check{c}(\varepsilon_\daleth)$ of combined (aggregated) SFNs and ranked as follows according to the maximum score values in Table 16;

Step-7 Under all the suggested generalized Einstein aggregation operators, Machine S_2 has the highest score value, therefore S_2 (Lockdown and provide food facility at home) is our best alternative with respect to offer attributes listing to tackle and prevent to novel coronavirus (COVID-2019).

6.2 Sensitivity analysis

In this subsection, we discuss the sensitivity analysis in variation of the value of the parameter Υ from 0 to 30 on the alternatives by using the proposed generalized Einstein aggregation operators to examine the distinct trends of the scores and ranking of the alternatives. Results obtained from proposed GSFEWA, GSFEWG, GSFEOWA and GSFEOWG aggregation operators are summarized in Table 16. These results obtained show the decision makers (DM,s) that they are able to select the values of Υ according to their preferences. If $\Upsilon = 1$, then the GSFEWA/GSFWG reduces to the SFEWA/SFEWG respectively. Also, it has been noted that $\Upsilon = 1$ means that the attitude of DM,s is neutral, and it is transpired that the overall score values of different alternates are increasing with an increase in Υ. Thus the management meaning of Υ is that the DM's different preferences had effects on the score values of alternatives, which lead to the different optimal alternatives. (can be seen in Table 17)

7 Comparison analysis

A comparison of the features of these proposed generalized Einstein aggregation operators with the designed MAGDM method is presented in this section to demonstrate the benefits of the developed methodology. The comparison is provided by comparing the characteristics of the different aggregation operators which are described in different methods. The logarithmic based aggregation operators were identified in the approach (Jin et al. 2019a). The method presented in (Ashraf and Abdullah 2019a) discussed the algebraic aggregation operators.

We give the good comparison of the proposed six Einstein aggregation with existing aggregation operators presented in (Ashraf and Abdullah 2019a; Ashraf et al. 2020a; Jin et al. 2019a), showing the strength to handle real-life DMPs with uncertainty. The results are seen in below tables. It is explained as follows:

(1) **Comparison with** Jin et al. (2019a) **logarithmic operators**

Collective SF information of (Jin et al. 2019a) is shown in Table 18.

The matrix for comparison after aggregation by proposed Einstein aggregation operators is shown in Table 19:

(2) **Comparison with** Ashraf and Abdullah (2019a) **algebraic aggregation operators**

Collective SF information of (Ashraf and Abdullah 2019a) is shown in Table 20.

The matrix for comparison after aggregation by proposed Einstein aggregation operators is shown in Table 21:

Hence, S_2 is the best alternative which is computed under list of attributes.

8 Comparison with TOPSIS approach

In this subsection, we propose the comparison study of the developed generalized Einstein aggregation opeators with the improved TOPSIS methodology proposed by Barukab et al. (2019). The spherical fuzzy information evaluted in Barukab et al. (2019) is given in Table 22, 23 and 24.

Collective spherical information using spherical fuzzy weighted averaging aggregation operator is calculated in Table 25:

The comparison ranking matrix after aggregation by proposed Einstein aggregation operators is shown as follows in Table 26:

From the outcomes of the proposed operators and the existing improved TOPSIS methodology, we conclude that ranking lists obtained from both the proposed method and the compared methods are same. Hence, Proposed generalized Einstein aggregation operators under the spherical fuzzy set environment is a generalized and novel approach to tackle uncertainty in DM problems. The proposed operators with the spherical fuzzy environment are a more flexible and effective to evaluate best alternative in real word problems.

9 Conclusion

In this paper, generalized spherical fuzzy Einstein weighted average, weighted geometric, ordered weighted average and ordered weighted geometric aggregation operators have been presented to aggregate the uncertainty in emergency situation of COVID-19 as a real life emergency decision making problem. The formal definitions and properties of these generalized Einstein aggregation operators have been respectively provided and explored. Their specific expressions are established via the operational laws of SFNs based on the Einstein t-norm and t-conorm. Based on the specific expressions, a new method for solving the MAGDM problem has been proposed. Based on these generalized spherical fuzzy Einstein aggregation operators, we designed an algorithms to tackle emergency situation of COVID-19 effectively by the physicians or administrators. Validation and effectiveness of the proposed designed algorithm is tested over existing techniques. Results shows that the proposed technique is reliable and effective to reduce/prevent the outbreak of COVID-19.

In future research, the other techniques of spherical FSs, like VIKOR, TODAM, Electric-I, II, and III with real life problems are investigated. Future work will also focus on applying the proposed method to solve practical MAGDM problems in manufacturing domain.

Limitation: A number of included studies were limited in terms data availability and methodological quality. Therefore, the reported findings should be interpreted cautiously within that context. Furthermore, our study was limited to the articles published in English. Considering the epicenter of COVID-19, Chinese literature should be included in future systematic reviews. We will continue to monitor the literature, and this method will be updated when new evidence emerges.

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Data availability No data were used to support this study.

Declarations

Conflict of Interest The authors declare that they have no conflict of interest regarding the publication of the research article.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

References

Ashraf S, Abdullah S, Mahmood T (2018) GRA method based on spherical linguistic fuzzy Choquet integral environment and its

application in multi-attribute decision-making problems. Math Sci 12(4):263–275
Ashraf S, Abdullah S (2019) Spherical aggregation operators and their application in multiattribute group decision-making. Int J Intell Syst 34(3):493–523
Ashraf S, Abdullah S, Mahmood T, Ghani F, Mahmood T (2019) Spherical fuzzy sets and their applications in multi-attribute decision making problems. J Intell Fuzzy Syst 36(3):2829–2844
Ashraf S, Abdullah S, Abdullah L (2019) Child development influence environmental factors determined using spherical fuzzy distance measures. Mathematics 7(8):661
Ashraf S, Abdullah S, Aslam M, Qiyas M, Kutbi MA (2019) Spherical fuzzy sets and its representation of spherical fuzzy t-norms and t-conorms. J Intell Fuzzy Syst 36(6):6089–6102
Ashraf S, Abdullah S, Mahmood T (2020) Spherical fuzzy Dombi aggregation operators and their application in group decision making problems. J Amb Intel Hum Comp 11:2731–2749
Ashraf S, Abdullah S, Almagrabi AO (2020) A new emergency response of spherical intelligent fuzzy decision process to diagnose of COVID19. Soft Comput. https://doi.org/10.1007/s00500-020-05287-8
Ashraf S, Abdullah S, Aslam M (2020) Symmetric sum based aggregation operators for spherical fuzzy information: Application in multi-attribute group decision making problem. J Intell Fuzzy Syst 38(4):5241–5255
Ashraf S, Abdullah S (2020) Emergency decision support modeling for COVID-19 based on spherical fuzzy information. Int J Intell Syst 35(11):1601–1645
Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Set Syst 20:87–96
Baležentis T, Štreimikienė D, Melnikienė R, Zeng S (2019) Prospects of green growth in the electricity sector in Baltic States: Pinch analysis based on ecological footprint. Resour Conserv Recy 142:37–48
Barukab O, Abdullah S, Ashraf S, Arif M, Khan SA (2019) A new approach to fuzzy topsis method based on entropy measure under spherical fuzzy information. Entropy 21(12):1231
Centers for Disease Control and Prevention, https://www.cdc.gov/coronavirus/2019-nCoV/index.html
Coronavirus COVID-19 Global Cases by the Center for Systems Science and Engineering (CSSE) at Johns Hopkins University (JHU). ArcGIS. Johns Hopkins CSSE. Retrieved 15 NOV 2020
Coronavirus COVID-19 situation of Pakistan and Worldwide, http://covid.gov.pk/stats/global
Cuong BC, Kreinovich V. Picture Fuzzy Sets-a new concept for computational intelligence problems. In2013 Third World Congress on Information and Communication Technologies (WICT 2013) 2013 (pp. 1-6). IEEE
Garg H, Munir M, Ullah K, Mahmood T, Jan N (2018) Algorithm for T-spherical fuzzy multi-attribute decision making based on improved interactive aggregation operators. Symmetry 10(12):670
Garg H (2016) Generalized intuitionistic fuzzy interactive geometric interaction operators using Einstein t-norm and t-conorm and their application to decision making. Comput Ind Eng 101:53–69
Garg H (2016) A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making. Int J Intell Syst 31(9):886–920
Garg H (2017) Generalized Pythagorean fuzzy geometric aggregation operators using Einstein t-norm and t-conorm for multicriteria decision-making process. Int J Intell Syst 32(6):597–630
Goala S, Dutta P (2018) Detection of area under potential threat via an advanced aggregation operator on generalized triangular fuzzy number. J Taibah Univ Sci 12(5):536–544
Hussain A, Ali MI, Mahmood T (2020) Pythagorean fuzzy soft rough sets and their applications in decision-making. J Taibah Univ Sci 14(1):101–113
Hwang CL, Yoon K (1981) Methods for multiple attribute decision making. Multiple attribute decision making. Springer, Berlin, pp 58–191
Jan N, Mahmood T, Zedam L, Abdullah L, Ullah K (2021) Analysis of double domination by using the concept of spherical fuzzy information with application. J Ambient Intell Human Comput. https://doi.org/10.1007/s12652-021-03370-y
Jin Y, Ashraf S, Abdullah S (2019) Spherical fuzzy logarithmic aggregation operators based on entropy and their application in decision support systems. Entropy 21(7):628
Jin Y, Wu H, Sun D, Zeng S, Luo D, Peng B (2019) A multi-attribute Pearson's picture fuzzy correlation-based decision-making method. Mathematics 7(10):999
Jin H, Ashraf S, Abdullah S, Qiyas M, Bano M, Zeng S (2019) Linguistic spherical fuzzy aggregation operators and their applications in multi-attribute decision making problems. Mathematics 7(5):413
Khan S, Abdullah S, Ashraf S (2019) Picture fuzzy aggregation information based on Einstein operations and their application in decision making. Math Sci. 13(3):213–229
Kutlu Gundogdu F, Kahraman C (2019) Extension of WASPAS with spherical fuzzy sets. Informatica 30(2):269–292
Kutlu Gündoğdu F, Kahraman C (2019) A novel VIKOR method using spherical fuzzy sets and its application to warehouse site selection. J Intell Fuzzy Syst 37(1):1197–1211
Qiyas M, Abdullah S, Liu Y, Naeem M (2021) Multi-criteria decision support systems based on linguistic intuitionistic cubic fuzzy aggregation operators. J Ambient Intell Human Comput 12:8285–8303
Rahman K, Abdullah S, Ali A, Amin F (2019) Interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging aggregation operator and their application to group decision making. Comp Intell Syst 5(1):41–52
Rahman K, Abdullah S, Ghani F (2019) Some new generalized interval-valued Pythagorean fuzzy aggregation operators using einstein t-norm and t-conorm. J Intell Fuzzy Syst 37(3):3721–3742
Rahman K, Abdullah S, Khan MSA (2020) Some interval-valued Pythagorean fuzzy Einstein weighted averaging aggregation operators and their application to group decision making. J Intell Syst 29(1):393–408
Rafiq M, Ashraf S, Abdullah S, Mahmood T, Muhammad S (2019) The cosine similarity measures of spherical fuzzy sets and their applications in decision making. J Intell Fuzzy Syst 36:6059–6073
Saaty TL (2008) Decision making with the analytic hierarchy process. Int J Serv Sci 1(1):83–98
Saha A, Majumder P, Dutta D, Debnath BK (2021) Multi-attribute decision making using q-rung orthopair fuzzy weighted fairly aggregation operators. J Amb Intel Hum Comp 12:8149–8171
WHO Coronavirus Disease (COVID-19) Dashboard 2021. https://covid19.who.int/
Yager RR. Pythagorean fuzzy subsets. In2013 joint IFSA world congress and NAFIPS annual meeting (IFSA/NAFIPS) 2013 Jun 24 (pp. 57–61). IEEE
Yager RR, Abbasov AM (2013) Pythagorean membership grades, complex numbers, and decision making. Int J Intell Syst 28(5):436–452
Zadeh LA (1965) Fuzzy sets. Inf control 8:338–353
Zeng S, Hussain A, Mahmood T, Ali MI, Ashraf S, Munir M (2019) Covering-based spherical fuzzy rough set model hybrid with TOPSIS for multi-attribute decision-making. Symmetry 11(4):547
Zeng S, Chen SM, Kuo LW (2019) Multiattribute decision making based on novel score function of intuitionistic fuzzy values and modified VIKOR method. Inf Sci 488:76–92
Zhang X, Xu Z (2014) Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. Int J Intell Syst 29(12):1061–1078
Zimmerman TG (1996) Personal area networks: near-field intrabody communication. IBM syst J 35(3.4):609–617

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