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General semi-implicit approximations with errors for common fixed points of nonexpansive-type operators and applications to Stampacchia variational inequality

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Abstract
It is meaningful and valuable to find common fixed points of different nonexpansive-type operators, which are associated with variational inequalities, integral equations, image process and other optimization problems in real life. The purpose of this paper is to suggest and consider a class of general semi-implicit iterative methods involving semi-implicit rule and inaccurate computing errors, which extend the iterative algorithm introduced by Ali et al. in 2020. Using Liu's lemma, we analyze convergence and stability of the new iterative approximations for common fixed points of three different nonexpansive-type operators. Furthermore, we provide convergence rates of the new iterations and some numerical examples to illustrate the efficiency and stability of the new iterative schemes. As an application of our main results presented in this paper, we use the proposed iterative schemes to solve the known Stampacchia variational inequality.

Keywords Convergence and stability · General semi-implicit iteration with errors · Common fixed point · Nonexpansive-type operator · Liu's lemma

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Abbreviations

JF	Iterative scheme introduced by Ali et al. (2020).
JFESD	General semi-implicit approximation with errors for three different nonexpansive-type operators.
JFSD	Semi-implicit iteration for three different nonexpansive-type operators.
JFES	Semi-implicit scheme with errors for a nonexpansive-type operator.
JFS	Semi-implicit scheme for a nonexpansive-type operator.
PMMI	Picard–Mann semi-implicit iteration with mixed errors for a nonexpansive-type operator (Li and Lan 2019).
PMI	Picard–Mann semi-implicit iterative process for a nonexpansive-type operator (Li and Lan 2019).
MANN	Mann iteration introduced by Mann (1953).
ISHIKAWA	Ishikawa iterative process due to Ishikawa (1974).
NOOR	Noor three-step iterative approximation scheme introduced by Noor (2007).
SAKURAI	Novel fixed point algorithm formulated by Sakurai and Iiduka (2014).
Iter.	The numbers of iteration.

1 Introduction

Let $\mathbb{X}$ be a Hilbert space and $K \subset \mathbb{X}$ be a nonempty closed convex bounded subset. For $i = 1, 2, 3$, suppose that $\Phi_i : K \to K$ is a nonexpansive-type operator with Lipschitz coefficient $\theta_i \in [0, 1]$, that is, $\|\Phi_i(x) - \Phi_i(y)\| \leq \theta_i \|x - y\|$ for each $x, y \in K$. In this paper, based on the iterative scheme (in short, JF) due to Ali et al. (2020), we introduce and investigate the following general semi-implicit (also named as implicit midpoint rule) approximation (in short, JFESD) with errors for Φ_i $(i = 1, 2, 3)$:

$$\begin{cases} \varrho_{n+1} = \Phi_1\left((1 - r_n)\,\varsigma_n + r_n\Phi_1(\varsigma_n) + r_n d_n\right), \\ \varsigma_n = \Phi_2\left(\dfrac{\vartheta_n + \varsigma_n}{2}\right) + e_n, \\ \vartheta_n = \Phi_3\left((1 - s_n)\,\dfrac{\vartheta_n + \varrho_n}{2} + s_n\Phi_3\left(\dfrac{\vartheta_n + \varrho_n}{2}\right)\right) + h_n, \end{cases} \tag{1.1}$$

where $\{r_n\}, \{s_n\} \subseteq [0, 1]$ are two real number sequences, and $\{d_n\}$, $\{e_n\}$ and $\{h_n\}$ are three errors to take into account some possible inexact computations of the nonexpansive-type operator points, which satisfy hypothesis **(H)**:

(i) $d_n = d_n' + d_n''$ with $\lim_{n\to\infty} \|d_n'\| = 0$ and $\sum_{n=0}^{\infty} \|d_n''\| < \infty$;
(ii) $\lim_{n\to\infty} \|e_n\| = 0$; (iii) $\sum_{n=0}^{\infty} \|h_n\| < \infty$.

Some special cases of the JFESD (1.1) can be found as follows:

(**Case 1**) While $d_n, e_n, h_n \equiv 0$ for all $n \in \mathbb{N}$, the JFESD (1.1) reduces to the following semi-implicit iteration (in short, JFSD) for three different nonexpansive-type operators:

$$\begin{cases} \varrho_{n+1} = \Phi_1\left((1 - r_n)\,\varsigma_n + r_n\Phi_1(\varsigma_n)\right), \\ \varsigma_n = \Phi_2\left(\dfrac{\vartheta_n + \varsigma_n}{2}\right), \\ \vartheta_n = \Phi_3\left((1 - s_n)\,\dfrac{\vartheta_n + \varrho_n}{2} + s_n\Phi_3\left(\dfrac{\vartheta_n + \varrho_n}{2}\right)\right). \end{cases} \tag{1.2}$$

(**Case 2**) If $\Phi_1 = \Phi_2 = \Phi_3 = \Phi$, then the JFESD (1.1) is equivalent to a class of semi-implicit scheme (in short, JFES) with errors for a nonexpansive-type operator as follows:

$$\begin{cases} \varrho_{n+1} = \Phi\left((1-r_n)\,\varsigma_n + r_n\Phi(\varsigma_n) + r_n d_n\right), \\ \varsigma_n = \Phi\left(\dfrac{\vartheta_n + \varsigma_n}{2}\right) + e_n, \\ \vartheta_n = \Phi\left((1-s_n)\,\dfrac{\vartheta_n + \varrho_n}{2} + s_n\Phi\left(\dfrac{\vartheta_n + \varrho_n}{2}\right)\right) + h_n. \end{cases} \tag{1.3}$$

(**Case 3**) When $d_n = e_n = h_n \equiv 0$ for each $n \in \mathbb{N}$, the JFES (1.3) becomes the following new semi-implicit iterative process (in short, JFS) for a nonexpansive-type operator:

$$\begin{cases} \varrho_{n+1} = \Phi\left((1-r_n)\,\varsigma_n + r_n\Phi(\varsigma_n)\right), \\ \varsigma_n = \Phi\left(\dfrac{\vartheta_n + \varsigma_n}{2}\right), \\ \vartheta_n = \Phi\left((1-s_n)\,\dfrac{\vartheta_n + \varrho_n}{2} + s_n\Phi\left(\dfrac{\vartheta_n + \varrho_n}{2}\right)\right). \end{cases} \tag{1.4}$$

Remark 1.1 (i) In regard to nonexpansive-type operator Φ_i for $i = 1, 2, 3$ in (1.1)–(1.4), one can see that Φ_i is nonexpansive if the Lipschitz coefficient $\theta_i = 1$; and when $\theta_i \in [0, 1)$, Φ_i becomes a contraction operator. Furthermore, if $\theta_i > 0$ for $i = 1, 2, 3$, then Φ_i is called θ_i-Lipschitzian continuous. In other words, for $i = 1, 2, 3$, nonexpansive-type operator with Lipschitz coefficient θ_i is said to be θ_i-Lipschitz continuous operator, here $\theta_i \le 1$.

(ii) We note that the above iterative processes (1.1)–(1.4) are brand new and not reported in the literature.

Example 1.1 It is well known that the projection operator $P_K : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$P_K(x) = \arg\min_{z\in K} \|z - x\|_2^2, \tag{1.5}$$

is a nonexpansive-type operator with Lipschitz coefficient 1 and it is widely used to solve variational inequality, which is an important branch of optimization and was introduced in the early sixties by the study of mechanics.

Remark 1.2 For some $i = 1, 2, 3$, when $\Phi_i = P_K$, the projection operator in Example 1.1, it is easy to see that the iterations (1.1)–(1.4) are still different from Algorithms 2.2–2.6 in Noor (2007).

As we all know, research of nonexpansive-type operators has a long history, one of the most important fields is to find fixed points via applying fixed point theory. In fact, many problems can be formulated as fixed point models or can be solved by fixed point theory. For example, the whole world has been profoundly impacted by the novel coronavirus since 2019 and it is imperative to depict the spread of the coronavirus. Panda et al. (2021) applied fractional derivatives to improve the 2019-nCoV/SARS-CoV-2 models, and by means of fixed point theory, existence and uniqueness of solutions of the models were proved. More applications of fixed point theory, such as equations, image process and other optimization problems, one can refer to Cacciapaglia and Sannino (2021); Ali et al. (2022); Panda et al. (2020); Harker and Pang (1990); Ali et al. (2020); Maldar (2021); Hanjing and Suantai (2020) and the reference therein.

In 1922, S. Banach first used the Picard iteration method to create a fixed point theorem in metric spaces, that is the famous Banach contraction principle. After that, many scholars introduced a lot of iteration methods, such as Mann iteration (in short, MANN) (Mann 1953), Ishikawa iterative process (in short, ISHIKAWA) (Ishikawa 1974), Noor three-step iterative approximation scheme (in short, NOOR) (Noor 2007; Noor and Yao 2007), novel fixed point algorithm (in short, SAKURAI) (Sakurai and Iiduka 2014) and so on, to approximate fixed points of various nonexpansive-type operators and obtained others fixed point theorems, see (Thakur et al. 2016) and the reference therein. Recently, to answer the question: Is it possible to define a new iterative scheme, whose rate of convergence is better than that of some known and leading iterative schemes, for generalized nonexpansive-type operators due to Hardy and Rogers? Ali et al. (2020) considered the JF as follows:

$$\begin{cases} x_{n+1} = \Phi\left((1-r_n)y_n + r_n\Phi(y_n)\right), \\ y_n = \Phi(z_n), \\ z_n = \Phi\left((1-s_n)x_n + s_n\Phi(x_n)\right), \end{cases} \tag{1.6}$$

where $r_n, s_n \in (0,1)$ are two sequences, and studied the conditions of weak and strong convergence for (1.6) to a generalized self-map Φ on a nonempty closed convex subset of a Banach space. The numerical examples in Ali et al. (2020) indicate that the JF (1.6) is much faster than some existing algorithms.

Remark 1.3 It is worth noting that the semi-implicit terms in the second and third equations of the JFS (1.4) replace the corresponding terms of the JF (1.6).

On the one hand, to solve nonlinear problems arising in mechanics, economics, optimization, differential equations and others mathematics and engineering problems, variational inequalities mentioned in Example 1.1 have become efficient tools and have been paid great attention by many scholars. See, for example, Daniele et al. (2003); Harker and Pang (1990); Noor and Yao (2007); Phannipa and Atid (2021); Noor (2007) and the reference therein.

In 1964, Stampacchia (1964) considered the following variational inequality problem of finding a point $u \in K$ such that

$$\langle F(u), v-u\rangle \geq 0, \quad \forall v \in K, \tag{1.7}$$

where K is a nonempty, closed and convex set of a real Hilbert space $\mathbb{X}$, and $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{X}$, $F: K \to K$ is a specific nonlinear operator. The set of solutions for the variational inequality (1.7) is denoted by $VI(K, F)$. As everyone knows that the problem (1.7) is equivalent to the following fixed point problem:

$$u = P_K(u - \kappa F(u)), \tag{1.8}$$

here κ is a positive constant and P_K is the projection operator formulated as in (1.5). There are many methods to solve the inequality problems of the form (1.7) (see Harker and Pang 1990), which include excellent numerical behaviors based on projection and contraction operator techniques.

For seeking common fixed points of nonexpansive-type operators and variational inequalities, Noor and Yao (2007) introduced some iterative schemes with projection operators and proved convergence of the iterative algorithms. To get strong convergence in more weak conditions, Phannipa and Atid (2021) introduced an iteration scheme and showed that the iteration scheme strongly converges to a common fixed point of nonexpansive-type operators and variational inequalities in Hilbert spaces. Very recently, based on the fundamental relation between variational inequality and fixed point problem presented in (1.8), Maldar (2021) reformulated the NOOR, JF (1.6) and some other iterative algorithms to approximate the common fixed points of generalized nonexpansive-type mappings and solutions of variational inequality problems.

On the other hand, the semi-implicit rule is a powerful numerical method for solving ordinary differential equations (Deuflhard 1985), differential algebraic equations (Schneider 1993). In addition, as an approximation method, some authors applied semi-implicit rule to iterative scheme of nonexpansive-type operators. Alghamdi et al. (2014) established a semi-implicit rule for nonexpansive-type operators and proved weak convergence of the iteration in Hilbert spaces, and used the algorithm to solve a nonlinear time-dependent evolution equation. Based on the semi-implicit rule, Luo and Cai (2017) and Aibinu et al. (2018) developed viscosity algorithms for nonexpansive-type operators and stated that these algorithms can strongly converge to a fixed point in smooth Banach spaces.

While calculating every iteration, errors will occur naturally, so it is worth of studying convergence of algorithms with errors. Liu (1995) first established ISHIKAWA and MANN with errors and proved that the iterations strongly converge to the unique solution of the accretive operator equations in Banach spaces. Following the work of Liu (1995), Chang et al. (2003) studied ISHIKAWA with mixed errors of nonexpansive-type operators, and found necessary and sufficient conditions for the iterative sequence to strongly converge to a fixed point in Banach spaces. Ni and Yao (2015) constructed a modified ISHIKAWA with errors for nonexpansive-type operator, and obtained strong convergence of the iterative algorithm in reflexive Banach spaces under suitable conditions.

By the importance of stable and unstable equilibria are really different in dynamics systems, the stability of equilibrium points should be investigated seriously. Lemaire (1996) may be the first one to investigate the stability of the iteration method for nonexpansive-type operators. In 2017, Alber (2017) proved weak and strong convergence and stability of some iteration schemes with perturbations in a Banach space. Lately, Li and Lan (2019) proposed convergence and stability analysis of new Picard–Mann iteration processes with errors for the semi-implicit rules of two different nonexpansive-type operators, and proved iterative approximation of solutions for a class of optimal control problems with elliptic boundary value constraint. Furthermore, Li and Lan (2019) gave numerical examples to show that the new Picard–Mann iteration process with mixed errors is more effective than the Picard–Mann semi-implicit iteration (in short, PMMI) with mixed errors, the Picard–Mann semi-implicit iterative process (in short, PMI) and other related iterative processes for a nonexpansive-type operator. And then, Ali et al. (2021) defined Picard's three-step iteration for approximating fixed points of Zamfirescu operators which contain contraction mappings, and proved the iteration is almost T-stable and compared convergence of the three-step iteration with that of some leading iterative processes.

On the basis of Ali et al. (2020), Li and Lan (2019) and other work mentioned above, to obtain a fast and stable iterative scheme of nonexpansive-type operators and to solve the variational inequality (1.7), we explored the JFESD (1.1) and its special case (1.2)–(1.4). Then, we analyze convergence and stability of the sequences generated by (1.1)–(1.4) in Hilbert spaces, and obtain convergence rate of the schemes (1.1)–(1.4). Finally, we provide some numerical examples to illustrate that our schemes are more effective than other aforesaid known iterative schemes and apply the general semi-implicit approximation (1.1) to solve Stampacchia variational inequality (1.7).

2 Main results

In this section, we introduce a useful definition and an important lemma, and prove convergence and stability of the JFESD (1.1). Furthermore, using our iteration scheme JFESD (1.1), an integral equation as a numerical example shall be solved.

Definition 2.1 Let $(\mathbb{X}, d)$ be a metric space, $\Phi : \mathbb{X} \to \mathbb{X}$ be a given operator and $\varrho_0 \in \mathbb{X}$. Assume that the iteration scheme $\{\varrho_n\}$ generated by

$$\varrho_{n+1} = f(\Phi, \varrho_n),$$

converges to a fixed point u of Φ, i.e., $u \in Fix(\Phi) := \{x \in \mathbb{X} | \, \Phi(x) = x\}$. Taking $\{\varsigma_n\}$ be an arbitrary sequence in $\mathbb{X}$ and setting $\epsilon_n = d\left(\varsigma_{n+1}, f(\Phi, \varsigma_n)\right)$ for $n \in \mathbb{N}$, then the fixed point iteration procedure $\{\varrho_n\}$ is called Φ-stable or stable with respect to Φ, and $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} \varsigma_n = u$.

Lemma 2.1 (Liu 1995) *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences meeting:*

$$a_{n+1} \le (1 - t_n)a_n + b_n + c_n$$

with $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$, and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

In the sequel, we give convergence and stability analysis of the JFESD (1.1) via using Lemma 2.1 and Definition 2.1.

Theorem 2.1 *Let K be a nonempty closed convex bounded subset of Hilbert space $\mathbb{X}$. If for $i = 1, 2, 3$, $\Phi_i : K \to K$ is a nonexpansive-type operator with Lipschitz coefficient $\theta_i \in [0, 1]$ satisfying $\sum_{i=1}^{3} \theta_i < 3$ and $Fix(\Phi_1 \cap \Phi_2 \cap \Phi_3) \neq \emptyset$,*

then the following statements hold:

(i) *The iterative sequence $\{\varrho_n\}$ generated by* JFESD (1.1) *converges to $\varrho^* \in Fix(\Phi_1 \cap \Phi_2 \cap \Phi_3)$ with convergence rate for every step (i.e., $n \in \mathbb{N}$):*

$$\rho_n = \theta_1\theta_2\theta_3 \cdot \frac{[1 - r_n(1 - \theta_1)]\,[1 - s_n(1 - \theta_3)]}{(2 - \theta_2)\,\{2 - \theta_3\,[1 - s_n(1 - \theta_3)]\}},$$

and there exists a constant $\rho \in (0, 1]$ such that $\lim_{n\to\infty} \rho_n < \rho$.

(ii) *For any sequence $\{w_n\} \subset \mathbb{X}$, $\lim_{n\to\infty} w_n = \varrho^*$ if and only if $\lim_{n\to\infty} \epsilon_n = 0$, here $\{\epsilon_n\}$ is generated by*

$$\begin{cases} \epsilon_n = \left\| w_{n+1} - \Phi_1\left((1 - r_n)\,\eta_n + r_n\Phi_1\eta_n + r_n d_n\right) \right\|, \\ \eta_n = \Phi_2\left(\dfrac{\eta_n + \xi_n}{2}\right) + e_n, \\ \xi_n = \Phi_3\left((1 - s_n)\,\dfrac{\xi_n + w_n}{2} + s_n\Phi_3\left(\dfrac{\xi_n + w_n}{2}\right)\right) + h_n. \end{cases} \tag{2.1}$$

Scilicet, the JFESD (1.1) *is Φ-stable.*

Proof First, we prove convergence of the iterative scheme JFESD (1.1). Let $\varrho^* \in Fix(\Phi_1 \cap \Phi_2 \cap \Phi_3)$. Then by (1.1), one has

$$\begin{aligned}
\left\|\vartheta_n - \varrho^*\right\| &\leq \left\|\Phi_3\left((1-s_n)\frac{\vartheta_n+\varrho_n}{2} + s_n\Phi_3\left(\frac{\vartheta_n+\varrho_n}{2}\right)\right) - \Phi_3\left(\varrho^*\right)\right\| + \|h_n\| \\
&\leq \theta_3\left\|(1-s_n)\frac{\vartheta_n+\varrho_n}{2} + s_n\Phi_3\left(\frac{\vartheta_n+\varrho_n}{2}\right) - \varrho^*\right\| + \|h_n\| \\
&\leq \theta_3\left(1-s_n\right)\left\|\frac{\vartheta_n+\varrho_n}{2} - \varrho^*\right\| + \theta_3^2 s_n\left\|\frac{\vartheta_n+\varrho_n}{2} - \varrho^*\right\| + \|h_n\| \\
&\leq \frac{\iota_n}{2}\left(\left\|\vartheta_n - \varrho^*\right\| + \left\|\varrho_n - \varrho^*\right\|\right) + \|h_n\|,
\end{aligned}$$

where $\iota_n = \theta_3\left(1 - s_n + \theta_3 s_n\right)$, this indicates that

$$\left\|\vartheta_n - \varrho^*\right\| \leq \frac{\iota_n}{2-\iota_n}\left\|\varrho_n - \varrho^*\right\| + \frac{2}{2-\iota_n}\|h_n\|. \tag{2.2}$$

Furthermore, it follows from the second formulation of (1.1) that

$$\begin{aligned}
\left\|\varsigma_n - \varrho^*\right\| &\leq \theta_2\left\|\frac{\vartheta_n+\varsigma_n}{2} - \varrho^*\right\| + \|e_n\| \\
&\leq \frac{\theta_2}{2}\left(\left\|\vartheta_n - \varrho^*\right\| + \left\|\varsigma_n - \varrho^*\right\|\right) + \|e_n\|,
\end{aligned}$$

this implies with (2.2) and $\iota_n \leq \theta_3$ for $n \in \mathbb{N}$ that

$$\begin{aligned}
\left\|\varsigma_n - \varrho^*\right\| &\leq \frac{\theta_2}{2-\theta_2}\left\|\vartheta_n - \varrho^*\right\| + \frac{2}{2-\theta_2}\|e_n\| \\
&\leq \frac{\theta_2\iota_n}{\left(2-\theta_2\right)\left(2-\iota_n\right)}\left\|\varrho_n - \varrho^*\right\| \\
&\quad + \frac{2\theta_2}{\left(2-\theta_2\right)\left(2-\theta_3\right)}\|h_n\| + \frac{2}{2-\theta_2}\|e_n\|.
\end{aligned} \tag{2.3}$$

Taking $\tau_n = \theta_1\left(1 - r_n + \theta_1 r_n\right)$, then $\tau_n \leq \theta_1$ for $n \in \mathbb{N}$ and by (1.1) and (2.3), and $\{r_n\}, \{s_n\} \subseteq [0, 1]$, now we know that for each $n \in \mathbb{N}$:

$$\begin{aligned}
\left\|\varrho_{n+1} - \varrho^*\right\| &\leq \theta_1\left\|(1-r_n)\varsigma_n + r_n\Phi_1\varsigma_n + r_n d_n - \varrho^*\right\| \\
&\leq \theta_1\left(1-r_n\right)\left\|\varsigma_n - \varrho^*\right\| + {\theta_1}^2 r_n\left\|\varsigma_n - \varrho^*\right\| + \theta_1\|r_n d_n\| \\
&\leq \tau_n\left\|\varsigma_n - \varrho^*\right\| + \theta_1 r_n(\left\|d_n'\right\| + \left\|d_n''\right\|) \\
&\leq \left(1-t_n\right)\left\|\varrho_n - \varrho^*\right\| + \frac{2\theta_2\theta_1}{\left(2-\theta_2\right)\left(2-\theta_3\right)}\|h_n\| \\
&\quad + \frac{2\theta_1}{2-\theta_2}\|e_n\| + \theta_1 r_n(\left\|d_n'\right\| + \left\|d_n''\right\|),
\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
t_n &= \frac{(2-\theta_2)(2-\iota_n)-\theta_2\tau_n\iota_n}{(2-\theta_2)(2-\iota_n)} \\
&\geq \frac{1}{4}\left\{[1+(1-\theta_2)][1+(1-\theta_3)]-\theta_1\theta_2\theta_3\right\} \\
&> 0, \\
1-t_n &= \frac{\theta_2\tau_n\iota_n}{(2-\theta_2)(2-\iota_n)} \geq 0,
\end{aligned}$$

which is because $\tau_n \in [0,\theta_1]$, $\iota_n \in [0,\theta_3]$, $\sum_{i=1}^{3}\theta_i < 3$, $\theta_i \in [0,1]$ for $i = 1,2,3$, and there exists $i \in \{1,2,3\}$ such that $\theta_i < 1$, i.e., $\theta_1\theta_2\theta_3 < 1$. Therefore, one has a lower bound $t_* > 0$ of $\{t_n\} \subseteq [0,1]$, that is $t_* = \liminf_{n\to\infty} t_n \in (0,1]$. Furthermore, we can easily know that $\sum_{n=0}^{\infty} t_n = \infty$. Thus it follows from (2.4) that

$$\begin{aligned}
\left\|\varrho_{n+1}-\varrho^*\right\| &\leq (1-t_n)\left\|\varrho_n-\varrho^*\right\| + t_n\left[\frac{2\theta_1}{t_*(2-\theta_2)}\|e_n\| + \frac{\theta_1 r_n}{t_*}\left\|d_n'\right\|\right] \\
&\quad + \left[\frac{2\theta_1\theta_2}{(2-\theta_2)(2-\theta_3)}\|h_n\| + \theta_1 r_n\left\|d_n''\right\|\right].
\end{aligned} \tag{2.5}$$

By Lemma 2.1, the inequality (2.5) yields with the boundedness of $\{r_n\}$ and the condition **(H)** that $\|\varrho_n - \varrho^*\|$ as $n \to \infty$, i.e., the sequence $\{\varrho_n\}$ generated by the JFESD (1.1) strongly converges to ϱ^* with convergence rate in every step:

$$\rho_n = \theta_1\theta_2\theta_3 \cdot \frac{[1-r_n(1-\theta_1)][1-s_n(1-\theta_3)]}{(2-\theta_2)\{2-\theta_3[1-s_n(1-\theta_3)]\}}.$$

Next, we prove the iterative scheme JFESD (1.1) is Φ-stable. First, we show that if $\lim_{n\to\infty}\epsilon_n \to 0$, then $\lim_{n\to\infty} w_n \to 0$. In fact, it follows from (2.1) that

$$\begin{aligned}
\epsilon_n &= \|w_{n+1} - \Phi_1((1-r_n)\eta_n + r_n\Phi_1\eta_n + r_n d_n)\| \\
&\geq \left\|w_{n+1}-w^*\right\| - \left\|\Phi_1((1-r_n)\eta_n + r_n\Phi_1\eta_n + r_n d_n) - w^*\right\|,
\end{aligned}$$

which implies with (2.5) that

$$\begin{aligned}
\left\|w_{n+1}-w^*\right\| &\leq \epsilon_n + \left\|\Phi_1((1-r_n)\eta_n + r_n\Phi_1\eta_n + r_n d_n) - w^*\right\| \\
&\leq (1-t_n)\left\|w_n-w^*\right\| + t_n\left[\frac{1}{t_*}\epsilon_n + \frac{2\theta_1}{t_*(2-\theta_2)}\|e_n\| + \frac{\theta_1 r_n}{t_*}\left\|d_n'\right\|\right] \\
&\quad + \left[\frac{2\theta_1\theta_2}{(2-\theta_2)(2-\theta_3)}\|h_n\| + \theta_1 r_n\left\|d_n''\right\|\right].
\end{aligned}$$

Hence, if $\lim_{n\to\infty}\epsilon_n \to 0$, then from Lemma 2.1 and the given conditions, we get $\lim_{n\to\infty}\|w_n - w^*\| \to 0$ and $\lim_{n\to\infty} w_n = w^*$.

Whereas, if $\lim_{n\to\infty} w_n = w^*$, by (2.4) and the assumptions, one knows that

$$\begin{aligned}
\epsilon_n &= \|w_{n+1} - \Phi_1((1-r_n)\eta_n + r_n\Phi_1\eta_n + r_n d_n)\| \\
&\leq (1-t_n)\left\|w_n-w^*\right\| + \frac{2\theta_1}{2-\theta_2}\|e_n\| + \theta_1 r_n\left\|d_n'\right\| \\
&\quad + \frac{2\theta_2\theta_1}{(2-\theta_2)(2-\theta_3)}\|h_n\| + \theta_1 r_n\left\|d_n''\right\| \to 0,
\end{aligned}$$

this shows that $\lim_{n\to\infty}\epsilon_n \to 0$. The proof is completed. □

From Theorem 2.1, it is easy to obtain the following results to the special cases (1.2)–(1.4) of the JFESD (1.1).

Corollary 2.1 *Suppose that* $\mathbb{X}$, K *and operator* Φ_i $(i = 1, 2, 3)$ *are the same as in* Theorem 2.1. *Then the sequence* $\{\varrho_n\}$ *generated by the* JFSD (1.2) *converges to* $\varrho^* \in Fix(\Phi_1 \cap \Phi_2 \cap \Phi_3)$ *with convergence rate for* $n \in \mathbb{N}$*:*

$$\rho_n = \theta_1\theta_2\theta_3 \cdot \frac{[1 - r_n(1 - \theta_1)]\,[1 - s_n(1 - \theta_3)]}{(2 - \theta_2)\,\{2 - \theta_3\,[1 - s_n(1 - \theta_3)]\}},$$

and there exists a positive constant ρ *such that* $\rho_n < \rho \leq 1$ *for any* $n \in \mathbb{N}$. *Moreover, for any sequence* $\{w_n\} \subset \mathbb{X}$, $\lim\limits_{n\to\infty} w_n = w^*$ *if and only if* $\lim\limits_{n\to\infty} \epsilon_n = 0$, *here* ϵ_n *is defined by*

$$\begin{cases} \epsilon_n = \left\| w_{n+1} - \Phi_1\left((1 - r_n)\,\eta_n + r_n\Phi_1\eta_n\right)\right\|, \\ \eta_n = \Phi_2\left(\dfrac{\eta_n + \xi_n}{2}\right), \\ \xi_n = \Phi_3\left((1 - s_n)\,\dfrac{\xi_n + w_n}{2} + s_n\Phi_3\left(\dfrac{\xi_n + w_n}{2}\right)\right), \end{cases}$$

i.e., the iteration scheme JFSD (1.2) *is* Φ*-stable.*

Corollary 2.2 *Let* $\mathbb{X}$ *and* K *be the same as in* Theorem 2.1. *If* $\Phi_1 = \Phi_2 = \Phi_3 = \Phi : K \to K$ *is nonexpansive-type operator with Lipschitz coefficient* $\theta \in [0, 1)$ *and* $Fix(\Phi) \neq \emptyset$, *then the sequence* $\{\varrho_n\}$ *generated by the* JFES (1.3) *converges to* $\varrho^* \in Fix(\Phi)$ *with convergence rate:*

$$\rho_n = \frac{\theta^3\,[1 - r_n(1 - \theta)]\,[1 - s_n(1 - \theta)]}{(2 - \theta)\,\{2 - \theta\,[1 - s_n(1 - \theta)]\}} \tag{2.6}$$

for every $n \in \mathbb{N}$ *and there exists a constant* $\rho \in (0, 1]$ *such that* $\rho_n < \rho$ *for* $n \in \mathbb{N}$. *Furthermore, letting a sequence* $\{\epsilon_n\}$ *be decided by*

$$\begin{cases} \epsilon_n = \left\| w_{n+1} - \Phi\left((1 - r_n)\,\eta_n + r_n\Phi\eta_n + r_n d_n\right)\right\|, \\ \eta_n = \Phi\left(\dfrac{\eta_n + \xi_n}{2}\right) + e_n, \\ \xi_n = \Phi\left((1 - s_n)\,\dfrac{\xi_n + w_n}{2} + s_n\Phi\left(\dfrac{\xi_n + w_n}{2}\right)\right) + h_n, \end{cases}$$

then for any sequence $\{w_n\} \subset \mathbb{X}$, $\lim_{n\to\infty} w_n = w^*$ *if and only if* $\lim\limits_{n\to\infty} \epsilon_n = 0$ *and the iteration scheme* JFES (1.3) *is called* Φ*-stable.*

Corollary 2.3 *Assume that* $\mathbb{X}$, K *and* Φ *are the same as in* Corollary 2.2. *then the sequence* $\{x_n\}$ *determined by the* JFS (1.4) *converges to* $\varrho^* \in Fix(\Phi)$ *with convergence rate* ρ_n *of* (2.6) *in every step. Furthermore, for any sequence* $\{w_n\} \subset \mathbb{X}$, $\lim_{n\to\infty} w_n = w^*$ *if and only if* $\lim_{n\to\infty} \epsilon_n = 0$, *where* ϵ_n *is confirmed as follows:*

$$\begin{cases} \epsilon_n = \left\| w_{n+1} - \Phi\left((1 - r_n)\,\eta_n + r_n\Phi\eta_n\right)\right\|, \\ \eta_n = \Phi\left(\dfrac{\eta_n + \xi_n}{2}\right), \\ \xi_n = \Phi\left((1 - s_n)\,\dfrac{\xi_n + w_n}{2} + s_n\Phi\left(\dfrac{\xi_n + w_n}{2}\right)\right), \end{cases}$$

that is, the iteration scheme JFS (1.4) *is* Φ*-stable.*

To verify our results, we give a numerical example as follows.

Example 2.1 We consider a known integral equation, which often arises in many physical problems, defined as

$$f(\varrho) = a + \int_0^{\varrho} k(\varrho, \varsigma) f(\varsigma) d\varsigma, \quad \forall \varrho \in [0, l], \tag{2.7}$$

where l and $a = f(0)$ are real constants, and $k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

From Eq. (2.7), one can easily decide an operator $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(f) := \Phi(f(\varrho)) = a + \int_0^{\varrho} k(\varrho, \varsigma) f(\varsigma) d\varsigma, \quad \forall \varrho \in [0, l], f \in \mathbb{R}, \tag{2.8}$$

and knows that if $\sup_{\varrho \in [0,l]} \int_0^{\varrho} \|k(\varrho, \varsigma)\| \, d\varsigma < 1$, then Φ is a nonexpansive-type operator. In fact, for any $f, g \in \mathbb{R}$, we have

$$\begin{aligned} \|\Phi(f) - \Phi(g)\| &= \sup_{\varrho \in [0,l]} \left\| \int_0^{\varrho} k(\varrho, \varsigma) \cdot [f(\varsigma) - g(\varsigma)] \, d\varsigma \right\| \\ &\leq \left(\sup_{\varrho \in [0,l]} \int_0^{\varrho} \|k(\varrho, \varsigma)\| \, d\varsigma \right) \|f - g\|. \end{aligned}$$

By Banach contraction mapping principle, the operator Φ has a fixed point, which is the solution of Eq. (2.7).

With regard to (2.7) and (2.8), taking $a = 1$, $l = \frac{9}{10}$ and $k(\cdot, \cdot) \equiv 1$, then by simple calculation, one can easily see that $\sup_{\varrho \in [0,l]} \int_0^{\varrho} \|k(\varrho, \varsigma)\| \, d\varsigma = \frac{9}{10} < 1$ and an exact solution of the following example of (2.7):

$$f(\varrho) = 1 + \int_0^{\varrho} f(\varsigma) d(\varsigma) \tag{2.9}$$

is $f(\varrho) = e^{\varrho}$ for every $\varrho \in \left[0, \frac{9}{10}\right]$, which is a fixed point of special operator Φ to (2.9). In subsequent work, the exact solution of the particular case (2.9) shall be numerically approximated via using our new iterative schemes JFESD (1.1) and JFSD (1.2). Assume $\Phi_1 = \Phi$, $\Phi_2 = \Phi_3 = I$, the identity operator, $r_n = 0.3$, $s_n = 0.5$, $d'_n = \frac{n^2}{10^n}$, $d''_n = \frac{n}{10^n}$, $e_n = \frac{1}{5^n}$, $h_n = \frac{10}{8^n}$ for each $n \in \mathbb{N}$, and the initial function $f(\varrho) = \varrho$ is given. Then the numerical solutions after some steps are displayed in Figs. 1 and 2, and mean square errors (in short, MSE) of JFESD (1.1) and JFSD (1.2) have been computed in Fig. 3. It is light to see that the iteration schemes JFESD (1.1) and JFSD (1.2) can fastly converge to the exact solution of (2.9), respectively, and although the convergence speed to JFESD (1.1) and JFSD (1.2) are not the same, the numbers of iteration (in short, Iter.) for converging to the precise solution are no more than twenty . These indicate the validity of Theorem 2.1 and Corollary 2.1.

3 Simulation and applications

In this section, using the new iterative schemes presented in this paper and based on Matlab 2020b and R 4.1, we will propose two numerical simulation examples for approximating fixed points of nonexpansive-type operators and to check on the efficiency of our methods. Furthermore, we apply the new scheme JFESD (1.1) to solve the variational inequality (1.7).

Fig. 1 Approximating solutions of (2.9) by the JFESD (1.1)

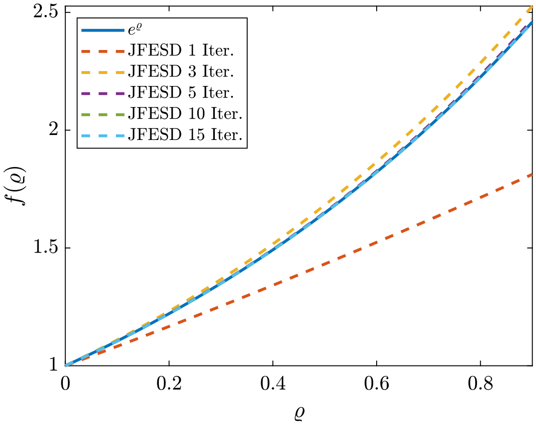
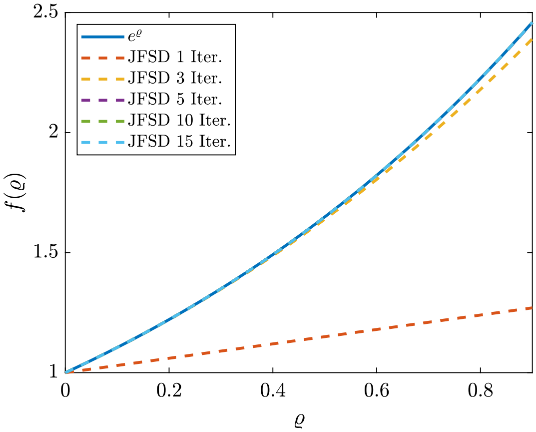


Fig. 2 Approximating solutions of (2.9) by the JFSD (1.2)



Example 3.1 Let $K = [0, +\infty]$, $\Phi(\varrho) = \sqrt{\varrho^2 - 6\varrho + 30}$, and for any $n \in \mathbb{N}$, $r_n = s_n = \frac{1}{2}$, $d'_n = \frac{n^2}{10^n}$, $d''_n = \frac{n}{10^n}$, $e_n = \frac{1}{5^n}$ and $h_n = \frac{10}{8^n}$.

In Li and Lan (2019), Li and Lan proved the operator Φ is a nonexpansive-type operator and showed that one fixed point of Φ is 5. Now we will use our new iteration schemes to approximate the fixed points of Φ. To make clear that our schemes are better than others, we generate 1000 random initial points between 0 and 20, and set the stop condition as $\|\varrho_{n+1} - \varrho_n\| \leq \varepsilon$, here $\varepsilon = 10^{-5}, 10^{-10}$. The numerical simulation results for two special cases of the JFESD (1.1): JFES (1.3) and JFS (1.4), and some others known schemes JF, PMMI, PMI, MANN, ISHIKAWA, NOOR and SAKURAI are appeared in Figs. 4 and 5.

In Fig. 4, horizontal axis means different iterative schemes, vertical axis is the Iter. of iterative schemes with different stopped conditions. From Fig. 4, one can easily see that the JFS (1.4) converges faster than other schemes under given stopped conditions, and they are nearly not affected by initial points. In Fig. 5, horizontal axis means different iterative schemes, vertical axis means the final approximated value, it is clear that the JFES (1.2) stably converges to a fixed point for the given nonexpansive-type operator.

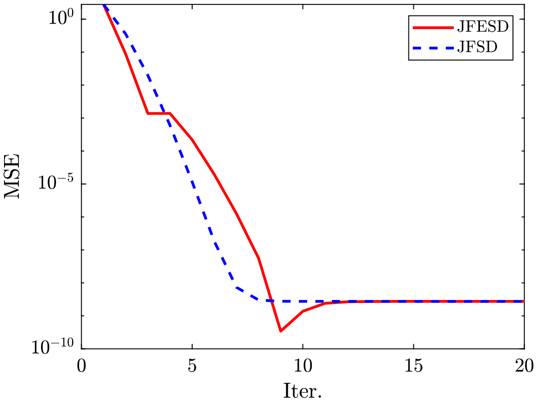


Fig. 3 MSE of the JFESD (1.1) and the JFSD (1.2) for (2.9)

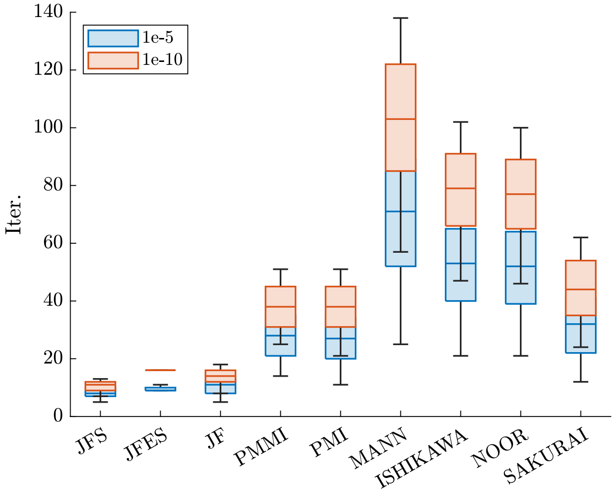


Fig. 4 Comparison of Iter. with random initial values and different tolerances for Example 3.1

Example 3.2 Take $K = [0, +\infty]$, $r_n = s_n = \frac{1}{2}$, $d'_n = \frac{n^2}{10^n}$, $d''_n = \frac{n}{10^n}$, $e_n = \frac{1}{5^n}$ and $h_n = \frac{10}{8^n}$ for all $n \in \mathbb{N}$, and define $\Phi_i = \Phi$ for $i = 1, 2, 3$ as

$$\Phi(\varrho) = \frac{1}{2}\left(\sin\varrho + \cos\varrho + \frac{1}{2}\varrho\right) + \frac{3\pi + 2}{4}, \tag{3.1}$$

where $\varrho \in [0, \infty)$.

From (3.1), we know that one fixed point of Φ is π and

$$\begin{aligned}
|\Phi(\varrho) - \Phi(\varsigma)| &\leq \frac{1}{4}|\varrho - \varsigma| + \frac{1}{2}|[\sin(\varrho) - \sin(\varsigma)] + [\cos(\varrho) - \cos(\varsigma)]| \\
&= \frac{1}{4}|\varrho - \varsigma| + \left|\cos\left(\frac{\varrho+\varsigma}{2}\right) - \sin\left(\frac{\varrho+\varsigma}{2}\right)\right| \cdot \left|\sin\left(\frac{\varrho-\varsigma}{2}\right)\right| \\
&\leq \frac{3}{4}|\varrho - \varsigma|,
\end{aligned}$$

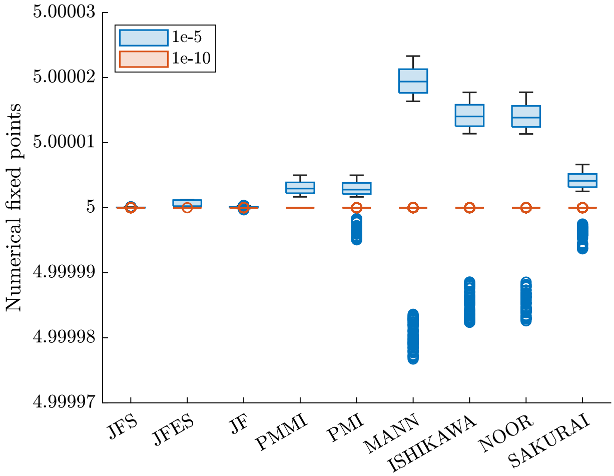


Fig. 5 Final approximations with different initial values and tolerances for Example 3.1

which shows that Φ is a nonexpansive-type operator. It follows from Theorem 2.1 that the JFESD (1.1) and its some special cases can approximate the fixed point of Φ. Indeed, taking $\varrho_0 = 18.5$, absolute errors of the iterative schemes JFES, JFS, JF, PMMI, MANN, ISHIKAWA, NOOR and SAKURAI for Example 3.2 are listed in Table 1, and one can readily see that the absolute error of our new iteration scheme JFS (1.4) becomes to 0 at the soonest. This implies that the JFS (1.4) is the best one of the compared eight schemes.

We note that from the fixed point problem (1.8), if the second ς_n in the first equation of the JFESD (1.1) is substituted by $P_K(\varsigma_n - \kappa F(\varsigma_n))$, where the projection operator P_K is the same as in (1.5), F is a nonlinear operator on K and κ is a positive constant, then the JFESD (1.1) is equivalent to the following iteration:

$$\begin{cases} \varrho_{n+1} = \Phi_1\left((1-r_n)\,\varsigma_n + r_n\Phi_1(P_K(\varsigma_n - \kappa F(\varsigma_n))) + r_n d_n\right), \\ \varsigma_n = \Phi_2\left(\dfrac{\vartheta_n + \varsigma_n}{2}\right) + e_n, \\ \vartheta_n = \Phi_3\left((1-s_n)\,\dfrac{\vartheta_n + \varrho_n}{2} + s_n\Phi_3\left(\dfrac{\vartheta_n + \varrho_n}{2}\right)\right) + h_n, \end{cases} \tag{3.2}$$

where $\{r_n\}$, $\{s_n\}$, $\{d_n\}$, $\{e_n\}$ and $\{h_n\}$ are the same sequences as in (1.1). Thus, the iterative process (3.2), as an application of the new iteration scheme JFESD (1.1), can be deemed to find common fixed points of operator Φ_i for $i = 1, 2, 3$ and solutions of Stampacchia variation inequality (1.7). This shall be proposed at the end of this section.

Theorem 3.1 *Let $\mathbb{X}$ be a Hilbert space endowed with the norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, and $K \subset \mathbb{X}$ be a nonempty closed convex bounded set. Suppose that the following conditions hold:*

(C_1) For $i = 1, 2, 3$, $\Phi_i : K \to K$ is nonexpansive-type operator with Lipschitz coefficient $\theta_i \in [0, 1]$ and $Fix(\Phi_1 \cap \Phi_2 \cap \Phi_3) \cap VI(K, F) \neq \emptyset$.
(C_2) $r_n \in (0, 1]$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} r_n \neq 0$ or $\lim_{n\to\infty} r_n$ does not exist .
(C_3) $F : K \to K$ is a μ-Lipschitzian continuous and σ-strongly monotone operator, i.e., if there exists a constant $\sigma > 0$ such that $\langle F(x) - F(y), x - y\rangle \geq \sigma\|x - y\|^2$ for all $x, y \in K$.

Table 1 Absolute errors of some iterative schemes for Example 3.2

Iter.	JFES	JFS	JF	PMMI	MANN	ISHIKAWA	NOOR	SAKURAI
0	1.536E+01	1.536E+01	1.536E+01	1.536E+01	1.536E+01	1.536E+01	1.536E+01	1.536E+01
1	5.326E−02	3.048E−03	3.287E−01	4.420E+00	9.998E+00	9.525E+00	9.075E+00	6.082E+00
2	9.065E−03	1.422E−06	7.125E−04	2.337E+00	6.845E+00	5.529E+00	5.507E+00	6.970E+00
3	1.956E−03	6.630E−10	1.566E−06	7.123E−01	4.183E+00	3.820E+00	3.840E+00	1.411E+00
4	3.577E−04	3.100E−13	3.440E−09	4.975E−02	3.206E+00	2.743E+00	2.719E+00	9.087E−01
5	6.095E−05	0	7.560E−12	9.311E−03	2.520E+00	1.793E+00	1.715E+00	1.182E−01
6	1.018E−05	0	1.021E−14	1.605E−03	1.882E+00	9.682E−01	8.939E−01	4.788E−02
7	1.714E−06	0	0	2.864E−04	1.265E+00	4.576E−01	4.193E−01	9.145E−03
8	2.966E−07	0	0	5.028E−05	7.271E−01	2.090E−01	1.912E−01	1.374E−03
9	5.304E−08	0	0	8.882E−06	3.515E−01	9.489E−02	8.659E−02	2.142E−04
10	9.791E−09	0	0	1.571E−06	1.489E−01	4.302E−02	3.913E−02	3.764E−05
15	2.750E−12	0	0	2.664E−10	1.198E−03	8.222E−04	7.358E−04	1.635E−08
20	0	0	0	3.997E−14	8.892E−06	1.571E−05	1.389E−05	1.127E−11
25	0	0	0	0	6.594E−08	3.000E−07	2.628E−07	1.021E−14
30	0	0	0	0	4.890E−10	5.731E−09	4.981E−09	0
35	0	0	0	0	3.630E−12	1.095E−10	9.454E−11	0
40	0	0	0	0	3.020E−14	2.090E−12	1.800E−12	0
45	0	0	0	0	0	3.997E−14	3.997E−14	0
50	0	0	0	0	0	0	0	0

(C_4) *The constant κ, which is the same constant as in (1.8), satisfies*

$$\kappa \in \begin{cases} (0, \infty), & if \ \ \theta_1 = 0. \\ \left(0, \dfrac{\sigma + \sqrt{\sigma^2 + \mu^2\theta_1^{-1}(1-\theta_1)}}{\mu^2}\right), & if \ \ \theta_1 \neq 0, \end{cases} \tag{3.3}$$

Then convergence and stability of the iteration (3.2) can be analyzed as follows:

(i) *The iterative sequence $\{\varrho_n\}$ defined by (3.2) converges to $Fix(\Phi_1 \cap \Phi_2 \cap \Phi_3) \cap VI(K, F)$ with convergence rate:*

$$\hat{\rho}_n = \theta_1\theta_2\theta_3 \cdot \frac{[1 - r_n(1 - \upsilon\theta_1)][1 - s_n(1 - \theta_3)]}{(2 - \theta_2)\{2 - \theta_3[1 - s_n(1 - \theta_3)]\}}$$

for any $n \in \mathbb{N}$, where $\upsilon = \sqrt{1 - 2\kappa\sigma + \kappa^2\mu^2}$, and there exists a constant $\hat{\rho} \in (0, 1]$ such that $\hat{\rho}_n < \hat{\rho}$.

(ii) *For any sequence $\{w_n\} \subset \mathbb{X}$, $\lim_{n\to\infty} w_n = w^*$ if and only if $\lim_{n\to\infty} \epsilon_n = 0$, here ϵ_n decided by*

$$\begin{cases} \epsilon_n = \|w_{n+1} - \Phi_1\left((1 - r_n)\,\eta_n + r_n\Phi_1\left(P_K\left(\eta_n - \kappa F(\eta_n)\right)\right) + r_n d_n\right)\|, \\ \eta_n = \Phi_2\left(\dfrac{\eta_n + \xi_n}{2}\right) + e_n, \\ \xi_n = \Phi_3\left((1 - s_n)\,\dfrac{\xi_n + w_n}{2} + s_n\Phi_3\left(\dfrac{\xi_n + w_n}{2}\right)\right) + h_n, \end{cases} \tag{3.4}$$

namely, the iteration scheme (3.2) is Φ-stable.

Proof By the condition (C_3), one gets

$$\mu\|x - y\|^2 \geq \|F(x) - F(y)\| \cdot \|x - y\| \geq \langle F(x) - F(y), x - y\rangle \geq \sigma\|x - y\|^2$$

for any $x, y \in K$, i.e., $\mu \geq \sigma$ and knows that $1 - 2\kappa\sigma + \kappa^2\mu^2 \geq 0$. Setting $\varrho^* \in Fix(\Phi_1 \cap \Phi_2 \cap \Phi_3) \cap VI(K, F)$, that is, $\varrho^* \in K$ is a common fixed point of operator Φ_i for $i = 1, 2, 3$ and a solution of Stampacchia variational inequality (1.7), then it follows that

$$\begin{aligned} &\left\|\varsigma_n - \varrho^* - \kappa\left[F(\varsigma_n) - F(\varrho^*)\right]\right\|^2 \\ &\quad = \left\|\varsigma_n - \varrho^*\right\|^2 - 2\left\langle\varsigma_n - \varrho^*, \kappa\left[F(\varsigma_n) - F(\varrho^*)\right]\right\rangle + \left\|\kappa\left[F(\varsigma_n) - F(\varrho^*)\right]\right\|^2 \\ &\quad \leq \left(1 - 2\kappa\sigma + \kappa^2\mu^2\right)\left\|\varsigma_n - \varrho^*\right\|^2, \end{aligned}$$

and so

$$\left\|\varsigma_n - \varrho^* - \kappa\left[F(\varsigma_n) - F(\varrho^*)\right]\right\| \leq \upsilon\left\|\varsigma_n - \varrho^*\right\|, \tag{3.5}$$

here $\upsilon = \sqrt{1 - 2\kappa\sigma + \kappa^2\mu^2}$. By (3.2), Example 1.1 and (3.5), now we know that

$$\begin{aligned} \left\|\varrho_{n+1} - \varrho^*\right\| &= \left\|\Phi_1\left((1 - r_n)\,\varsigma_n + r_n\Phi_1\left(P_K(\varsigma_n - \kappa F(\varsigma_n))\right) + r_n d_n\right) - \varrho^*\right\| \\ &\leq \theta_1\Big[(1 - r_n)\left\|\varsigma_n - \varrho^*\right\| + \|r_n d_n\| \\ &\quad + r_n\left\|\Phi_1\left(P_K(\varsigma_n - \kappa F(\varsigma_n))\right) - \Phi_1\left(P_K(\varrho^* - \kappa F(\varrho^*))\right)\right\|\Big] \\ &\leq \theta_1(1 - r_n)\left\|\varsigma_n - \varrho^*\right\| + \theta_1 r_n(\left\|d_n'\right\| + \left\|d_n''\right\|) \\ &\quad + r_n\theta_1^2\left\|\varsigma_n - \varrho^* - \kappa\left(F(\varsigma_n) - F(\varrho^*)\right)\right\| \\ &\leq \hat{\tau}_n\left\|\varsigma_n - \varrho^*\right\| + \theta_1 r_n(\left\|d_n'\right\| + \left\|d_n''\right\|), \end{aligned} \tag{3.6}$$

where $\hat{\tau}_n = \theta_1 [(1 - r_n) + \upsilon\theta_1 r_n]$, as the condition (C_4) intends that $\hat{\tau}_n = 0 < 1$ is always right if $\theta_1 = 0$. Otherwise, it is easy to see that $\upsilon < \theta_1^{-1}$ when $\theta_1 \neq 0$ and $\hat{\tau}_n < \theta_1$ by the condition (C_2). Thus, it follows from the condition (C_1) that there exists a constant $\hat{t}_* = \liminf_{n\to\infty} \hat{t}_n \in (0, 1]$, where $\hat{t}_n = \frac{(4-2\theta_2-2\iota_n)+\theta_2\iota_n(1-\hat{\tau}_n)}{(2-\theta_2)(2-\iota_n)}$. Hence, for each $n \in \mathbb{N}$, replacing τ_n, t_* and $1 - t_n$ in (2.4) by $\hat{\tau}_n$, $\hat{t}_*$ and $1 - \hat{t}_n = \frac{\theta_2\hat{\tau}_n\iota_n}{(2-\theta_2)(2-\iota_n)}$ in several, where ι_n is the same as in (2.2), then similar to the proof of Theorem 2.1, it follows from (2.3), (3.6), $\hat{\tau}_n < \theta_1$ and Lemma 2.1 that $\|\varrho_n - \varrho^*\|$ as $n \to \infty$, i.e., the sequence $\{\varrho_n\}$ strongly converges to ϱ^* with the following convergence rate in every step:

$$\hat{\rho}_n = \theta_1\theta_2\theta_3 \cdot \frac{[1 - r_n(1 - \upsilon\theta_1)][1 - s_n(1 - \theta_3)]}{(2 - \theta_2)\{2 - \theta_3[1 - s_n(1 - \theta_3)]\}}.$$

The rest proof can be immediately obtained from the proof of Theorem 2.1 and so it is omitted. □

Remark 3.1 (i) One can easily see that the difference between the JFESD (1.1) and the iteration (3.2) is only the second ς_n in the first equation of (1.1), which depends on the equivalence of Stampacchia variational inequality (1.7) and the fixed point problem (1.8). That is to say, solving the inequality (1.7) is equivalent to finding common fixed points of nonexpansive-type operators Φ_1, Φ_2, Φ_3 and P_K.

(ii) As projection operator is a nonexpansive-type operator, and the conditions (C_1) and (C_2) in Theorem 3.1 separately mean $\theta_i = 1$ for $i = 1, 2, 3$ and the lower bound of $\hat{t}_n$ exists, the operator Φ_i with Lipschitz coefficient $\theta_i \in [0, 1]$ in (3.2) can be replaced by projection operator for each $i = 1, 2, 3$, and so one can get a new iteration scheme with errors for the variational inequality (1.7) as follows:

$$\begin{cases} \varrho_{n+1} = P_K\left((1 - r_n)\,\varsigma_n + r_n P_K(P_K(\varsigma_n - \kappa F\varsigma_n)) + r_n d_n\right), \\ \varsigma_n = P_K\left(\dfrac{\vartheta_n + \varsigma_n}{2}\right) + e_n, \\ \vartheta_n = P_K\left((1 - s_n)\,\dfrac{\vartheta_n + \varrho_n}{2} + s_n P_K\left(\dfrac{\vartheta_n + \varrho_n}{2}\right)\right) + h_n, \end{cases} \tag{3.7}$$

and one can achieve convergence and stability of the scheme (3.7) under some suitable conditions.

(iii) Furthermore, some variational inequalities of the form (1.7), for example, Noor (general) variational inequality introduced and studied by Noor (2007), can be solved via the new approximation method of the form JFESD (1.1) or its special cases (3.2) and (3.7). However, it follows from Remark 1.2, Figs. 4 and 5 in Example 3.1 and Table 1 of Example 3.2 that the convergence and stable of our approximation methods presented in this paper are better than those of the Noor (2007).

4 Conclusions

In this paper, we introduced a class of general semi-implicit approximations with errors and proved that the general iterative approximations converge to common fixed points of three different nonexpansive-type operators in Hilbert spaces. In addition, we studied stability of the iterative approximations, obtained convergence and validated our iterative schemes based on some numerical examples, which show that the new iterative methods presented in this paper have better convergence rate and stability. Finally, to solve Stampacchia variational inequality (1.7), we applied the new iterative methods to approximate common fixed points of nonexpansive-type operators and the projection operator associated with the inequality (1.7).

However, the following two **open questions** are worthy of future research:

(1) Nonexpansive-type operators include various forms, does the general iteration scheme JFESD (1.1) also converge to fixed points of generalized nonexpansive operators (Thakur et al. 2016; Ali et al. 2020), asymptotically quasi-nonexpansive type operators (Chang et al. 2003), totally quasi-D-asymptotically nonexpansive operator (Ni and Yao 2015), and other nonexpansive operators?

(2) Many iteration schemes are composed of Picard and Mann iteration schemes, so whether can all iteration schemes based on Picard and Mann iterations converge to fixed points of nonexpansive-type operators? And how about are convergence and stability when errors and semi-implicit rule are considered?

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Code availability We will provide under request.

Declarations

Conflict of interest All authors declare no conflicts of interest in this paper.

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References

Aibinu MO, Pillay P, Olaleru JO, Mewomo OT (2018) The implicit midpoint rule of nonexpansive mappings and applications in uniformly smooth Banach spaces. J Nonlinear Sci Appl 11(12):1374–1391

Alber YI (2017) On the stability of iterative approximations to fixed points of nonexpansive mappings. J Math Anal Appl 328(2):958–971

Alghamdi MA, Alghamdi MA, Shahzad N, Xu HK (2014) The implicit midpoint rule for nonexpansive mappings. Fixed Point Theory Appl 96:9

Ali F, Ali J, Nieto JJ (2020) Some observations on generalized non-expansive mappings with an application. Comput Appl Math 39(2):20

Ali F, Ali J, Rodríguez-López R (2021) Approximation of fixed points and the solution of a nonlinear integral equation. Nonlinear Funct Anal Appl 26(5):869–885

Ali W, Turab A, Nieto JJ (2022) On the novel existence results of solutions for a class of fractional boundary value problems on the cyclohexane graph. J Inequal Appl 5:19

Cacciapaglia G, Sannino F (2021) Evidence for complex fixed points in pandemic data. Front Appl Math Stat 7:659580

Chang SS, Cho YJ, Zhou YY (2003) Iterative sequences with mixed errors for asymptotically quasi-nonexpansive type mappings in Banach spaces. Acta Math Hungar 100(1–2):147–155

Daniele P, Giannessi F, Maugeri A (2003) Equilibrium Problems and Variational Models. Kluwer, Boston

Deuflhard P (1985) Recent progress in extrapolation methods for ordinary differential equations. SIAM Rev 27(4):505–535

Hanjing A, Suantai S (2020) A fast image restoration algorithm based on a fixed point and optimization method. Math MDPI 8(3):378

Harker PT, Pang JS (1990) Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications. Math Prog 48(2):161–220

Ishikawa S (1974) Fixed points by a new iteration method. Proc Am Math Soc 44(1):147–150

Lemaire B (1996) Stability of the iteration method for non expansive mappings. Serdica Math J 22(3):331–340

Li TF, Lan HY (2019) On new Picard-Mann iterative approximations with mixed errors for implicit midpoint rule and applications. J Funct Spaces 2019:13 (**Art. ID 4042965**)

Liu LS (1995) Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces. J Math Anal Appl 194(1):114–125

Luo P, Cai G (2017) The viscosity iterative algorithms for the implicit midpoint rule of nonexpansive mappings in uniformly smooth Banach spaces. J Inequal Appl 154:12

Maldar S (2021) Iterative algorithms of generalized nonexpansive mappings and monotone operators with application to convex minimization problem. J Appl Math Comput (Published online: 15 July 2021) https://doi.org/10.1007/s12190-021-01593-y

Mann WR (1953) Mean value methods in iteration. Proc Am Math Soc 4(3):506–510

Ni RX, Yao JC (2015) The modified Ishikawa iterative algorithm with errors for a countable family of Bregman totally quasi-D-asymptotically nonexpansive mappings in reflexive Banach spaces. Fixed Point Theory Appl 35:24

Noor MA (2007) General variational inequalities and nonexpansive mappings. J Math Anal Appl 331(2):810–822

Noor MA, Yao YH (2007) Three-step iterations for variational inequalities and nonexpansive mappings. Appl Math Comput 190(2):1312–1321

Panda SK, Abdeljawad T, Ravichandran C (2020) Novel fixed point approach to Atangana-Baleanu fractional and L^p-Fredholm integral equations. Alexandria Eng J 59(4):1959–1970

Panda SK, Atangana A, Nieto JJ (2021) New insights on novel coronavirus 2019-nCoV/SARS-CoV-2 modelling in the aspect of fractional derivatives and fixed points. Math Biosci Eng 18(6):8683–8726

Phannipa W, Atid K (2021) An approximation method for solving fixed points of general system of variational inequalities with convergence theorem and application. IAENG Int J Appl Math 51(3):751–756

Sakurai K, Iiduka H (2014) Acceleration of the Halpern algorithm to search for a fixed point of a nonexpansive mapping. Fixed Point Theory Appl 202:11

Schneider C (1993) Analysis of the linearly implicit mid-point rule for differential-algebra equations. Electron Trans Numer Anal 1:1–10

Stampacchia G (1964) Formes bilinéaires coercitives sur les ensembles convexes (French). C R Acad Sci Paris 258:4413–4416

Thakur BS, Thakur D, Postolache M (2016) A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings. Appl Math Comput 275:147–155