

Generalized Persistence Diagrams *

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Abstract

We generalize the persistence diagram of Cohen-Steiner, Edelsbrunner, and Harer to the setting of constructible persistence modules valued in a symmetric monoidal category. We call this the *type \mathcal{A}* persistence diagram of a persistence module. If the category is also abelian, then we define a second *type \mathcal{B}* persistence diagram. In addition, we show that both diagrams are stable to all sufficiently small perturbations of the module.

1 Introduction

Let $f : \mathbb{M} \rightarrow \mathbb{R}$ be a Morse function on a compact manifold \mathbb{M} . The function f filters \mathbb{M} by sublevel sets $\mathbb{M}_{f \leq r} = \{x \in \mathbb{M} \mid f(x) \leq r\}$. Apply homology with coefficients in a field and we call the resulting object F a *constructible persistence module of vector spaces*. The *persistence diagram* and the *barcode* are two invariants of a persistence module obtained as follows.

- By Images: Edelsbrunner, Letscher, and Zomorodian [ELZ02] define the *persistent homology group* F_s^t , for $s < t$, as the image of $F(s < t)$. Cohen-Steiner, Edelsbrunner, and Harer [CSEH07] define the *persistence diagram* of F as a finite set of points in the plane above the diagonal satisfying the following property. For each $s < t$, the number of points in the upper-left quadrant defined by (s, t) is the rank of F_s^t .
- By Indecomposables: The module F is isomorphic to a finite direct sum of indecomposable persistence modules $F \cong F_1 \oplus \cdots \oplus F_n$. Any two ways of writing F as a sum of indecomposables are the same up to a reordering of the indecomposables. Furthermore, each indecomposable F_i is an *interval persistence module*. That is, there are a

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pair of values $r < t$, where t may be infinite, such that $F_i(s)$ is a copy of the field for all values $r \leq s < t$ and zero elsewhere.¹ Zomorodian and Carlsson define the *barcode* of F as its list of indecomposables [ZC05]. See also Carlsson and de Silva [CdS10].

A barcode translates to a persistence diagram by plotting the left endpoint versus the right endpoint of each interval persistence module. A persistence diagram translates to a barcode by turning each point (s, t) into an interval persistence module starting at s and ending at t . In this way, the persistence diagram is equivalent to a barcode. However, the two definitions are very different in philosophy.

Suppose the homology of each sublevel set $\mathbb{M}_{f \leq r}$ is calculated using integer coefficients. Then the resulting object F is a *constructible persistence module of finitely generated abelian groups*. However, an indecomposable persistence module of finitely generated abelian groups need not look anything like an interval persistence module. For example, the module in Figure 4 is indecomposable. Indecomposables are hard to interpret especially under perturbations to the module.

We generalize the persistence diagram of Cohen-Steiner, Edelsbrunner, and Harer to the setting of constructible persistence modules F valued in a symmetric monoidal category \mathcal{C} with images. The category of sets, the category of vector spaces, and the category of finitely generated abelian groups are examples of such categories. We call this diagram the *type \mathcal{A} persistence diagram* of F . If \mathcal{C} is also abelian, then we define a second *type \mathcal{B} persistence diagram* of F . The category of vector spaces and the category of abelian groups are examples of abelian categories. The type \mathcal{B} persistence diagram of F may contain less information than the type \mathcal{A} persistence diagram of F . However, the advantage of a type \mathcal{B} diagram is a stronger statement of stability. Depending on \mathcal{C} , our persistence diagrams may not be a complete invariant of a persistence module.

Persistence is motivated by data analysis and data is noisy. A small perturbation to a persistence module should not result in a drastic change to its persistence diagram. We use the standard *interleaving distance* to measure differences between persistence modules [CCSG⁺09]. We define a new metric we call *erosion distance* to measure differences between persistence diagrams. In Theorem 8.2, we show that if the interleaving distance between two constructible persistence modules valued in an abelian category \mathcal{C} is ϵ , then the erosion distance between their type \mathcal{B} persistence diagrams is at most ϵ . We call this *continuity* of type \mathcal{B} persistence diagrams. If \mathcal{C} is simply a symmetric monoidal category, then Theorem 8.1 is a weaker one-way statement of continuity for type \mathcal{A} persistence diagrams. We call this *semicontinuity* of type \mathcal{A} persistence diagrams. These theorems show that the information contained in both diagrams is stable to all sufficiently small perturbations of the module.

Cohen-Steiner, Edelsbrunner, and Harer define a stronger metric on the set of persistence diagrams they call *bottleneck distance*. They show that for two Morse functions $f, g : \mathbb{M} \rightarrow \mathbb{R}$, the bottleneck distance between their persistence diagrams is at most $\max |f - g|$. They do this by looking at the 1-parameter family of persistence modules obtained from the linear interpolation $h : \mathbb{M} \times [0, 1] \rightarrow \mathbb{R}$ taking $h_0 = f$ to $h_1 = g$. Using the Box Lemma, which is a local statement of stability, they track each point in the persistence diagram of h_0 all the

¹The interval persistence module F_i is fully described by the half open interval $[s, t)$.

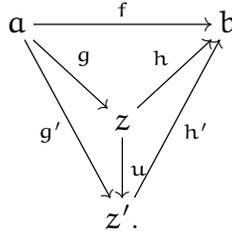
way to the persistence diagram of h_1 . Theorem 8.2 resembles the Box Lemma and assuming \mathcal{C} has colimits, there is a way to construct a 1-parameter 1-Lipschitz family of persistence modules between any two interleaved persistence modules [BdSN17]. This suggests that bottleneck stability might extend to type \mathcal{B} persistence diagrams. We leave the issue of bottleneck stability for future investigations.

2 Persistence Modules

Let (\mathcal{C}, \square) be an essentially small symmetric monoidal category with images. By essentially small, we mean that the collection of isomorphism classes of objects in \mathcal{C} is a set. A symmetric monoidal category is, roughly speaking, a category \mathcal{C} with a binary operation \square on its objects and an identity object $e \in \mathcal{C}$ satisfying the following properties:

- (Symmetry) $a \square b \cong b \square a$, for all objects $a, b \in \mathcal{C}$
- (Associativity) $a \square (b \square c) \cong (a \square b) \square c$, for all objects $a, b, c \in \mathcal{C}$
- (Identity) $a \square e \cong a$, for all objects $a \in \mathcal{C}$.

See [Wei13, page 114] for a precise definition of a symmetric monoidal category. By images, we mean that for every morphism $f : a \rightarrow b$, there is a monomorphism $h : z \rightarrow b$ and a morphism $g : a \rightarrow z$ such that $f = h \circ g$. Furthermore, for a monomorphism $h' : z' \rightarrow b$ and a morphism $g' : a \rightarrow z'$ such that $f = h' \circ g'$, there is a unique morphism $u : z \rightarrow z'$ such that the following diagram commutes:



See [Mit65, page 12] for a discussion of images.

Definition 2.1: A **persistence module** is a functor $F : (\mathbb{R}, \leq) \rightarrow \mathcal{C}$ out of the poset of real numbers.

Let $S = \{s_1 < \dots < s_n\}$ be a finite set of real numbers. Let $e \in \mathcal{C}$ be an identity object.

Definition 2.2: A persistence module F is **S-constructible** if

- for $p \leq q < s_1$, $F(p \leq q)$ is the identity on e
- for $s_i \leq p \leq q < s_{i+1}$, $F(p \leq q)$ is an isomorphism
- for $s_n \leq p \leq q$, $F(p \leq q)$ is an isomorphism.

We say F is *constructible* if there is a finite set S such that F is S -constructible. Note that if F is S -constructible and T -constructible, then it is also $(S \cup T)$ -constructible.

We draw examples from the following five essentially small symmetric monoidal categories with images.

Example 2.1: Let \mathbf{FinSet} be the category of finite sets. \mathbf{FinSet} is a symmetric monoidal category under finite colimits (disjoint unions). A constructible persistence module valued in this category is often called a *merge tree* [MBW13].

The following four categories have more structure: they are abelian (see [Wei13, page 124]) and Krull-Schmidt (see Appendix A). In short, an abelian category is a category that behaves like the category of abelian groups. Finite products and coproducts are the same. Every morphism has a kernel and a cokernel. Every monomorphism is the kernel of some morphism, and every epimorphism is the cokernel of some morphism. The symmetric monoidal operation \square is the direct sum \oplus .

Example 2.2: Let \mathbf{Vec} be the category of finite dimensional k -vector spaces, for some fixed field k . Each vector space $\mathbf{a} \in \mathbf{Vec}$ is isomorphic to $k_1 \oplus k_2 \oplus \cdots \oplus k_n$, where n is the dimension of \mathbf{a} . Note that every short exact sequence $0 \rightarrow \mathbf{a} \rightarrow \mathbf{b} \rightarrow \mathbf{c} \rightarrow 0$ splits. That is, $\mathbf{b} \cong \mathbf{a} \oplus \mathbf{c}$.

Example 2.3: Let \mathbf{Ab} be the category of finitely generated abelian groups. An indecomposable of \mathbf{Ab} is isomorphic to the infinite cyclic group \mathbb{Z} or to a primary cyclic group $\mathbb{Z}/p^m\mathbb{Z}$, for a prime p and a positive integer m . By the fundamental theorem of finitely generated abelian groups, each object is uniquely isomorphic to

$$\mathbb{Z}^n \oplus \frac{\mathbb{Z}}{p_1^{m_1}\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p_2^{m_2}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p_k^{m_k}\mathbb{Z}},$$

for some $n \geq 0$ and primary cyclic groups $\mathbb{Z}/p_i^{m_i}\mathbb{Z}$. Not every short exact sequence in this category splits. Consider the following short exact sequence

$$0 \longrightarrow \frac{\mathbb{Z}}{2\mathbb{Z}} \xrightarrow{\times 2} \frac{\mathbb{Z}}{4\mathbb{Z}} \xrightarrow{/} \frac{\mathbb{Z}}{2\mathbb{Z}} \longrightarrow 0.$$

Of course $\mathbb{Z}/4\mathbb{Z}$ is not isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. A finitely generated abelian group is simple iff it is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for p prime. That is, $\mathbb{Z}/p\mathbb{Z}$ has no subgroups other than 0 and itself.

Example 2.4: Let \mathbf{FinAb} be the category of finite abelian groups. An indecomposable of \mathbf{FinAb} is isomorphic to a primary cyclic group $\mathbb{Z}/p^m\mathbb{Z}$, for prime p and a positive integer m . By the fundamental theorem of finitely generated abelian groups, each object is uniquely isomorphic to

$$\frac{\mathbb{Z}}{p_1^{m_1}\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p_2^{m_2}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p_k^{m_k}\mathbb{Z}}.$$

As shown in the previous example, not every short exact sequence in this category splits.

Example 2.5: Let $\text{Rep}(\mathbb{N})$ be the category of functors from the commutative monoid of natural numbers $\mathbb{N} = \{0, 1, \dots\}$ to Vec . We think of \mathbb{N} as a category with a single object and an endomorphism for each $n \in \mathbb{N}$ where $n \circ m$ is $n + m$. A functor in $\text{Rep}(\mathbb{N})$ is completely determined by where it sends 1. $\text{Rep}(\mathbb{N})$ is therefore equivalent to the category whose objects are endomorphisms $A : \mathfrak{a} \rightarrow \mathfrak{a}$ in Vec and whose morphisms $f : A \rightarrow B$ are maps $\hat{f} : \mathfrak{a} \rightarrow \mathfrak{b}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{a} & \xrightarrow{\hat{f}} & \mathfrak{b} \\ A \downarrow & & \downarrow B \\ \mathfrak{a} & \xrightarrow{\hat{f}} & \mathfrak{b}. \end{array}$$

We represent each object of $\text{Rep}(\mathbb{N})$ by a square matrix of elements in k . Suppose k is algebraically closed. Then such a matrix decomposes into a Jordan normal form

$$\begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_n \end{pmatrix}$$

where each Jordan block is of the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}.$$

The indecomposables of $\text{Rep}(\mathbb{N})$ are Jordan blocks. An object of $\text{Rep}(\mathbb{N})$ is simple iff its a Jordan block of dimension one.

Not every short exact sequence in $\text{Rep}(\mathbb{N})$ splits. Let $A : k \rightarrow k$ be given by (λ) , let $B : k^2 \rightarrow k^2$ be given by $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, and let $f : A \rightarrow B$ be given by $\hat{f}(x) = (x, 0)$. The quotient $C = B/\text{im}f$ is isomorphic to A . This gives us a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{\quad} C \longrightarrow 0$$

that does not split because B is not isomorphic to $(\lambda) \oplus (\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

Let $\text{PMod}(C)$ be the full subcategory of the functor category $[(\mathbb{R}, \leq), C]$ consisting of constructible persistence modules. Henceforth, all persistence modules are constructible.

3 Interleaving Distance

There is a natural distance between persistence modules. For $\varepsilon \in \mathbb{R}$, let

$$\text{Shift}^\varepsilon : (\mathbb{R}, \leq) \rightarrow (\mathbb{R}, \leq)$$

be the poset map that sends r to $r + \varepsilon$. If $F \in \text{PMod}$ is S -constructible, then $F \circ \text{Shift}^\varepsilon$ is $(S + \varepsilon)$ -constructible. Thus Shift^ε gives rise to a functor

$$\Delta^\varepsilon : \text{PMod}(\mathbb{C}) \rightarrow \text{PMod}(\mathbb{C}).$$

For each $\varepsilon \geq 0$, there is a canonical morphism $\sigma_F^\varepsilon : F \rightarrow \Delta^\varepsilon(F)$ given by $\sigma_F^\varepsilon(r) = F(r \leq r + \varepsilon)$.

Definition 3.1: Two modules $F, G \in \text{PMod}(\mathbb{C})$ are ε -interleaved if there are morphisms $\phi : F \rightarrow \Delta^\varepsilon(G)$ and $\psi : G \rightarrow \Delta^\varepsilon(F)$ such that $\sigma_F^{2\varepsilon} = \Delta^\varepsilon(\psi) \circ \phi$ and $\sigma_G^{2\varepsilon} = \Delta^\varepsilon(\phi) \circ \psi$.

Any two persistence modules F and G are constructible with respect to a common set $T = \{t_1 < \dots < t_m\}$. Both F and G are therefore constant over the half-open intervals $[t_i, t_{i+1})$ and $[t_m, \infty)$. As a consequence, if there is an interleaving between F and G , then there is a minimum interleaving between F and G .

Definition 3.2: The **interleaving distance** $d_I(F, G)$ between two persistence modules is the minimum over all $\varepsilon \geq 0$ such that F and G are ε -interleaved. If F and G are not interleaved, let $d_I(F, G) = \infty$.

Example 3.1: Let $f : M \rightarrow \mathbb{R}$ be a Morse function on a compact manifold M . The function f filters M by sublevel sets $M_{f \leq r}$. Apply homology with coefficients in k and the resulting object is in $\text{PMod}(\text{Vec})$. Apply homology with integer coefficients and the resulting object is in $\text{PMod}(\text{Ab})$. Apply homology with coefficients in a finite abelian group G and the resulting object is in $\text{PMod}(\text{FinAb})$. Suppose $\varepsilon > |f - g|$. Then $M_{f \leq r} \subseteq M_{g \leq r + \varepsilon} \subseteq M_{f \leq r + 2\varepsilon}$ implying, by functoriality of homology, an ε -interleaving between the two persistence modules.

Remark 3.1: The idea of interleavings appears in [CSEH07] but it is not named until [CCSG⁺09]. Since then, interleavings have been abstracted to other settings [MBW13, BS14, Cur14, BdSS15, Les15, DSMP16].

4 Persistence Diagrams

We now generalize the persistence diagram of Cohen-Steiner, Edelsbrunner, and Harer.

Definition 4.1: Define (Dgm, \supseteq) as the poset of all half-open intervals $[q, r) \subset \mathbb{R}$, for $q < r$, and all half-infinite intervals $[q, \infty) \subset \mathbb{R}$. The poset relation is the containment relation.

Let $S = \{s_1 < \dots < s_n\}$ be a finite set of real numbers and \mathcal{G} an abelian group. In the setting of Cohen-Steiner, Edelsbrunner, and Harer, the group \mathcal{G} is the integers.

Definition 4.2: A map $X : \mathbf{Dgm} \rightarrow \mathcal{G}$ is **S-constructible** if for every $J \supseteq I$ such that $J \cap S = I \cap S$, $X(I) = X(J)$.

We say a map $X : \mathbf{Dgm} \rightarrow \mathcal{G}$ is *constructible* if it is S-constructible for some set S. In the setting of Cohen-Steiner, Edelsbrunner, and Harer, X is the rank function.

Definition 4.3: A map $Y : \mathbf{Dgm} \rightarrow \mathcal{G}$ is **S-finite** if $Y(I) \neq e$ implies $I = [s_i, s_j]$ or $I = [s_i, \infty)$.

We say a map $Y : \mathbf{Dgm} \rightarrow \mathcal{G}$ is *finite* if it is T-finite for some set T.

Definition 4.4: A **persistence diagram** is a finite map $Y : \mathbf{Dgm} \rightarrow \mathcal{G}$.

We visualize the poset \mathbf{Dgm} as the set of points in the extended plane $\mathbb{R} \times \mathbb{R} \cup \{\infty\}$ above the diagonal. We visualize a persistence diagram Y by marking each $I \in \mathbf{Dgm}$ for which $Y(I) \neq [e]$ with the group element $Y(I)$. See Figures 2, 3, 4, 5, and 6.

In order to define a morphism between persistence diagrams, we require more structure on the abelian group \mathcal{G} . Let (\mathcal{G}, \preceq) be an abelian group with a translation invariant partial ordering on its elements. That is if $\mathbf{a} \preceq \mathbf{b}$, then $\mathbf{a} + \mathbf{c} \preceq \mathbf{b} + \mathbf{c}$ for any $\mathbf{c} \in \mathcal{G}$. Let $\mathbf{e} \in \mathcal{G}$ be the additive identity.

Definition 4.5: A **morphism** $Y_1 \rightarrow Y_2$ of persistence diagrams is the relation

$$\sum_{J \in \mathbf{Dgm}: J \supseteq I} Y_1(J) \preceq \sum_{J \in \mathbf{Dgm}: J \supseteq I} Y_2(J),$$

for each $I \in \mathbf{Dgm}$ such that $Y_1(I) \neq \mathbf{e}$.

Let $\mathbf{PDgm}(\mathcal{G})$ be the poset of persistence diagrams valued in (\mathcal{G}, \preceq) .

Theorem 4.1 (Möbius Inversion Formula): For any S-constructible map $X : \mathbf{Dgm} \rightarrow \mathcal{G}$, there is an S-finite map $Y : \mathbf{Dgm} \rightarrow \mathcal{G}$ satisfying the Möbius inversion formula

$$X(I) = \sum_{J \in \mathbf{Dgm}: J \supseteq I} Y(J),$$

for each $I \in \mathbf{Dgm}$.

Proof. Let $S = \{s_1 < \dots < s_n\}$. Define

$$Y([s_i, s_j]) = X([s_i, s_j]) - X([s_i, s_{j+1}]) + X([s_{i-1}, s_{j+1}]) - X([s_{i-1}, s_j]) \quad (1)$$

$$Y([s_i, \infty)) = X([s_i, \infty)) - X([s_{i-1}, \infty)). \quad (2)$$

Here we interpret s_0 as any value less than s_1 and s_{n+1} as any value greater than s_n . Define $Y(I) = \mathbf{e}$ for all other $I \in \mathbf{Dgm}$. Let us check that Y satisfies the Möbius inversion formula.

Fix an interval $I \in \text{Dgm}$. Suppose $I = [s_i, s_j]$. We have

$$\begin{aligned}
\sum_{J \in \text{Dgm}: J \supseteq I} Y(J) &= \sum_{k=j}^n \sum_{h=1}^i Y([s_h, s_k]) + \sum_{h=1}^i Y([s_h, \infty)) \\
&= \sum_{k=j}^n \sum_{h=1}^i \left[X([s_h, s_k]) - X([s_h, s_{k+1}]) + X([s_{h-1}, s_{k+1}]) - X([s_{h-1}, s_k]) \right] \\
&\quad + \sum_{h=1}^i \left[X([s_h, \infty)) - X([s_{h-1}, \infty)) \right] \\
&= \sum_{k=j}^n \left[X([s_i, s_k]) - X([s_i, s_{k+1}]) \right] + X([s_i, \infty)) \\
&= X([s_i, s_j]).
\end{aligned}$$

Suppose I is of the form $[s_i, \infty)$. We have

$$\begin{aligned}
\sum_{J \in \text{Dgm}: J \supseteq I} Y(J) &= \sum_{h=1}^i Y([s_h, \infty)) \\
&= \sum_{h=1}^i \left[X([s_h, \infty)) - X([s_{h-1}, \infty)) \right] \\
&= X([s_i, \infty)).
\end{aligned}$$

Suppose I is not of the form $[s_i, s_j]$. Then there is an $I' \in \text{Dgm}$ of the form $[s_i, s_j]$ or $[s_i, \infty)$ such that $I' \cap S = I \cap S$. We have

$$\sum_{J \in \text{Dgm}: J \supseteq I} Y(J) = \sum_{J \in \text{Dgm}: J \supseteq I'} Y(J) = X(I') = X(I).$$

□

The persistence diagram Y of Cohen-Steiner, Edelsbrunner, and Harer is the Möbius inversion of the rank function X .

Remark 4.1: The Möbius inversion formula applies to any constructible map from a poset to an abelian group. See [Rot64, BG75, Lei12]. This suggests a notion of a persistence diagram for constructible persistence modules not just over (\mathbb{R}, \leq) but over more general posets. See [BS14, BdSS15].

5 Erosion Distance

The interleaving distance suggests a natural metric between persistence diagrams. For $\varepsilon \geq 0$, let

$$\text{Grow}^\varepsilon : \text{Dgm} \rightarrow \text{Dgm}$$

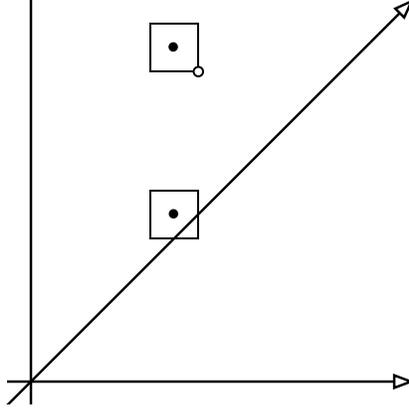


Figure 1: The ε -erosion $\nabla^\varepsilon(Y)$ (circle) of a persistence diagram Y (dots) slides each point of Y to the lower-right corner of the square of side length 2ε centered at that point. Points close to the diagonal disappear into the diagonal. Note that $\nabla^\varepsilon(Y) \rightarrow Y$.

be the poset map that sends each $[p, q]$ to $[p - \varepsilon, q + \varepsilon]$ and each $[p, \infty)$ to $[p - \varepsilon, \infty)$. For a morphism $Y_1 \rightarrow Y_2$ in $\text{PDgm}(\mathcal{G})$, we have $Y_1 \circ \text{Grow}^\varepsilon \rightarrow Y_2 \circ \text{Grow}^\varepsilon$. Thus Grow^ε gives rise to a functor

$$\nabla^\varepsilon : \text{PDgm}(\mathcal{G}) \rightarrow \text{PDgm}(\mathcal{G})$$

given by precomposition with Grow^ε . For each $\varepsilon \geq 0$, we have $\nabla^\varepsilon(Y) \rightarrow Y$. The persistence diagram $\nabla^\varepsilon(Y)$ is visualized as the persistence diagram Y with all its points shifted towards the diagonal by a distance $\sqrt{2}\varepsilon$. See Figure 1.

Definition 5.1: An ε -erosion between two persistence diagrams $Y_1, Y_2 \in \text{PDgm}(\mathcal{G})$ is a pair of morphisms $\nabla^\varepsilon(Y_2) \rightarrow Y_1$ and $\nabla^\varepsilon(Y_1) \rightarrow Y_2$.

Any two persistence diagrams are finite with respect to a common set $\mathbb{T} = \{t_1 < \dots < t_n\}$. As a consequence, if there is an ε -erosion between Y_1 and Y_2 , then there is a minimum ε for which there is an ε -erosion.

Definition 5.2: The erosion distance $d_E(Y_1, Y_2)$ is the minimum over all $\varepsilon \geq 0$ such that there is an ε -erosion between Y_1 and Y_2 . If there is no ε -erosion, let $d_E(Y_1, Y_2) = \infty$.

Proposition 5.1: Let $X : \text{Dgm} \rightarrow \mathcal{G}$ be a constructible map and let $Y : \text{Dgm} \rightarrow \mathcal{G}$ be a finite map that satisfies the Möbius inversion formula

$$X(I) = \sum_{J \in \text{Dgm}: J \supseteq I} Y(J),$$

for each $I \in \text{Dgm}$. Then

$$X \circ \text{Grow}^\varepsilon(I) = \sum_{J \in \text{Dgm}: J \supseteq I} \nabla^\varepsilon(Y)(J),$$

for each $I \in \text{Dgm}$. In other words, Grow^ε commutes with the Möbius inversion formula.

Proof. We have

$$\begin{aligned} \sum_{J \in \text{Dgm}: J \supseteq I} \nabla^\varepsilon(Y)(J) &= \sum_{J \in \text{Dgm}: J \supseteq I} Y \circ \text{Grow}^\varepsilon(J) \\ &= X \circ \text{Grow}^\varepsilon(I) \end{aligned}$$

□

Remark 5.1: The erosion distance first appears in [EMP11] which is an early attempt to develop a theory of persistence for maps from a surface to the Euclidean plane.

6 Grothendieck Groups

We are interested in two abelian groups: the Grothendieck group \mathcal{A} of an essentially small symmetric monoidal category and the Grothendieck group \mathcal{B} of an essentially small abelian category. See [Wei13] for an introduction to the two Grothendieck groups. Note that every abelian category is a symmetric monoidal category under the direct sum \oplus and the additivity identity is the zero object.

6.1 Symmetric Monoidal Category

Let \mathbf{C} be an essentially small monoidal category. The set $\mathcal{J}(\mathbf{C})$ of isomorphism classes in \mathbf{C} is a commutative monoid under \square . We write the isomorphism class of an object $\mathbf{a} \in \mathbf{C}$ as $[\mathbf{a}] \in \mathcal{J}(\mathbf{C})$, the binary operation in $\mathcal{J}(\mathbf{C})$ as $[\mathbf{a}] + [\mathbf{b}] = [\mathbf{a} \square \mathbf{b}]$, and the additive identity of $\mathcal{J}(\mathbf{C})$ as $[\mathbf{e}]$.

Definition 6.1.1: The **Grothendieck group** $\mathcal{A}(\mathbf{C})$ of \mathbf{C} is the group completion of the commutative monoid $\mathcal{J}(\mathbf{C})$.

Explicitly, an element of $\mathcal{A}(\mathbf{C})$ is of the form $[\mathbf{a}] - [\mathbf{b}]$ with addition coordinatewise, and $[\mathbf{a}] = [\mathbf{c}]$ iff $[\mathbf{a}] + [\mathbf{d}] = [\mathbf{c}] + [\mathbf{d}]$, for some element $[\mathbf{d}] \in \mathcal{J}(\mathbf{C})$. If \mathbf{C} is additive and Krull-Schmidt (see Appendix A), then each object in \mathbf{C} is isomorphic to a unique direct sum of indecomposables. This means $\mathcal{A}(\mathbf{C})$ is the free abelian group generated by the set of isomorphism classes of indecomposables. The Grothendieck group $\mathcal{A}(\mathbf{C})$ has a natural translation-invariant partial ordering. We define $[\mathbf{a}] \preceq [\mathbf{b}]$ iff $[\mathbf{b}] - [\mathbf{a}] \in \mathcal{J}(\mathbf{C})$. If $[\mathbf{a}] \preceq [\mathbf{b}]$, then $[\mathbf{a}] + [\mathbf{c}] \preceq [\mathbf{b}] + [\mathbf{c}]$ for any $[\mathbf{c}] \in \mathcal{A}(\mathbf{C})$. See [Wei13, page 72] for an introduction to translation-invariant partial orderings on Grothendieck groups.

Example 6.1.1: Every finite set is a finite disjoint union of the singleton set. We have

$$\mathcal{A}(\text{FinSet}) \cong \mathbb{Z}.$$

Example 6.1.2: Every finite dimensional vector space is isomorphic to a finite direct sum of k . We have

$$\mathcal{A}(\text{Vec}) \cong \mathbb{Z}.$$

Example 6.1.3: An indecomposable of \mathbf{Ab} is the free cyclic group or a primary cyclic group. We have

$$\mathcal{A}(\mathbf{Ab}) \cong \mathbb{Z} \oplus \bigoplus_{(m,p)} \mathbb{Z},$$

over all primes p and positive integers m .

Example 6.1.4: An indecomposable of \mathbf{FinAb} is a primary cyclic group. We have

$$\mathcal{A}(\mathbf{FinAb}) \cong \bigoplus_{(m,p)} \mathbb{Z}$$

over all primes p and positive integers m .

Example 6.1.5: An indecomposable of $\mathbf{Rep}(\mathbb{N})$ is a Jordan block. We have

$$\mathcal{A}(\mathbf{Rep}(\mathbb{N})) \cong \bigoplus_{(m,\lambda)} \mathbb{Z},$$

over all positive integers m and elements λ in the field k .

6.2 Abelian Category

Suppose \mathbf{C} is an essentially small abelian category. We say two elements $[b]$ and $[a] + [c]$ in $\mathcal{A}(\mathbf{C})$ are related, written $[b] \sim [a] + [c]$, if there is a short exact sequence $0 \rightarrow a \rightarrow b \rightarrow c \rightarrow 0$.

Definition 6.2.1: The **Grothendieck group** $\mathcal{B}(\mathbf{C})$ of \mathbf{C} is the quotient group $\mathcal{A}(\mathbf{C})/\sim$. That is, $\mathcal{B}(\mathbf{C})$ is the abelian group with one generator for each isomorphism classes $[a]$ in \mathbf{C} and one relation $[b] \sim [a] + [c]$ for each short exact sequence $0 \rightarrow a \rightarrow b \rightarrow c \rightarrow 0$.

Let $\pi: \mathcal{A}(\mathbf{C}) \rightarrow \mathcal{B}(\mathbf{C})$ be the quotient map. Note that $\pi(\mathcal{J}(\mathbf{C}))$ is a commutative monoid that generates $\mathcal{B}(\mathbf{C})$. This allows us to define a translation-invariant partial ordering on $\mathcal{B}(\mathbf{C})$ as follows. We define $[a] \preceq [b]$ iff $[b] - [a] \in \pi(\mathcal{J}(\mathbf{C}))$. If $[a] \preceq [b]$, then $[a] + [c] \preceq [b] + [c]$ for any $[c] \in \mathcal{B}(\mathbf{C})$. The quotient map π is a poset map.

Example 6.2.1: Every short exact sequence in \mathbf{Vec} splits. We have

$$\mathcal{B}(\mathbf{Vec}) \cong \mathbb{Z}.$$

The quotient map $\pi: \mathcal{A}(\mathbf{Vec}) \rightarrow \mathcal{B}(\mathbf{Vec})$ is the identity.

Example 6.2.2: Every primary cyclic group $\mathbb{Z}/p^m\mathbb{Z}$ fits into a short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \frac{\mathbb{Z}}{p^m\mathbb{Z}} \rightarrow 0.$$

This means $[\mathbb{Z}] \sim [\mathbb{Z}] + \left[\frac{\mathbb{Z}}{p^m\mathbb{Z}}\right]$ and therefore $0 \sim \left[\frac{\mathbb{Z}}{p^m\mathbb{Z}}\right]$. We have

$$\mathcal{B}(\mathbf{Ab}) \cong \mathbb{Z}.$$

The quotient map $\pi : \mathcal{A}(\mathbf{Ab}) \rightarrow \mathcal{B}(\mathbf{Ab})$ forgets the torsion part of every finitely generated abelian group.

Example 6.2.3: Every primary cyclic group $\mathbb{Z}/\mathfrak{p}^m\mathbb{Z}$ fits into a short exact sequence

$$0 \rightarrow \frac{\mathbb{Z}}{\mathfrak{p}\mathbb{Z}} \rightarrow \frac{\mathbb{Z}}{\mathfrak{p}^m\mathbb{Z}} \rightarrow \frac{\mathbb{Z}}{\mathfrak{p}^{m-1}\mathbb{Z}} \rightarrow 0.$$

This means

$$\left[\frac{\mathbb{Z}}{\mathfrak{p}^m\mathbb{Z}} \right] \sim \mathfrak{m} \left[\frac{\mathbb{Z}}{\mathfrak{p}\mathbb{Z}} \right].$$

Furthermore, $\frac{\mathbb{Z}}{\mathfrak{p}\mathbb{Z}}$ is a simple object so it can not be broken by a short exact sequence. We have

$$\mathcal{B}(\mathbf{FinAb}) \cong \bigoplus_{\mathfrak{p}} \mathbb{Z}$$

over all \mathfrak{p} prime. The quotient map $\pi : \mathcal{A}(\mathbf{FinAb}) \rightarrow \mathcal{B}(\mathbf{FinAb})$ takes each primary cyclic group $\left[\frac{\mathbb{Z}}{\mathfrak{p}^m\mathbb{Z}} \right]$ to \mathfrak{m} in the \mathfrak{p} factor of $\mathcal{B}(\mathbf{FinAb})$.

Example 6.2.4: Every Jordan block fits into a short exact sequence. For example,

$$0 \rightarrow (\lambda) \rightarrow \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \rightarrow \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \rightarrow 0$$

and

$$0 \rightarrow (\lambda) \rightarrow \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \rightarrow (\lambda) \rightarrow 0.$$

This means

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \sim 3(\lambda).$$

Futhermore, each one-dimensional Jordan block (λ) is simple so it can not be broken by a short exact sequence. We have

$$\mathcal{B}(\mathbf{Rep}(\mathbb{N})) \cong \bigoplus_{\lambda \in \mathfrak{k}} \mathbb{Z}.$$

The quotient map $\pi : \mathcal{A}(\mathbf{Rep}(\mathbb{N})) \rightarrow \mathcal{B}(\mathbf{Rep}(\mathbb{N}))$ takes each Jordan block of dimension $\mathfrak{m} \in \mathbb{N}$ with eigenvalue $\lambda \in \mathfrak{k}$ to \mathfrak{m} in the λ factor of $\mathcal{B}(\mathbf{Rep}(\mathbb{N}))$.

7 Diagram of a Module

Fix an essentially small symmetric monoidal category \mathbf{C} with images. We now assign to each persistence module $F \in \mathbf{PMod}(\mathbf{C})$ a persistence diagram $F_{\mathcal{A}} \in \mathbf{PDgm}(\mathcal{A}(\mathbf{C}))$. If \mathbf{C} is also abelian, then we assign to F a second persistence diagram $F_{\mathcal{B}} \in \mathbf{PDgm}(\mathcal{B}(\mathbf{C}))$.

We start by constructing a map

$$dF_{\mathcal{J}} : \text{Dgm} \rightarrow \mathcal{J}(\mathbf{C}).$$

Recall $\mathcal{J}(\mathbf{C})$ is the commutative monoid of isomorphism classes of objects in \mathbf{C} . Suppose F is $\mathbf{S} = \{s_1 < \dots < s_n\}$ -constructible. Then there is a $\delta > 0$ such that $s_{i-1} < s_i - \delta$, for each $1 < i \leq n$. Choose a value $s' > s_n$. Define

$$dF_{\mathcal{J}}(I) = \begin{cases} [\text{im } F(\mathfrak{p} < s_i - \delta)] & \text{for } I = [\mathfrak{p}, s_i] \\ [\text{im } F(\mathfrak{p} < s')] & \text{for } I = [\mathfrak{p}, \infty) \\ [\text{im } F(\mathfrak{p} < \mathfrak{q})] & \text{for all other } I = [\mathfrak{p}, \mathfrak{q}). \end{cases}$$

Note that if F is also \mathbf{T} -constructible, then $dF_{\mathcal{J}}$ constructed using \mathbf{T} is the same as $dF_{\mathcal{J}}$ constructed using \mathbf{S} . Now compose with the inclusion map $\mathcal{J}(\mathbf{C}) \hookrightarrow \mathcal{A}(\mathbf{C})$ and we have an \mathbf{S} -constructible map

$$dF_{\mathcal{A}} : \text{Dgm} \rightarrow \mathcal{A}(\mathbf{C}).$$

Suppose \mathbf{C} is abelian. Then by composing with the quotient map $\pi : \mathcal{A}(\mathbf{C}) \rightarrow \mathcal{B}(\mathbf{C})$, we have an \mathbf{S} -constructible map

$$dF_{\mathcal{B}} : \text{Dgm} \rightarrow \mathcal{B}(\mathbf{C}).$$

Definition 7.1: The **type \mathcal{A} persistence diagram** of F is the Möbius inversion

$$F_{\mathcal{A}} : \text{Dgm} \rightarrow \mathcal{A}(\mathbf{C})$$

of $dF_{\mathcal{A}} : \text{Dgm} \rightarrow \mathcal{A}(\mathbf{C})$.

Definition 7.2: The **type \mathcal{B} persistence diagram** of F is the Möbius inversion

$$F_{\mathcal{B}} : \text{Dgm} \rightarrow \mathcal{B}(\mathbf{C})$$

of $dF_{\mathcal{B}} : \text{Dgm} \rightarrow \mathcal{B}(\mathbf{C})$.

Note that if F is \mathbf{S} -constructible, then both $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$ are \mathbf{S} -finite persistence diagrams.

Proposition 7.1 (Positivity): For each $I \in \text{Dgm}$, $[e] \preceq F_{\mathcal{B}}(I)$.

Proof. Suppose F is $\mathbf{S} = \{s_1 < \dots < s_n\}$ -constructible. We need only show the inequality for intervals I of the form $[s_i, s_j)$ and $[s_i, \infty)$. For all other I , $F_{\mathcal{B}}(I) = [e]$.

Suppose $I = [s_i, s_j)$. Consider the following subdiagram of F , for a sufficiently small $\delta > 0$:

$$\begin{array}{ccc} F(s_{i-1}) & \xrightarrow{F(s_{i-1} < s_i)} & F(s_i) \\ \downarrow & & \downarrow F(s_i < s_j - \delta) \\ F(s_{j+1} - \delta) & \xleftarrow{F(s_j - \delta < s_{j+1} - \delta)} & F(s_j - \delta). \end{array}$$

Here we interpret s_0 as any value less than s_1 and s_{n+1} as any value greater than s_n . By Equation 1,

$$F_{\mathcal{B}}([s_i, s_j]) = dF_{\mathcal{B}}([s_i, s_j]) - dF_{\mathcal{B}}([s_i, s_{j+1}]) + dF_{\mathcal{B}}([s_{i-1}, s_{j+1}]) - dF_{\mathcal{B}}([s_{i-1}, s_j])$$

Observe

$$\begin{aligned} dF_{\mathcal{B}}([s_i, s_j]) - dF_{\mathcal{B}}([s_i, s_{j+1}]) &= [\text{im } F(s_i < s_j - \delta)] \\ &\quad - \left[\frac{\text{im } F(s_i < s_j - \delta)}{\text{im } F(s_i < s_j - \delta) \cap \ker F(s_j - \delta < s_{j+1} - \delta)} \right] \\ &= [\text{im } F(s_i < s_j - \delta)] - [\text{im } F(s_i < s_j - \delta)] \\ &\quad + [\text{im } F(s_i < s_j - \delta) \cap \ker F(s_j - \delta < s_{j+1} - \delta)] \\ &= [\text{im } F(s_i < s_j - \delta) \cap \ker F(s_j - \delta < s_{j+1} - \delta)]. \end{aligned}$$

Here the intersection is interpreted as the pullback of the two subobjects. By a similar argument,

$$dF_{\mathcal{B}}([s_{i-1}, s_{j+1}]) - dF_{\mathcal{B}}([s_{i-1}, s_j]) = -[\text{im } F(s_{i-1} < s_j - \delta) \cap \ker F(s_j - \delta < s_{j+1} - \delta)].$$

Note that

$$\text{im } F(s_{i-1} < s_j - \delta) \cap \ker F(s_j - \delta < s_{j+1} - \delta)$$

is a subobject of

$$\text{im } F(s_i < s_j - \delta) \cap \ker F(s_j - \delta < s_{j+1} - \delta).$$

Therefore

$$F_{\mathcal{B}}([s_i, s_j]) = \left[\frac{\text{im } F(s_i < s_j - \delta) \cap \ker F(s_j - \delta < s_{j+1} - \delta)}{\text{im } F(s_{i-1} < s_j - \delta) \cap \ker F(s_j - \delta < s_{j+1} - \delta)} \right] \succeq [e].$$

Suppose $I = [s_i, \infty)$. Then by a similar argument using Equation 2, we have

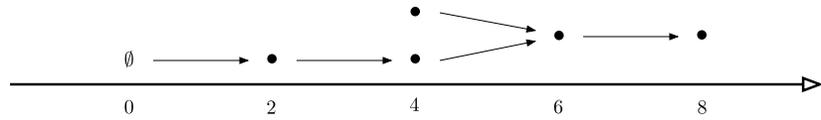
$$F_{\mathcal{B}}([s_i, \infty)) = \left[\frac{\text{im } F(s_i < s_{n+1})}{\text{im } F(s_{i-1} < s_{n+1})} \right] \succeq [e].$$

□

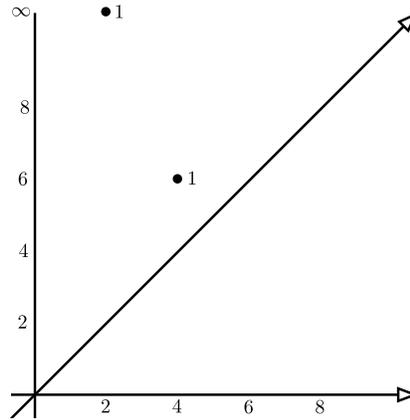
Example 7.1: See Figure 2 for an example of a persistence module in $\text{PMod}(\text{FinSet})$ and its type \mathcal{A} persistence diagram. Note that FinSet is not an abelian category so it does not have a type \mathcal{B} persistence diagram.

Example 7.2: See Figure 3 for an example of a persistence module in $\text{PMod}(\text{Vec})$ and its type \mathcal{A} and type \mathcal{B} persistence diagrams. Note that the quotient map $\pi : \mathcal{A}(\text{Vec}) \rightarrow \mathcal{B}(\text{Vec})$ is an isomorphism and therefore the two diagrams are the same.

Example 7.3: See Figure 4 for an example of a persistence module in $\text{PMod}(\text{Ab})$ and its type \mathcal{A} persistence diagram. Note that the quotient map $\pi : \mathcal{A}(\text{C}) \rightarrow \mathcal{B}(\text{C})$ forgets torsion and therefore the type \mathcal{B} persistence diagram is, for this example, zero.



(a) Persistence module



(b) Type \mathcal{A} persistence diagram

Figure 2: Here we have an example of a persistence module in $\text{PMod}(\text{FinSet})$ and its type \mathcal{A} persistence diagram.

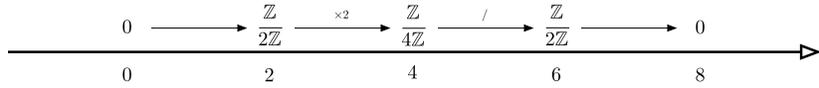
Example 7.4: See Figure 5 for an example of a persistence module in $\text{PMod}(\text{FinAb})$ and its type \mathcal{A} and type \mathcal{B} persistence diagrams.

Example 7.5: See Figure 6 for an example of a persistence module in $\text{PMod}(\text{Rep}(\mathbb{N}))$ and its type \mathcal{A} and type \mathcal{B} persistence diagrams.

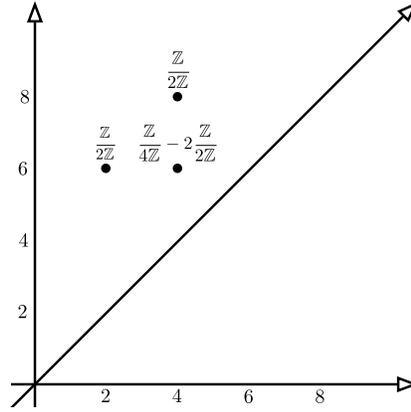
8 Stability

We now relate the interleaving distance between persistence modules to the erosion distance between their persistence diagrams.

For the first theorem, we make a simplifying assumption on \mathbf{C} that makes it possible to chase diagrams. We assume that \mathbf{C} is concrete and that its images are concrete. That is, \mathbf{C} embeds into the category \mathbf{Set} and an image of a morphism in \mathbf{C} is the image of the corresponding set map. Note that all our examples satisfy this criteria. By the Freyd-Mitchell embedding theorem [Wei95, page 28], an essentially small abelian category \mathbf{C} embeds into the category of \mathbf{R} -modules, for some ring \mathbf{R} , and the image of a morphism in \mathbf{C} is the image under the corresponding set map. Therefore, all essentially small abelian categories satisfy our criteria.

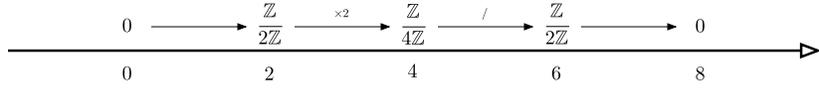


(a) Persistence module

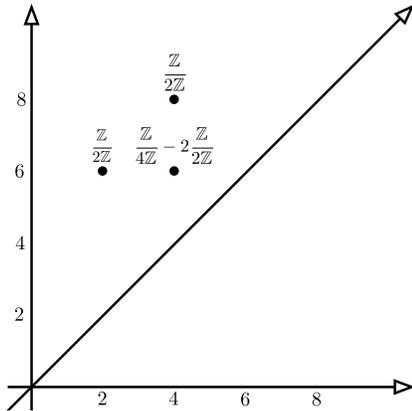


(b) Type \mathcal{A} persistence diagram

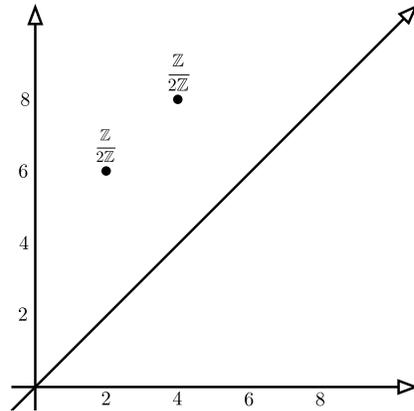
Figure 4: Here we have an example of a persistence module in $\text{PMod}(\text{Ab})$ and its type \mathcal{A} persistence diagram. The map from 4 to 6 is the quotient of $\mathbb{Z}/4\mathbb{Z}$ by the image of the previous map.



(a) Persistence module

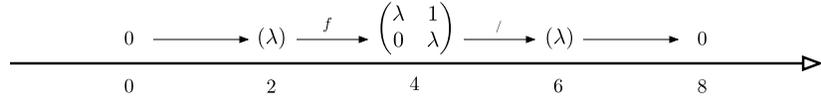


(b) Type \mathcal{A} persistence diagram

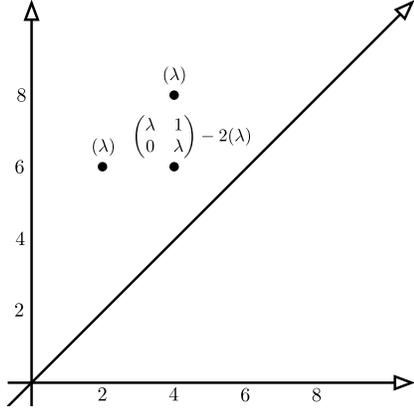


(c) Type \mathcal{B} persistence diagram

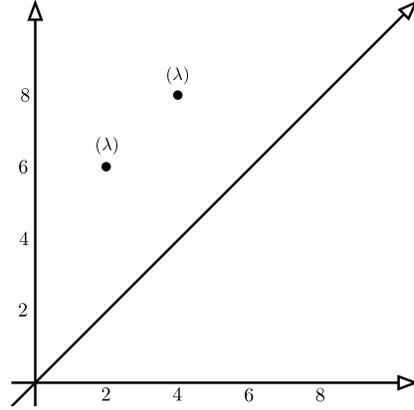
Figure 5: Here we have an example of a persistence module in $\text{PMod}(\text{FinAb})$ and its type \mathcal{A} and type \mathcal{B} persistence diagrams. This is the same example module as in Figure 4.



(a) Persistence module



(b) Type \mathcal{A} persistence diagram



(c) Type \mathcal{B} persistence diagram

Figure 6: Here we have an example of a persistence module in $\text{PMod}(\text{Ab})$ and its type \mathcal{A} and type \mathcal{B} persistence diagrams. The map from 4 to 6 is the quotient by the image of f .

Theorem 8.1 (Semicontinuity): Let \mathcal{C} be an essentially small symmetric monoidal category with images. Suppose $F \in \text{PMod}(\mathcal{C})$ is $S = \{s_1 < \dots < s_n\}$ -constructible and let

$$\rho = \frac{1}{4} \min_{1 < i \leq n} (s_i - s_{i-1}).$$

For any second persistence module $G \in \text{PMod}(\mathcal{C})$ such that $\varepsilon = d_I(F, G) < \rho$, there is a morphism

$$\nabla^\varepsilon(F_{\mathcal{A}}) \rightarrow G_{\mathcal{A}}$$

in $\text{PDgm}(\mathcal{A}(\mathcal{C}))$.

Proof. Let $\phi : F \rightarrow \Delta^\varepsilon(G)$ and $\psi : G \rightarrow \Delta^\varepsilon(F)$ be an ε -interleaving. For each $I \in \text{Dgm}$ such that $F_{\mathcal{A}}(I) \neq [e]$, we must show

$$d_{F_{\mathcal{A}}} \circ \text{Grow}^\varepsilon(I) \preceq d_{G_{\mathcal{A}}}(I).$$

By constructibility, it is sufficient to show this inequality for $I = [s_i + \varepsilon, s_j - \varepsilon]$ and $I =$

$[s_i + \varepsilon, \infty)$. Suppose $I = [s_i + \varepsilon, s_j - \varepsilon)$. Consider the following commutative diagram:

$$\begin{array}{ccc}
F(s_i) & \xrightarrow{F(s_i < s_j - \delta)} & F(s_j - \delta) \\
\downarrow \phi(s_i) & & \uparrow \psi(s_j - \varepsilon - \delta) \\
G(s_i + \varepsilon) & \xrightarrow{G(s_i + \varepsilon < s_j - \varepsilon - \delta)} & G(s_j - \varepsilon - \delta) \\
\downarrow \psi(s_i + \varepsilon) & & \uparrow \phi(s_j - 2\varepsilon - \delta) \\
F(s_i + 2\varepsilon) & \xrightarrow{F(s_i + 2\varepsilon < s_j - 2\varepsilon - \delta)} & F(s_j - 2\varepsilon - \delta).
\end{array} \tag{3}$$

By \mathcal{S} -constructibility of F , the two vertical compositions are isomorphisms. By a diagram chase, we see that

$$dF_{\mathcal{A}}([s_i, s_j]) = dG_{\mathcal{A}}([s_i + \varepsilon, s_j - \varepsilon]).$$

This proves the claim. Suppose I is of the form $[s_i, \infty)$, then

$$dF_{\mathcal{A}}([s_i, \infty)) = dG_{\mathcal{A}}([s_i + \varepsilon, \infty))$$

by a similar commutative diagram. □

Semicontinuity is saying there is an open neighborhood of F in the metric space of persistence modules such that for each G in this open neighborhood, $F_{\mathcal{A}}$ lives on in $G_{\mathcal{A}}$. However, semicontinuity is unsatisfying in two interesting ways. First, the ε must be smaller than ρ which is half the injectivity radius of \mathcal{S} in \mathbb{R} . Second, $\nabla^\varepsilon(F_{\mathcal{A}}) \rightarrow G_{\mathcal{A}}$ but we can not prove the converse $\nabla^\varepsilon(G_{\mathcal{A}}) \rightarrow F_{\mathcal{A}}$. The fundamental limitation here is that not all short exact sequences in \mathcal{C} split.

Theorem 8.2 (Continuity): Let \mathcal{C} be an essentially small, concrete, abelian category. For any two persistence modules $F, G \in \text{PMod}(\mathcal{C})$, we have

$$d_E(F_{\mathcal{B}}, G_{\mathcal{B}}) \leq d_I(F, G).$$

Proof. Let $\varepsilon = d_I(F, G)$. For each $I \in \text{Dgm}$ such that $F_{\mathcal{A}}(I) \neq [e]$, we must show

$$dF_{\mathcal{A}} \circ \text{Grow}^\varepsilon(I) \preceq dG_{\mathcal{A}}(I)$$

and for each $I \in \text{Dgm}$ such that $G_{\mathcal{B}}(I) \neq [e]$, we must show

$$dG_{\mathcal{A}} \circ \text{Grow}^\varepsilon(I) \preceq dF_{\mathcal{A}}(I).$$

We will prove the first inequality and the second inequality follows by simply interchanging the roles of F and G in the proof.

Suppose F is $\mathcal{S} = \{s_1 < \dots < s_n\}$ -constructible. By constructibility, it is sufficient to show the first inequality for I of the form $[s_i + \varepsilon, s_j - \varepsilon)$ and $[s_i + \varepsilon, \infty)$. Suppose

$I = [s_i + \varepsilon, s_j - \varepsilon]$. Let $\phi : F \rightarrow \Delta^\varepsilon(G)$ and $\psi : G \rightarrow \Delta^\varepsilon(F)$ be an ε -interleaving. Consider the following commutative diagram:

$$\begin{array}{ccc}
F(s_i) & \xrightarrow{F(s_i < s_j - \delta)} & F(s_j - \delta) \\
\downarrow \phi(s_i) & & \uparrow \psi(s_j - \varepsilon - \delta) \\
G(s_i + \varepsilon) & \xrightarrow{G(s_i + \varepsilon < s_j - \varepsilon - \delta)} & G(s_j - \varepsilon - \delta).
\end{array} \tag{4}$$

By commutativity,

$$\text{im } F(s_i < s_j - \delta) \cong \frac{\text{im } G(s_i + \varepsilon < s_j - \varepsilon - \delta)}{\text{im } G(s_i + \varepsilon < s_j - \varepsilon - \delta) \cap \ker \psi(s_j - \varepsilon - \delta)}.$$

Therefore

$$\begin{aligned}
dF_{\mathcal{B}}([s_i < s_j]) &= dG(s_i + \varepsilon < s_j - \varepsilon) - [\ker \psi(s_j - \varepsilon - \delta)] \\
&\preceq dG_{\mathcal{B}}([s_i + \varepsilon < s_j - \varepsilon])
\end{aligned}$$

This proves the claim. Suppose $I = [s_i, \infty)$. Then

$$dF_{\mathcal{B}}([s_i < \infty)) \preceq dG_{\mathcal{B}}([s_i + \varepsilon < \infty)).$$

by a similar commutative diagram. □

9 Concluding Remarks

Torsion in data. We hope our theory will allow for the study of torsion in data. For example, let $P \subset \mathbb{R}^n$ be a finite set of points. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function dependent on P , for example $f(x) = \min_{p \in P} \|x - p\|_2$. Apply homology with integer coefficients to the sublevel set filtration induced by f and we have a constructible persistence module $F \in \text{PMod}(\text{Ab})$. Its type \mathcal{A} persistence diagram is measuring torsion in data and semicontinuity applies. If continuity is required, then we may look at the type \mathcal{B} persistence diagram of F . However, the type \mathcal{B} persistence diagram forgets all torsion. Perhaps a better approach is to apply homology with coefficients in a finite abelian group. Then the resulting persistence module is in $\text{PMod}(\text{FinAb})$ and its type \mathcal{B} diagram encodes simple torsion.

Time series. The flexibility we offer in choosing \mathcal{C} should allow for the encoding of more structure in data. Consider time series data. Suppose $P = \{p_1, \dots, p_k\}$ is a finite sequence of points in \mathbb{R}^n . There is more to P than its shape. The forward shift $p_i \rightarrow p_{i+1}$ along the sequence should induce dynamics on the shape of P at each scale. The algebraic object of study is not clear, but it will certainly have more structure than a vector space or an abelian group.

Non-constructible modules. Suppose we are given an infinite set of points $P \subset \mathbb{R}^n$. Then the resulting persistence module, as constructed above, is not constructible. Is there a persistence diagram for a non-constructible persistence module?

This question is addressed by [CdSGO16] for $\mathbf{C} = \mathbf{Vec}$. They define a persistence diagram for a non-constructible persistence module as a *rectangular measure* $\mu : \mathbf{Rect} \rightarrow \mathbb{N}$, where \mathbf{Rect} is the poset of all pairs $J \supset I$ in \mathbf{Dgm} , satisfying a certain additivity condition. Our type \mathcal{B} diagram should generalize to a rectangular measure. For \mathbf{C} abelian, we may use an argument similar to the one in the proof of Proposition 7.1 to assign an element of $\mathcal{B}(\mathbf{C})$ to each $J \supset I$ without making use of constructibility. Is this assignment a rectangular measure?

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A Krull-Schmidt

We now provide a compact treatment of Krull-Schmidt categories. The following ideas are classical and may be found in many books, for example [AF92].

A category \mathbf{C} is *additive* if all its hom-sets are abelian, composition is bilinear, and finite products and finite coproducts are the same. The (co)product of the empty set is the *zero object* of \mathbf{C} . Suppose \mathbf{C} is additive.

Definition A.1: A non-zero object $\mathbf{a} \in \mathbf{C}$ is **indecomposable** if it is not the direct sum of two non-zero objects.

Definition A.2: An additive category \mathbf{C} is **Krull-Schmidt** if each object $\mathbf{a} \in \mathbf{C}$ is isomorphic to a finite direct sum $\mathbf{a} \cong \mathbf{a}_1 \oplus \mathbf{a}_2 \oplus \cdots \oplus \mathbf{a}_n$ and each ring of endomorphisms $\text{End}_{\mathbf{C}}(\mathbf{a}_i)$ is *local*. That is, $0 \neq 1$ and if $f_1 + f_2 = 1$, then f_1 or f_2 is invertible.

Suppose \mathbf{C} is Krull-Schmidt.

Proposition A.1: An object $\mathbf{a} \in \mathbf{C}$ is indecomposable iff its endomorphism ring $\text{End}(\mathbf{a})$ is local.

Proof. Suppose $\mathbf{a} \in \mathbf{C}$ is decomposable. That is, there is an isomorphism $\mathbf{i} : \mathbf{a} \rightarrow \mathbf{a}_1 \oplus \mathbf{a}_2$ such that $\mathbf{a}_1, \mathbf{a}_2 \neq 0$. Define $\pi_1 : \mathbf{a}_1 \oplus \mathbf{a}_2 \rightarrow \mathbf{a}_1 \oplus \mathbf{a}_2$ as the endomorphism that sends the first factor to zero and $\pi_2 : \mathbf{a}_1 \oplus \mathbf{a}_2 \rightarrow \mathbf{a}_1 \oplus \mathbf{a}_2$ as the endomorphism that sends the second factor to zero. Then the two maps $\rho_1, \rho_2 : \mathbf{a} \rightarrow \mathbf{a}$, where $\rho_1 = \mathbf{i}^{-1} \circ \pi_1 \circ \mathbf{i}$ and $\rho_2 = \mathbf{i}^{-1} \circ \pi_2 \circ \mathbf{i}$, are both non-isomorphisms in $\text{End}_{\mathbf{C}}(\mathbf{a})$. However, $\rho_0 + \rho_1 : \mathbf{a} \rightarrow \mathbf{a}$ is an isomorphism. We have a contradiction of locality.

Suppose $\mathbf{a} \in \mathbf{C}$ is indecomposable. Then, by definition of a Krull-Schmidt category, $\text{End}_{\mathbf{C}}(\mathbf{a})$ is a local ring. \square

Proposition A.2: Each object $\mathbf{a} \in \mathbf{C}$ is isomorphic to a finite direct sum of indecomposables.

Proof. By definition of a Krull-Schmidt category, $\mathbf{a} \cong \mathbf{a}_1 \oplus \mathbf{a}_2 \oplus \cdots \oplus \mathbf{a}_n$ where each $\text{End}_{\mathbf{C}}(\mathbf{a}_i)$ is a local ring. By Proposition A.1, each \mathbf{a}_i is indecomposable. \square

Theorem A.1 (Krull-Schmidt): Suppose an object $\mathbf{c} \in \mathbf{C}$ is isomorphic to $\mathbf{a}_1 \oplus \mathbf{a}_2 \oplus \cdots \oplus \mathbf{a}_m$ and $\mathbf{b}_1 \oplus \mathbf{b}_2 \oplus \cdots \oplus \mathbf{b}_n$, where each \mathbf{a}_i and \mathbf{b}_j are indecomposable. Then $m = n$, and there is a permutation $\mathbf{p} : [m] \rightarrow [n]$ such that $\mathbf{a}_i \cong \mathbf{b}_{\mathbf{p}(i)}$.

Proof. By definition of an additive category, we have canonical projections $\pi_i : \bigoplus_i \mathbf{a}_i \rightarrow \mathbf{a}_i$ and $\rho_j : \bigoplus_j \mathbf{b}_j \rightarrow \mathbf{b}_j$ and canonical inclusions $\mu_i : \mathbf{a}_i \rightarrow \bigoplus_i \mathbf{a}_i$ and $\nu_j : \mathbf{b}_j \rightarrow \bigoplus_j \mathbf{b}_j$. Furthermore $\mu_j \circ \pi_i$ and $\nu_j \circ \rho_i$ are the identity on \mathbf{a}_i and \mathbf{b}_i , respectively, iff $i = j$. Let $\mathbf{f} : \mathbf{a}_1 \oplus \mathbf{a}_2 \oplus \cdots \oplus \mathbf{a}_m \rightarrow \mathbf{b}_1 \oplus \mathbf{b}_2 \oplus \cdots \oplus \mathbf{b}_n$ be an isomorphism.

Define $\mathbf{h}_j : \mathbf{a}_1 \rightarrow \mathbf{a}_1$ as $\mathbf{h}_j = \pi_1 \circ \mathbf{f}^{-1} \circ \nu_j \circ \rho_j \circ \mathbf{f} \circ \mu_1$. Let $\mathbf{h} = \sum_j \mathbf{h}_j : \mathbf{a}_1 \rightarrow \mathbf{a}_1$. Observe \mathbf{h} is an isomorphism. By locality, there is an index j such that \mathbf{h}_j is an isomorphism. This means $\mathbf{a}_1 \cong \mathbf{b}_j$ and we specify $\mathbf{p}(1) = j$. Repeat. \square