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AN INTUITIVE INTERPRETATION OF SYSTEMS OF FOUR-VALUED LOGIC

NICHOLAS RESCHER

It is well known that, despite their formalistic usefulness (in proving the independence of axioms, for example), the *interpretation* of systems of many-valued logic is a matter of considerable difficulty. The aim of the present paper is to present an approach to some of the familiar systems of four-valued logic using a very natural intuitive interpretation of the truth-value assignments.

It would appear on first thought that the most plausible and tempting possibility of interpreting the truth-values of four-valued logic would be somewhat as follows:

Truth-value	Interpretation I	Interpretation II
1	necessarily true	true
2	contingently true	probably true
3	contingently false	probably false
4	necessarily false	false

It is plain that both 1 and 2 must be taken as *designated* truth-values in this scheme; for we wish to keep to the customary meaning of a "tautology" (i.e., a propositional schema always taking designated truth-values) as a schema that is "uniformly *true* for every truth-value assignment to its constituents."

In both instances, negation would be characterized by the familiar matrix:

	$\sim$
1	*4*
2	*3*
3	*2*
4	*1*

But the same vitiating difficulty arises alike in both cases with regard to the matrix for conjunction. Consider what entry is to correspond to *2* & *3*.

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In Interpretation II, it is obvious that this cannot be specified as simply *2*, or as simply *3*, since *both* will clearly be possibilities, and indeed in the instance of "p & $\sim p$," *4* would be the appropriate entry. Again, in Interpretation I, the entry corresponding to *2* & *3* will in general be *3*, but will have to be *4* if—again as in the case of "p & $\sim p$"—the conjuncts are mutually exclusive. The root difficulty is that in the case of either interpretation it is impossible to carry through a truth-functional specification of a matrix for conjunction.

The approach to be presented is based on a modification of Interpretation I which—by an appropriate but minor change in the intended meaning of the truth-values—*removes* the possibility that conjunctions corresponding to the truth-value compounds *2* & *3* or *3* & *2* can possibly be self-consistent, i.e., can yield *3* as well as *4*.

Let it be supposed that there are just two possible alternatives (i.e., incompatible states of affairs), the *actual* state X, and the *possible alternative* state Y. To any proposition whatever we will assign the truth-value *1*, *2*, *3*, or *4* according as it is:

(1) true in X and in Y (i.e., is necessarily true).
(2) true in X but not in Y(i.e., is actually but not necessarily true).[1]
(3) false in X but true in Y (i.e., is actually but not necessarily false).
(4) false in both X and in Y(i.e., is necessarily false).

This interpretation of the truth-values[2] clearly gives rise to the following matrices for negation and conjunction:

	$\sim$	& *1*	*2*	*3*	*4*
1	*4*	*1*	*2*	*3*	*4*
2	*3*	*2*	*2*	*4*	*4*
3	*2*	*3*	*4*	*3*	*4*
4	*1*	*4*	*4*	*4*	*4*

We note that this characterization of negation and conjunction in fact coincides with that of Lewis and Langford.[3]

Even after our proposed interpretation of the four truth-values is given, there still remain several distinct, more or less pluasible and "natural" truth-value characterizations for the modality of possibility '$\Diamond$', and derivatively for a strict implication relationship defined in the familiar way:

$$p \rightarrow q \equiv \mathrm{Df} \sim \Diamond (p \ \& \sim q)$$

The available possibilities seem to be pretty well exhausted by the following list:

	(A)	(B)	(C)	(D)	(E)
1	*1*	*1*	*2*	*1*	*2*
2	*1*	*2*	*2*	*1*	*2*
3	*1*	*2*	*2*	*1*	*2*
4	*4*	*4*	*4*	*3*	*3*

Here the adoption of (A) leads to a system of many-valued truth-tables coinciding with Lewis' "Group III."[4] Alternative (C) leads to "Group IV." (D) leads to "Group I." Cases (B) and (E) are not considered by Lewis. Contrariwise, Lewis' "Group II" and "Group V" are incompatible with our proposed interpretation of the truth-values.[5]

It would appear that, in terms of our semantical approach to the interpretation of the truth-values, alternative (A)—and thus "Group III"—is the most suitable choice of a matrix for the possibility-modality. It is based on a quite plausible construction of the notion of possibility, the convention that: *A proposition asserting the possibility of some thesis is necessarily true if true, necessarily false if false, and true just in case the thesis at issue is not necessarily false*. None of the other alternatives (B) - (E) embodies a set of principles that can readily be defended—as alternative (A) can be—on the basis of traditional conceptions of the matter. Still, the other alternatives are certainly not wholly devoid of claims to consideration.

In general, the result of our discussion may thus be summarized by saying that it has presented a semantical approach to the intuitive interpretation of four-valued logic, and that this mechanism makes possible the interpretation of three of the classical "Groups" of Lewis and Langford, as well as several cognate systems.

NOTES

1. Both *1* and *2* are of course designated truth-values.

2. This approach has been evolved, albeit with crucial modifications, from a proposal put forward by A. N. Prior in a paper on "Many-Valued and Modal Systems: An Intuitive Approach," *The Philosophical Review*, vol. 64 (1955), pp. 626-630.

3. C. I. Lewis and C. H. Langford, *Symbolic Logic*, 1932 (reprinted 1959). See Appendix II. All references here to "Lewis and Langford" are to this appendix which was, however, written by Lewis.

4. Since all the theorems of Lewis' S5 are tautologies of system (A) = "Group III," we can claim that our intuitive interpretation of the four-valued truth tables—like that of Prior mentioned in footnote 2 above—furnishes derivatively an intuitive interpretation of this system of modal logic.

5. The system of alternative (B) is, as its characterization makes evident, intermediate between the systems of alternatives (A) and (C)—that is, between Lewis' "Group III" and "Group IV." This system may well repay further study. Suffice it here to remark that it resembles "Group IV" in that neither of Lewis' axioms A7 ($\sim\Diamond p \rightarrow \sim p$) nor B7 (*modus ponens*) are forthcoming as tautologies.

University of Pittsburgh
Pittsburgh, Pennsylvania